

КРАЙОВА ЗАДАЧА ДЛЯ АБСТРАКТНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ОПЕРАТОРОМ ІНВОЛЮЦІЇ

Вивчається нелокальна задача для диференціально-операторних рівнянь з інволюцією. Встановлено спектральні властивості та умови існування і єдиності розв'язку. Наведено достатні умови при яких система кореневих функцій суттєво несамоспряженого оператора задачі утворює базис Рісса.

We study a nonlocal problem for differential-operator equations of order 2 with involution. The spectral properties of the operator of this problem are analyzed and the conditions for the existence and uniqueness of its solution are established. It is also proved that the system of root functions essentially a nonself-adjoint operator of the analyzed problem forms a Riesz basis.

Statement of the Problem.

Suppose that H is a separable Hilbert space, $A : H \rightarrow H$ is a positive self-adjoint operator with point spectrum $\sigma_p(A) = \{z_k \in \mathbf{R} : z_k \sim \beta k^\alpha, \alpha, \beta > 0, k = 1, 2, \dots\}$, $V(A) \equiv \{v_k \in H : k = 1, 2, \dots\}$ is a system of eigenfunctions that form an orthonormalized basis in the space H , $H(A^s) = \{v \in H : A^s v \in H\}$, $s \geq 0$, $(u, v; H(A^s)) \equiv (A^s u, A^s v; H)$, $\|y; H(A^s)\| \equiv \|A^s y; H\|$, $H_1 \equiv L_2((0, 1), H) = \{u(t) : (0, 1) \rightarrow H, \|u(t); H\| \in L_2(0, 1)\}$, $D_x : H_1 \rightarrow H_1$ is a strong derivative in the space H_1 , i.e. $\left\| \frac{u(x + \Delta x) - u(x)}{\Delta x} - D_x u; H \right\| \rightarrow 0$, $(\Delta x \rightarrow 0)$, $I : L_2(0, 1) \rightarrow L_2(0, 1)$ is operator of involution:

$$\begin{aligned} Iu(x) &\equiv u(1-x), (u(x) \in L_2(0, 1)), \\ p_0 &\equiv \frac{1}{2}(E + I), \quad p_1 \equiv \frac{1}{2}(E - I), \\ L_{2,j}(0, 1) &\equiv \{y \in L_2(0, 1) : y = p_j y\}, \\ H_{1,j} &\equiv \{y(x) \in H_1 : y(x) \equiv p_j y(x)\}, j = 0, 1, \\ H_2 &\equiv \{y(x) \in H_1 : D_x^2 y \in H_1, A^2 y \in H_1\}, \\ \|y; H_2\|^2 &\equiv \|D_x^2 y; H_1\|^2 + \|A^2 y; H_1\|^2, \\ L(H(A^m); H(A^q)) &\text{ is algebra of bounded linear operators } A : H(A^m) \rightarrow H(A^q), \\ (m, q \geq 0), \quad H(A^0) &= H, \quad L(H(A^m)) \equiv L(H(A^m); H(A^m)), \quad H^1 \equiv H\left(A^{\frac{3}{2}}\right), \end{aligned}$$

$$\begin{aligned} H^2 &\equiv H\left(A^{\frac{1}{2}}\right), \quad B^j \in L(H^2), \quad B(x) \equiv \sum_{r=1}^q B_r \left(x - \frac{1}{2}\right)^{2r-1}, \\ B_r &\in L(H_2), \quad B_r v_k = \end{aligned}$$

$\frac{1}{r!} b_{r,k} v_k, B^j v_k = b_k^j v_k, B(x) v_k = b_k(x) v_k, b_{r,k}, b_k^j \in \mathbf{R}, r = 1, 2, \dots; q, j = 0, 1; k = 1, 2, \dots$. Consider the following problem:

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y + B(x)(y(x) - y(1-x)) = f(x), \tag{1}$$

$$l_1 y \equiv y(0) - y(1) = h_1, \tag{2}$$

$$l_2 y \equiv B^0 D_x y(0) + B^1 D_x y(1) = h_2, \tag{3}$$

$$f(x) \in H_1, h_1 \in H^1, h_2 \in H^2.$$

We interpret the solution [16,21] of problem (1), (3) as a function $y(x) \in H_2$ satisfying the equalities

$$\begin{aligned} \|L(D_x, A)y - f; H_1\| = 0, \quad \|l_1 y - h_1; H^1\| = 0, \\ \|l_2 y - h_2; H^2\| = 0. \end{aligned}$$

Differential equation (1) includes operator of involution. First study the properties of involution operator launched C.Babbage (see [5]). In paper [13] T.Carleman introduced the concept of operator shift – a generalization of the concept of involution $Iy(x) \equiv y(1-x)$, $x \in L_2(0, 1)$. Exploration partial differential equations with involution are devoted [2, 3, 4, 7, 11, 13, 22, 32, 37, 37].

