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ON THE THEORY OF GENERALIZED EVEN TOEPLITZ KERNELS ON THE FINITE INTERVAL

Доведено інтегральне зображення парних узагальнених ядер Тепліца на скінченному інтервалі. Це доведення базується на спектральній теорії відповідного диференціального оператора, що діє в гільбертовому просторі, побудованому за таким ядром.

A proof of integral representation of the generalized even Toeplitz kernels on the finite interval is given. This proof is based on the spectral theory of corresponding differential operator which acts in the Hilbert space constructed from the kernel of this sort.

Introduction. In the article [6] M.G. Krein using the method of directional functionals obtained the integral representation positive definite kernels $K(x,y)$ $(x, y \in R^1)$. Yu.M. Berezansky in [1] developed general approach to the integral representation of positive definite kernels, which is based on the theory of generalized eigenfunction expansion of differential (and other) operators in space constructed from $K(x,y)(x,y \in R^1)$. This approach gives a possibility to prove the integral representation Toeplitz kernels [see 2]. In the article [7] the author is considering the integral representation of even Toeplitz kernels. In [5] the integral representation of generalized Toeplitz kernels is proved. In this article we are proving integral representation of generalized even Toeplitz kernels. This proof is based on the books [1-4] and the article [5].

Let $I = (-l, l)$, $0 < l < \infty$ and $I \times I \ni (x, y) \mapsto K(x, y) \in \mathbb{R}^1$ be a bounded measurable (with respect to Lebesgue measure $dxdy$) even real-valued kernel. Recall that this kernel K is called positive definite if for every $f \in C^{\infty}_{fin}(I)$,

$$\iint\limits_{I\times I} K(x,y)f(y)f(x)dxdy \geq 0.$$

It is obvious that in this inequality, it is possible to take f to be continuous with compact support or integrable on I with respect to dx, etc.

This kernel is called a even Toeplitz kernel if the even real-valued function $(-2l, 2l) \ni t \mapsto k(t) \in C^1$ exists such that

$$K(x,y) = k(x-y), \quad x,y \in I$$

(such a function k is said to be a positive definite function). For even Toeplitz kernel the following integral representation

$$K(x,y) = k(x-y) = \int\limits_{\mathbb{R}^1_+} \cos\sqrt{\lambda}(x-y)d\sigma(\lambda), \qquad x, y \in I$$

where $d\sigma(\lambda)$ is a nonnegative bounded Borel measure on $\mathbb{R}^1_+$. In the case $I = \mathbb{R}^1$, this measure is determined by K uniquely (see [6], p. 284).

Formulation of result. Let I be an interval of the for $I = (-l, l)$ and let $I_1 = I \cap [0,\infty)$, $I_2 = I \cap (-\infty, 0)$. Denote $\forall\ \alpha, \beta = 1, 2$,

$$I_{\alpha\beta} = \{t = x - y \mid x \in I_\alpha,\ y \in I_\beta\}, \qquad (1)$$

i. e. $I_{11} = I_{22} = (-l, l)$, $I_{12} = [0, 2l)$, $I_{21} = (-2l, 0)$.

Consider a bounded even positive definite kernel

$$I \times I \ni (x,y) \mapsto K(x,y) \in \mathbb{R}^1.$$

This even kernel is, by definition, a generalized Toeplitz (e.g.T.) kernel, if there exist four continuous functions $I_{\alpha\beta} \ni t \to k_{\alpha\beta}(t) \in \mathbb{R}^1$ such that

$$K(x,y) = k_{\alpha\beta}(x-y) \qquad (2)$$
$$(x,y) \in I_\alpha \times I_\beta,\ \alpha,\beta = 1,2.$$

Any positive definite kernel is Hermitian ($K(x,y)=\overline{K(y,x)},\ (x,y)\in I\times I$), therefore representation (2) gives:

$$\begin{aligned} k_{\alpha\alpha}(t)=\overline{k_{\alpha\alpha}(-t)},\quad t\in I_{\alpha\alpha},\ \alpha=1,2;\\ k_{12}(t)=\overline{k_{21}(-t)},\quad t\in I_{12}. \end{aligned} \tag{3}$$

For every $\alpha,\beta=1,2$, the restriction $K\restriction(I_\alpha\times I_\beta)$ is a continuous function $k_{\alpha\beta}(x-y)$ hence the function k is continuous on $I\times I$.