Properties spectral problems for ordinary differential and functional- differential equations with involution investigated in the works [1, 9- 10, 17- 20, 23- 25, 29- 31, 34- 36, 38] and [5, 6, 8, 12, 27, 33, 39] respectively.

Solutions of spectral problem

$$L(D_x, A)y = \lambda y(x), \quad (4)$$

$$\lambda \in C, l_1 y = 0, l_2 y = 0,$$

consider as a product $y(x) = u(x)v_k$, $k = 1, 2, \dots$, $u(x) \in W_2^2(0, 1)$.

To determine the functions $u(x)$ obtain spectral problem

$$L_k(D_x)u \equiv -D_x^2 u(x) + z_k^2 u + b_k(x)(u(x) + u(1-x)) = \lambda u(x), \quad (5)$$

$$l_{1,k}u \equiv u(0) - u(1) = 0, \quad (6)$$

$$l_{2,k}u \equiv b_k^0 D_x u(0) + b_k^1 D_x u(1) = 0. \quad (7)$$

Auxiliary spectral problems.

Consider the particular case the problem (5) – (7) if the specified conditions $B(x) = 0$, $b_k^0 = -b_k^1 = 1$.

$$-D_x^2 u(x) + z_k^2 u = \lambda u(x), \quad (8)$$

$$u(0) - u(1) = 0, \quad D_x u(0) - D_x u(1) = 0. \quad (9)$$

Lemma 1. Let $B^0 = -B^1 = E$, $B(x) = 0$. Then problem (8), (9) have point spectrum $\sigma_k \equiv \{\lambda_{k,n} \in \mathbf{R} : \lambda_{k,n} = (2\pi n)^2 + z_k^2, n = 0, 1, \dots\}$ and system of eigenfunctions

$$T \equiv \left\{ t_n^s \in L_2(0, 1) : t_0^0(x) = 1, \right.$$

$$t_n^0(x) = \sqrt{2} \cos 2\pi n x,$$

$$\left. t_n^1(x) = \sqrt{2} \sin 2\pi n x, n \in N \right\}$$

forms a orthonormalized basis in spaces $L_2(0, 1)$.

Property of spectral problem (6) – (8).

We now consider the operator $L_{0,k} : L_2(0, 1) \rightarrow L_2(0, 1)$ of the problem (6) – (8)

$$L_{0,k}u \equiv -D_x^2 u + z_k^2 u, u \in D(L_{0,k}),$$

$$D(L_{0,k}) \equiv \{u \in W_2^2(0, 1) : l_{1,k}u = 0, l_{2,k}u = 0\}.$$

Let

$$v_{k,0}^{0,0}(x) \equiv 1, v_{k,n}^{0,0}(x) \equiv \sqrt{2} \cos 2\pi n x, \quad (10)$$

$$v_{k,n}^{1,0}(x) \equiv \sqrt{2} (1 + \beta_k (2x - 1)) \sin 2\pi n x, \quad (11)$$

$$\beta_k \equiv (b_k^0 - b_k^1)^{-1} (b_k^0 + b_k^1). \quad (12)$$

You can check that

$$L_{0,k}v_{k,n}^{1,0}(x) = \lambda_{k,n}v_{k,n}^{1,0} + \xi_{k,n}^0 v_{k,n}^{0,0}, \quad (13)$$

$$\xi_{k,n}^0 = -8\pi n \beta_k, n = 0, 1, 2, \dots$$

Hence

$V(L_{0,k}) \equiv \{v_{k,n}^{s,0}(x) ; s = 0, 1, n = 0, 1, 2, \dots\}$ are the system of root functions of the operator $L_{0,k}$ in the sense of equality (13).

Lemma 2. Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator $L_{0,k}$ of problem (8), (6), (7) have point spectrum σ_k , and system $V(L_{0,k})$ of root functions complete and minimal in $L_2(0, 1)$.

Proof. We now consider the adjoint spectral problem

$$-D_x^2 w(x) + z_k^2 w = \mu w(x), \mu \in C,$$

$$b_k^1 w(0) + b_k^0 w(1) = 0, \quad (14)$$

$$D_x w(0) - D_x w(1) = 0.$$

The operator of this adjoint problem (14) have point spectrum σ_k , and system of root functions

$$w_{k,0}^{1,0}(x) \equiv 1 - \beta_k (2x - 1), \quad (15)$$

$$w_{k,n}^{0,0}(x) \equiv \sqrt{2} \sin 2\pi n x,$$

$$w_{k,n}^{1,0}(x) \equiv \sqrt{2} (1 - \beta_k (2x - 1)) \cos 2\pi n x. \quad (16)$$

You can check that

$$L_{0,k}v_{k,n}^{1,0}(R_0) = \lambda_{k,n}v_{k,n}^{1,0} + \xi_{k,n}^0v_{k,n}^{0,0}, \quad (17)$$

$$\xi_{k,n}^0 = 8\pi n\beta_k, n = 1, 2, \dots$$

Hence, the system of root functions $V(L_{0,k})$ of the operator $L_{0,k}$ possesses a unique biorthogonal system $W(L_{0,k}) \equiv \{w_{k,n}^{s,0}(x); s = 0, 1; n = 1, 2, \dots\}$ are the system of root functions of the problem (14) in the sense of equality (17).

$$(v_{k,n}^{r,0}, w_{k,q}^{s,0}; L_2(0,1)) = \delta_{r,s}\delta_{k,q},$$

$$(r, s = 0, 1; q, n = 0, 1, \dots).$$

Consider the operators $R_{0,k}, S_{0,k} : L_2(0,1) \rightarrow L_2(0,1)$, $R_{0,k}t_{k,n}^p \equiv v_{k,n}^{p,0}$, $R_{0,k} = E + S_{0,k}$, $p = 0, 1; n = 0, 1, \dots$.

From the definition of the operator $R_{0,k}$ and the completeness of system $V(L_{0,k})$ in space $L_2(0,1)$ we get $S_{0,k} : L_{2,0}(0,1) \rightarrow 0$, $S_{0,k} : L_{2,1}(0,1) \rightarrow L_{2,0}(0,1)$.

To prove that the system $V(L_{0,k})$ forms a Riesz basis (see [15, 26]) in $L_2(0,1)$, it is sufficient, according to formula $R_{0,k} = E + S_{0,k}$, to show that the operator $S_{0,k} : L_2(0,1) \rightarrow L_2(0,1)$ is bounded.

Let ω be an arbitrary element from the space $L_2(0,1)$. We represent ω as a Fourier series in the system T .

$$\omega = \omega_0^0 t_0^0 + \sum_{m=1}^{\infty} \omega_m^0 t_m^0 + \omega_m^1 t_m^1,$$

$$\omega_m^j = (\omega, t_m^j; L_2(0,1)).$$

According to the definition of the operator $S_{0,k}$, we find

$$S_{0,k}\omega = \beta_k(2x-1) \sum_{m=0}^{\infty} \omega_m^0 \sqrt{2} \cos 2\pi mx.$$

Using the ratio

$$\|S_{0,k}\omega; L_2(0,1)\| =$$

$$= \left\| \beta_k(2x-1), \sum_{m=0}^{\infty} \omega_m^0 \sqrt{2} \cos 2\pi mx; L_2(0,1) \right\|,$$

we estimate it

$$\|S_{0,k}\omega; L_2(0,1)\| \leq |\beta_k| \|\omega; L_2(0,1)\|.$$

Hence, $R_{0,k} = E + S_{0,k} \in L(L_2(0,1))$ and $(R_{0,k}^{-1})^* = E - S_{0,k}^* \in L(L_2(0,1))$.

So using theorem N.K.Bary (see theorem 6.2.1 [15]) we obtain the following statement.

Theorem 1. Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator $L_{0,k}$ of problem (8), (6), (7) have point spectrum σ_k , and system $V(L_{0,k})$ of root functions in the sense of equality (12) forms a Riesz basis in $L_2(0,1)$.