Theorem. *For every real-valued generalized Toeplitz even kernel, the following integral representation takes place:*

$$K(x,y)=$$
$$=\int_{\mathbb{R}^1_+}\cos\sqrt{\lambda}(x-y)\sum_{\alpha,\beta=1}^{2}k_\alpha(x)k_\beta(y)d\sigma(\lambda), \tag{4}$$
$$(x,y)\in I\times I.$$

Here k_α is the characteristic function of the interval I_α, $\alpha=1,2$, and $d\sigma(\lambda)=(d\sigma_{\alpha\beta}(\lambda))^2_{\alpha,\beta=1}$ is finite nonnegative matrix-valued Borel "spectral"measure on $\mathbb{R}^1_+$ ($d\sigma_{11}(\lambda)$ and $d\sigma_{22}(\lambda)$ are nonnegative finite scalar measures, $d\sigma_{12}(\lambda)=\overline{d\sigma_{21}(\lambda)}$ has bounded variation on $\mathbb{R}^1_+$). Conversely, every even kernel of form (4) with a finite nonnegative measure $d\sigma(\lambda)$ is a even real-valued a generalized Toeplitz kernel.

Proof of theorem. Using a given e.g.T kernel K we introduce a quasiscalar product

$$\langle f,g\rangle_{H_k}=\iint_{I\times I}K(x,y)f(y)\overline{g(x)}dxdy,\ \ f,g\in L^2, \tag{5}$$

where $L^2=L^2(I,dx)$, dx is the Lebesgue measure. Identifying all $f\in L^2$ for which $\langle f,f\rangle_{H_k}=0$ with zero and then completing the set of the corresponding classes

$$\hat f=\{h\in L^2\mid\langle f-h,f-h\rangle_{H_k}=0\},\ \ f\in L^2 \tag{6}$$

we obtain a space H_k in which our operators will act. Vectors from H_k are denoted by $F,\mathcal{Y},\ldots$.

Consider the rigging (chain)

$$W^{-2}_{2,0}(I)\supset L^2\supset W^2_{2,0}(I), \tag{7}$$

where $W^2_{2,0}(I)$ is the subspace of the Soboler space $W^2_2(I)$ consisting of functions $u\in W^2_2(I)$ for which $u(0)=0;\ u'(0)=0$. This rigging is quasinuclear, i.e. the imbedding $W^2_{2,0}(I)\circlearrowleft L^2$ is quasinuclear. Using (7) it is possible to construct a quasinuclear rigging

$$H_{k,-}\supset H_k\supset H_{k,+}, \tag{8}$$

where the space $H_{k,+}$ consists of classes $\hat u,\ u\in W^2_{2,0}(I)$, with the corresponding scalar product. This scalar product $\langle\hat u,\hat v\rangle_{H_k}$, is equal to $(u_N,v_N)_{W^2_{2,0}(I)}$, where u_N is a special unique vector from $W^2_{2,0}(I)$ belonging to $\hat u$. For details of the above construction see [3, Chapter 5, §5, Subsect 5.1] or in [1, Chapter 8, §1].

Denote by $k_{\alpha\beta}(x,y)$ the characteristic function of set $I_\alpha\times I_\beta$ and introduce the kernels

$$\begin{aligned} K_{\alpha\beta}(x,y)=k_{\alpha\beta}(x,y)K(x,y),\\ (x,y)\in I\times I,\ \alpha,\beta=1,2. \end{aligned} \tag{9}$$

Using (2) we can write:

$$K(x,y)=\sum_{\alpha,\beta=1}^{2}K_{\alpha\beta}(x,y)=$$
$$=\sum_{\alpha,\beta=1}^{2}k_{\alpha\beta}(x,y)k_{\alpha\beta}(x-y),\ (x,y)\in I\times I. \tag{10}$$