Further, we introduce operator $L_{1,k} : L_2(0,1) \rightarrow L_2(0,1)$ of the problem (5) – (7):

$$L_{1,k}u \equiv L_k(D_x)u, u \in D(L_{1,k}),$$

$$D(L_{1,k}) \equiv \{u \in W_2^2(0,1) : l_{1,k}u = 0, l_{2,k}u = 0\}.$$

By the direct substitution we can show that the functions $v_{k,n}^{0,1}(x) = \sqrt{2} \cos 2\pi nx$, is eigenfunctions of operator $L_{1,k}$:

$$v_{k,n}^{0,1}(x) = \sqrt{2} \cos 2\pi nx \in D(L_{1,k}),$$

$$L_{1,k}v_{k,n}^{0,1}(x) = \lambda_{k,n}v_{k,n}^{0,1}(x), k = 1, 2, \dots$$

Root function of operator $L_{1,k}$, defined by relation

$$v_{k,n}^{1,1}(x) \equiv \sqrt{2} \sin 2\pi nx + v_{k,n}^0(x) + v_{k,n}^1(x) + v_{k,n}^2(x), \quad (18)$$

$$v_{k,n}^0(x) \equiv \sqrt{2}\beta_k(2x-1) \sin 2\pi nx,$$

$$\beta_k \equiv (b_k^0 - b_k^1)^{-1}(b_k^0 + b_k^1), \quad (19)$$

$$v_{k,n}^1(x) \equiv \sqrt{2}\beta_{k,n}^1(2x-1) \sin 2\pi nx, \quad (20)$$

$$v_{k,n}^2(x) \equiv \sqrt{2} \sum_{j=1}^{q+1} c_{2j+1,n,k} \frac{1}{(2j+1)!} \left(x - \frac{1}{2}\right)^{2j+1} \sin 2\pi n x + \sqrt{2} \sum_{j=0}^q c_{2j,n,k} \frac{1}{(2j)!} \left(x - \frac{1}{2}\right)^{2j} \cos 2\pi n x. \quad (21)$$

Support functions $v_{k,n}^1(x), v_{k,n}^2(x)$ so choose

$$(L_k(Dx) - \lambda_{k,n}) v_{k,n}^{1,1}(x) = \xi_{k,n}^1 v_{k,n}^{0,1}(x) = 0, \quad (22)$$

$$l_{1,k} (v_{k,n}^1(x) + v_{k,n}^2(x)) = 0, \quad (23)$$

$$l_{2,k} (v_{k,n}^1(x) + v_{k,n}^2(x)) = 0. \quad (24)$$

Substitute equity (21) ratio (22) and equal define parameters $c_{r,n,k}$

$$c_{2s,n,k} = \frac{1}{2\pi n} \left(b_{s-1,k} \pm \frac{1}{(4\pi n)^2} b_{s-1,k} + \dots \pm \frac{1}{(4\pi n)^{2q-2s+2}} b_{q,k} \right), \quad (25)$$

$$c_{2s+1,n,k} = -\frac{1}{2(\pi n)^2} \left(b_{s,k} \pm \frac{1}{(4\pi n)^2} b_{s+1,k} + \dots \pm \frac{1}{(4\pi n)^{2q-2s}} b_{q,k} \right). \quad (26)$$

Taking into account that $v_{k,n}^1(x) + v_{k,n}^2(x) \in L_{2,0}(0,1)$, $\sqrt{2} \sin 2\pi n x + v_{k,n}^0(x) = v_{k,n}^{1,0}(x) \in D(L_{0,k})$ obtain equality

$$l_{1,k} v_{k,n}^{1,1} = l_{1,k} (v_{k,n}^{1,0} + v_{k,n}^1 + v_{k,n}^2) = 0.$$

If

$$D_x v_{k,n}^1(0) + D_x v_{k,n}^2(0) \equiv 2\sqrt{2} \beta_{k,n}^1 +$$

$$+ 2\sqrt{2} \sum_{j=1}^{q+1} c_{2j+1,n,k} \frac{1}{(2j)!} \left(\frac{1}{2}\right)^{2j} = 0,$$

$$\beta_{k,n}^1 = - \sum_{j=1}^{q+1} c_{2j+1,n,k} \frac{1}{(2j)!} \left(\frac{1}{2}\right)^{2j} \text{ then}$$

$$l_{2,k} v_{k,n}^{1,1} = l_{2,k} v_{k,n}^{1,0} + l_{2,k} (v_{k,n}^1 + v_{k,n}^2) = l_{2,k} (v_{k,n}^1 + v_{k,n}^2) = 0.$$