Representation () permit to rewrite expression (5) in the form

$$\langle f,g\rangle_{H_k}=\sum_{\alpha,\beta=1}^{2}\iint_{I_\alpha\times I_\beta}k_{\alpha\beta}(x-y)f(y)\overline{g(x)}dxdy, \tag{11}$$
$$f,g\in L^2(I,dx).$$

Introduce now operators connected with our problem. Denote by $C^\infty_0(I)$ the set of all functions u,u' from $C^\infty(I)$ which are equal to zero in some neighborhoods of the points $-\infty,0,\infty$. On such finite functions, we define the operator

$$\begin{aligned} \operatorname{Dom}(A')=C^\infty_0(I)\ni u\to A'u=\\ =-\frac{d^2}{dx}=:(\Im u)(x). \end{aligned} \tag{12}$$

Lemma. The operator A' Hermitian with respect to quasiscalar product (5) i. e.

$$\langle A'u, v\rangle_{H_k} = \langle u, A'v\rangle_{H_k}, \quad u, v \in C_0^\infty(I) \quad (13)$$

and

$$\langle A'u, u\rangle_{H_k} \geq 0. \quad (14)$$

Proof. Using representation (11) for (5) we get

$$\langle A'u, v\rangle_{H_k} = \\ = \sum_{\alpha,\beta=1}^{2} \iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u''(y)v(x)dxdy, \quad (15) \\ u, v \in C_0^\infty(I).$$

Fix some α, β and function $u, v \in C^\infty(I)$, and consider the integral

$$-\iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u''(y)v(x)dxdy. \quad (16)$$

Extend in an arbitrary way the function $k_{\alpha\beta}(x,y)$ from $I_{\alpha\beta}$ onto $\mathbb{R}^1$ as a bounded function and extend the functions $u(y)$ and $v(x)$ to be zero for $y \in \mathbb{R}^1 \backslash I_\beta,\ x \in \mathbb{R}^1 \backslash I_\alpha$. Because the functions u, u' from $C_0^\infty(I)$ are equal to zero in some neighborhoods of the points $-\infty, 0, \infty$, these extended u, v belongs to $C_{fin}^\infty(I)$ and we can rewrite integral (16) in following way:

$$-\iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u''(y)v(x)dxdy = \\ = -\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} k_{\alpha\beta}(x-y)u''(y)v(x)dxdy = \\ = -\int_{\mathbb{R}^1} k_{\alpha\beta}(t)\left(\int_{\mathbb{R}^1} u''(x-t)v(x)dx\right)dt = \quad (17) \\ = -\int_{\mathbb{R}^1} k_{\alpha\beta}(t)\left(\int_{\mathbb{R}^1} u(x-t)v''(x)dx\right)dt = \\ = -\iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u(y)v''(x)dxdy$$

(we used above two integration by parts formula). Applying equality (17) to each term (15) give (13).

As so each term (15)

$$-\iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u''(y)u(x)dxdy = \\ = -\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} k_{\alpha\beta}(x-y)u''(y)u(x)dxdy = \\ = -\int_{\mathbb{R}^1} k_{\alpha\beta}(t)\left(\int_{\mathbb{R}^1} u''(x-t)u(x)dx\right)dt = \\ = \int_{\mathbb{R}^1} k_{\alpha\beta}(t)\left(\int_{\mathbb{R}^1} u'(x-t)u'(x)dx\right)dt = \\ = \iint_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)u'(y)u'(x)dxdy \geq 0,$$

we get (14). $\square$

The Hermitness of A' in $\langle \cdot, \cdot\rangle_{H_k}$ gives that this operator in a natural way can be extended to smooth classes (6); $A'\hat{u} = (A'u)^\wedge$. So, as a result we have, in the Hilbert space H_k, densely defined Hermitian operator A', and $A' \geq 0$, let A be its closure, $A = (A')^\wedge$. Operator A be self-ajoint, as so g.T. kernels are real-valued and bounded. For the application of the spectral projection theorem to operator A, it is necessary to construct an extension of rigging (8).

Turn $C_0^\infty(I)$ into a linear topological space by introducing the convergence $C_0^\infty(I) \ni u_n \to u \in C_0^\infty(I)$ which is uniform for the functions and all their derivatives that have uniformly bounded supports.

Consider the space of classes

$$D = (C_0^\infty(I))^\wedge \quad (18)$$

and endow it with the quotient topology via the map $u \to \hat{u}$. As result, we construct an extension of chain (8)

$$H_{k,-} \supset H_k \supset H_{k,+} \supset D; \quad (19)$$

the in bedding $D \circlearrowleft H_{k,+}$ is dense and continuous.

Chain (19) is standardly connected with the operator $A:\ D \subset \mathrm{Dom}(A)$ and the restriction $A \restriction D$ acts continuously from D into $H_{k,+}$.

In our general situation when K may be degenerate we apple some corollary of this theorem (Theorem 5.1 from [3], Chapter 5, §5); we use only is special case for a single selfadjoint operator and the spaces $\mathcal{Y}_+ = \mathcal{Y}_0 = \mathcal{Y}_- = L^2$, $H_+ = W^2_{2,0}(\mathbb{R}^1, dx)$. The above mentioned Theorem 5.1 in the necessary special case asserts the following.

Proposition 1. For the kernel K, the following representation holds:

$$K = \int\limits_{\mathbb{R}^1_+} \Omega(\lambda) d\rho(\lambda). \tag{20}$$

Here $\Omega(\lambda) \in H_- \otimes H_-$, $H_- = W^{-2}_{2,0}(\mathbb{R}^1, dx)$, is an elementary positive definite kernel and the norm $\|\Omega(\lambda)\|_{H_-\otimes H_-}$ is bounded with respect to λ; the measure ρ is a Borel nonnegative finite measure on the axis $\mathbb{R}^1_+$. The integral in (20) convergent in the norm of the space $H_- \otimes H_-$.

The positive definiteness of the kernel $\Omega(\lambda)$, $\lambda \in \mathbb{R}^1_+$, mean that $\forall\, u \in H_+ = W^2_{2,0}(\mathbb{R}^1, dx)$

$$(\Omega(\lambda), u \otimes u)_{L^2 \otimes L^2} \geq 0. \tag{21}$$

The elementary character of $\Omega(\lambda)$ consists in validity of the following equality:

$$\begin{gathered}(\Omega(\lambda), v \otimes (A'u))_{L^2\otimes L^2} = \\ = (\Omega(\lambda), (A'v) \otimes u)_{L^2\otimes L^2} = \lambda(\Omega(\lambda), v \otimes u), \\ u, v \in C^\infty_0(\mathbb{R}^1).\end{gathered} \tag{22}$$

Observe that, in terms of the tensor product, expression (5) has the form

$$\langle f, g\rangle_{H_k} = (K, g \otimes f)_{L^2\otimes L^2}, \quad f, g \in L^2.$$

Proof of Proposition 1. In the beginning we will consider nondegenerated kernel K. In this case we obtain representation (20) from the spectral projection theorem (see [4] chapter 15, §2, Theorem 2.1). We apply this theorem to the operator B standardly connected with the chain (19), now $\hat{f} = f$ and, therefore $D = C^\infty_0(I)$.

Using this theorem, it is possible to assert that the following statement takes place.

Proposition 2. On the axis $\mathbb{R}^1_+$ there exists a Borel nonnegative finite measure ρ for which the following Parseval equality holds:

$$\begin{gathered}\langle u, v\rangle_{H_k} = \int\limits_{\mathbb{R}^1_+} \langle P(\lambda)u, v\rangle_{H_k} d\rho(\lambda), \\ u, v \in H_{k,+} = W^{-2}_2(\mathbb{R}^1, q(x)dx).\end{gathered}$$

Here $P(\lambda)$ is defined, for ρ-almost every $\lambda \in \mathbb{R}^1_+$ and it is an operator-valued function values of which are operators from $H_{k,+}$ into $H_{k,-}$. The corresponding Hilbert-Schmidt norm $\|P(\lambda)\|_{H.S.} \leq 1$.