Hence, $v_{k,n}^{1,1}(x) \in D(L_{1,k})$,

$$L_{1,k} v_{k,n}^{1,1}(x) \equiv \lambda_{k,n} v_{k,n}^{1,1}(x) + \xi_{k,n}^1 v_{k,n}^{0,1}(x), \quad (27)$$

here

$$\xi_{k,n}^1 = 8\pi n (\beta_n + \beta_n^1), k = 1, 2, \dots, n = 0, 1, \dots$$

Consider the system functions $V_k \equiv \{v_{k,n}(x), v_{k,n}^{0,1}(x) \in L_2(0,1), k = 1, 2, \dots, n = 0, 1, \dots\}$,

here

$$v_{k,n}(x) \equiv \sqrt{2} \cos 2\pi n x + v_{k,n}^0(x) + v_{k,n}^1(x), v_{k,n}^{0,1}(x) = \sqrt{2} \sin 2\pi n x, k = 1, 2, \dots, n = 0, 1, \dots \quad (28)$$

Lemma 3. Let $b_k^0 \neq b_k^1, k = 1, 2, \dots$ Then the system functions (28) forms a Riesz basis in space $L_2(0,1)$.

Lemma proved just as Theorem 1.

Lemma 4. Let $b_k^0 \neq b_k^1, k = 1, 2, \dots$ Then the system functions V_k and $V(L_{1,k})$ a square close in space $L_2(0,1)$.

Proof. A formula (19), (25), (26) that follows.

So get inequality

$$\sum_{n=0}^{\infty} \|v_{k,n}^{1,1} - v_{k,n}, L_2(0,1)\|^2 = \sum_{n=0}^{\infty} \|v_{k,n}^2, L_2(0,1)\|^2 < \infty.$$

Of the results [14] follows that the system V_k is complete and minimal in space $L_2(0,1)$.

Hence, the system of root functions $V(L_{1,k})$ of the operator $L_{1,k}$ possesses a unique bi-orthogonal system $W(L_{1,k})$

$$(v_{k,n}^{r,1}, w_{q,n}^{s,1}; L_2(0,1)) = \delta_{r,s} \delta_{k,q}, (r, s = 0, 1; q, k = 0, 1, \dots)$$

So using theorem N.K.Bary (see theorem 6.2.3 [15]) we obtain the following statement.

Theorem 2. Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator $L_{1,k}$ of problem (5) – (7) have point spectrum σ_k , and system $V(L_{1,k})$ of root functions in the sense of equality (27) forms a Riesz basis in space $L_2(0, 1)$.

The spectral problem (1), (2).

We now consider the operator $L : H_1 \rightarrow H_1$ of problem (1), (2):

$$Ly \equiv L(D_x, A)y, y \in D(L),$$

$$D(L) \equiv \{y \in H_2 : l_1 y = 0, l_2 y = 0\}.$$

Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator L of problem (1), (2) have point spectrum

$$\sigma \equiv \{\lambda_{k,n} \in R : \lambda_{k,n} \equiv 4n^2\pi^2 + z_k^2, \\ n = 0, 1, \dots; k = 1, 2, \dots\},$$

and system of root functions

$$V(L) \equiv \{v_{k,n}^s(L) \in H_1 : v_{k,n}^s(L) = v_{k,n}^{s,1}(x)v_k, \\ s = 0, 1; n = 0, 1, 2, \dots; k = 1, \dots\},$$

$$v_{k,0}^0(L) \equiv v_k, v_{k,n}^0(L) \equiv \sqrt{2} \sin 2\pi n x v_k,$$

$$v_{k,n}^1(L) \equiv v_{k,n}^{1,1}(x)v_k, n, k = 1, 2, \dots.$$

System $V(L)$ of root functions of the operator L possesses a unique biorthogonal system

$$W(L) \equiv \{w_{p,m}^s \in H_1 : w_{p,m}^s \equiv w_{p,m}^{s,1}(x)v_m, \\ p = 1, 2, \dots; m = 0, 1, \dots; s = 0, 1\}$$

in the sense of equality

$$(v_{k,m}^j, w_{p,n}^s; H_1) = \delta_{j,s} \delta_{k,p} \delta_{m,n}.$$

Hence, we obtain the following statement.

Lemma 5. Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator L of problem (1), (2) have complete and minimal in H_1 system of root functions $V(L_0)$.

Then $\|R_{1,k}\omega; L_2(0, 1)\| \leq C \|\omega; L_2(0, 1)\|$, $\|(R_{1,k}^{-1})\omega; L_2(0, 1)\| \leq C \|\omega; L_2(0, 1)\|$, $C > 0$.

Further, we introduce operator

$$B \equiv (B^0 + B^1)(B^0 - B^1)^{-1}.$$

Theorem 3. Let $B \in L(H^2)$, $B_r \in L(H_2)$, $r = 1, 2, \dots, q$. Then the operator L of problem (1), (2), have system of root functions $V(L)$ forms a Riesz basis in H_1 .

Proof. Let

$$T_1 \equiv \{t_{k,n}^s \in H_1 : t_{k,n}^j \equiv t_n^j v_k, t_n^j \in T, \\ v_k \in V(A), j = 0, 1, n = 0, 1, \dots, k = 1, 2, \dots\}.$$

Consider the operators $R_1, S_1 : H_1 \rightarrow H_1$, $R_1 t_{k,n}^s \equiv v_{k,n}^{s,1}$, $R_1 = E + S_1$.

From the definition of the operator R_1 for any $g = \sum_{s,k,m} g_{k,m}^s t_{k,m}^s \in H_1$, $g_{k,m}^s = (g, t_{k,m}^s; H_1)$ we get

$$R_1 g = \sum_{j,k,m} g_{k,m}^j v_{k,m}^{j,1} \in H_1,$$

$$(R_1^{-1})^* \sum_{j,k,m} g_{k,m}^j t_{k,m}^j = \sum_{j,k,m} g_{k,m}^j w_{k,m}^{j,1}$$

$$\|R_1 g, H_1\| \leq \max \|E + S_{1,k}, L(H_1)\| \|g, H_1\| =$$

$$= C_1 \|g, H_1\|,$$

$$\|(R_1^{-1})^* g, H_1\| \leq \max \|E - (S_{1,k})^*, L(H_1)\|$$

$$\|g, H_1\| = C_2 \|g, H_1\|, k = 1, 2, \dots.$$

So using theorem N.K.Bary (see theorem 6.2.1 [15]) we obtain the following statement of the theorem 3.

3. Property of problem (1), (2).

Replaced condition (2) on equivalent terms

$$l_1 y \equiv y(0) - y(1) = h_1, \\ l_3 y \equiv D_x y(0) - D_x y(1) + \\ + B(D_x y(0) + D_x y(1)) = h_3. \quad (29)$$

Here $h_3 \equiv 2(B^0 - B^1)^{-1} h_1$.

Consider the particular case the problem (1), (29) if the specified conditions $B = 0$, $B_0 = 0$

$$-D_x^2 y(x) + A^2 y = g(x), \quad (30)$$

$$y(0) - y(1) = g_1, \\ D_x y(0) - D_x y(1) = g_2, g_j \in H^j, j = 1, 2. \quad (31)$$

Theorem 4. Let $B = 0, B_0 = 0$. Then for any $g \in H_1, g_1 \in H^1, g_2 \in H^2$, there exists a unique solution of problem (30), (31).

Proof. We seek the solution of this problem in the form $y = u + v$, there u is the solution of the problem

$$\begin{aligned} -D_x^2 u(x) + A^2 u &= g(x), y(0) - y(1) = 0, \\ D_x y(0) - D_x y(1) &= 0 \end{aligned} \quad (32)$$

and v is the solution of the problem

$$\begin{aligned} -D_x^2 v(x) + A^2 v(x) &= 0, v(0) - v(1) = g_1, \\ D_x v(0) - D_x v(1) &= g_2. \end{aligned} \quad (33)$$

Consider the problem (32). We expand the functions $u(x), g(x)$ in a series in the orthonormalized T_1 basis in the space H_1 :

$$u = \sum_{s,k,m} u_{k,m}^s t_{k,m}^s, \quad u_{k,m}^s = (u, t_{k,m}^s; H_1),$$

$$g = \sum_{s,k,m} g_{k,m}^s t_{k,m}^s, \quad g_{k,m}^s = (g, t_{k,m}^s; H_1).$$

We estimate a number

$$-D_x^2 u = \sum_{s,k,m} (2\pi m)^2 ((2\pi m)^2 + z_k^2)^{-1} g_{k,m}^s t_{k,m}^s,$$

$$\|D_x^2 u; H_1\| \leq \|g; H_1\|,$$

$$A^2 u = \sum_{s,k,m} z_k^2 ((2\pi m)^2 + z_k^2)^{-1} g_{k,m}^s t_{k,m}^s,$$

$$\|A^2 u; H_1\| \leq \|g; H_1\|.$$

Hence

$$\|u; H_2\| \leq \sqrt{2} \|g; H_1\|. \quad (34)$$

Consider the problem (33). Further, we introduce operators, $Y_j(x, A) \equiv e^{Ax} + (-1)^j e^{A(1-x)} \in L(H^2; H_2), j = 0, 1$. The solution of the differential equation (33) has the form

$$v(x) = Y_0(x, A)\varphi_0 + Y_1(x, A)\varphi_1 \quad (35)$$

where φ_0, φ_1 are unknown.

To determine the, $\varphi_0, \varphi_1 \in H^1$ we substitute expression (35) in the condition (33) and obtain

$$\varphi_1 = \frac{1}{2} W_1(0, A)^{-1} g_1, \varphi_0 = \frac{1}{2} W_1(0, A)^{-1} A^{-1} g_2.$$

Hence,

$$\begin{aligned} v &= \frac{1}{2} W_1(x, A) W_1(0, A)^{-1} g_1 + \\ &+ \frac{1}{2} W_0(x, A) W_1(0, A)^{-1} A^{-1} g_2, \end{aligned}$$

$$\|v; H_2\|^2 \leq C \left(\|g_1; H^1\|^2 + \|g_2; H^2\|^2 \right). \quad (36)$$

Therefore follows from inequalities (34), (36) inequality

$$\begin{aligned} \|y; H_2\|^2 &\leq C_1 \left(\|g; H_1\|^2 + \right. \\ &\left. + \|g_1; H^1\|^2 + \|g_2; H^2\|^2 \right). \end{aligned}$$

We now return to the original problem (1), (2). Consider in connection problem as the sum $y = y_0 + y_1, y_j \in H_{1,j} \cap H_{2,j}, j = 0, 1$.

To determine the unknowns $y_j \in H_{1,j}$ get the problem

$$-D_x^2 y_1(x) + A^2 y_1(x) = f_1(x), f_1(x) \in H_{1,1},$$

$$y_1(0) - y_1(1) = h_1, D_x y_1(0) - D_x y_1(1) = 0,$$

$$-D_x^2 y_0(x) + A^2 y_0(x) = f_0(x) - 2B(x)y_1(x),$$

$$f_0(x) \in H_{1,0},$$

$$y_0(0) - y_0(1) = 0, D_x y_0(0) - D_x y_0(1) =$$

$$= h_3 - B(D_x y_1(0) + D_x y_1(1)),$$

$$\|y; H_2\|^2 \leq C \left(\|f; H_1\|^2 + \|h_1; H^1\|^2 + \right.$$

$$\left. + \|h_2; H^2\|^2 \right), (C > 0).$$

For unknown functions $y_j \in H_{1,j}$ get that problem is a particular case of the problem (21), (22). Hence the statement is correct.

Theorem 5. Let $B \in L(H^1)$, $B(x) \in L(H_2)$. Then for any $f \in H_1, h_1 \in H^1, h_2 \in H^2$, there exists a unique solution of problem (1), (2) and fair inequality

$$\|y; H_2\|^2 \leq C \left(\|f; H_1\|^2 + \|h_1; H^1\|^2 + \|h_2; H^2\|^2 \right), (C > 0).$$

Conclusion.

Investigated the spectral properties essentially a nonself-adjoint operator nonlocal problems for abstract differential equation with involution.

Studied the problem solution is built on a number of root functions

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