The operator $P(\lambda)$ "prosects" onto generalized eigenvectors of the operator B corresponding to the "eigenvalue" λ in the following sense: $\forall u \in H_{k,+}$

$$\begin{gathered}\langle P(\lambda)u, A'v\rangle_{H_k} = \lambda\langle P(\lambda)u, v\rangle_{H_k}, \\ v \subset D = C^\infty_0(\mathbb{R}^1) \;\; (B \restriction C^\infty_0(\mathbb{R}^1) = A').\end{gathered}$$

This operator is nonnegative with respect to chain (19), i.e.

$$\langle P(\lambda)u, u\rangle_{H_k} \geq 0, \quad u \in H_{k,+}.$$

Then Proposition 1 follows from Proposition 2 in the case of a nondegenerate K see [5].

Note that, for a degenerate e.g.T. kernel K, the proof of Proposition 1 is the same as above but technically it is more complicated.

The proof of Theorem 1 is based on Proposition 1 and the following assertion.

Let $\mathcal{Y} \in \mathbb{R}^1$ and $\xi \in W^{-2}_2(\mathcal{Y})$ be a generalized solution of the equation ($\Im\xi = \lambda\xi$, $\Im = \Im^+$ is given (12), $\lambda \in \mathbb{R}^1_+$), i.e. the following equality holds:

$$(\xi, \Im v)_{L^2(\mathcal{Y})} = \lambda(\xi, v)_{L^2(\mathcal{Y})}, \quad v \in C^\infty_0(\mathcal{Y}). \tag{23}$$

Then, automatically, $\xi \in C^\infty(\bar{\mathcal{Y}})$ and has the form

$$\begin{gathered}\xi(x) = C\cos\sqrt{\lambda}x + C\sin\sqrt{\lambda}x, \\ x \in \bar{\mathcal{Y}},\; \lambda \in \mathbb{R}^1_+,\end{gathered} \tag{24}$$

where $C \in \mathbb{R}^1$ is some constant.

This result is a special case of the theorem about smoothness, up to the boundary, of a generalized solution of ordinary differential equation (see [4], Chapter 16, §6, Theorem 6.1).

Proof of Theorem. Denote by $H_{\alpha,+}$ the subspace of $H_+ = W^2_{2,0}(I)$ consisting of functions from H_+, which are equal to zero on $I\backslash I_\alpha$, and let

$$H_{\alpha\beta,+} = H_{\alpha,+} \otimes H_{\beta,+} \subset H_+ \otimes H_+, \ \alpha,\beta{=}1,2.$$

Let

$$H_{\alpha,-} \supset L^2(I_\alpha) \supset H_{\alpha,+}, \quad \alpha = 1,2 \qquad (25)$$

be the rigging connected with the spaces $L^2(I_\alpha)$ and $H_{\alpha,+}$.

Fix $\lambda \in \mathbb{R}^1_+$ and denote by $\Omega_{\alpha\beta}(\lambda)$ the restriction of the generalized function $\Omega(\lambda) \in H_- \otimes H_-$ to $H_{\alpha\beta,+}$, i.e.

$$\begin{aligned}(\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes \bar{u}_\beta)_{L^2(I_\alpha)\otimes L^2(I_\beta)} = \\ = (\Omega(\lambda), v_\alpha \otimes \bar{u}_\beta)_{L^2\otimes L^2}, \\ v_\alpha \in H_{\alpha,+},\ u_\beta \in H_{\beta,+},\ \alpha,\beta = 1,2. \quad (26)\end{aligned}$$

Evidently, we have the equality

$$\begin{aligned}(\Omega(\lambda), v \otimes \bar{u})_{L^2\otimes L^2} = \\ = \sum_{\alpha,\beta=1}^{2} (\Omega_{\alpha\beta}(\lambda), k_\alpha(x)v(x)k_\beta(y)u(y))_{L^2(I_\alpha)\otimes L^2(I_\beta)}, \\ u, v \in H_+. \qquad (27)\end{aligned}$$

We will find the expression for $\Omega_{\alpha\beta}(\lambda)$; below $\alpha,\beta = 1,2$ are fixed. Note at first that the bilinear form

$$\begin{aligned}H_{\beta,+} \otimes H_{\alpha,+} \ni \langle u_\beta, v_\alpha\rangle \mapsto a(u_\beta, v_\alpha) := \\ := (\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes \bar{u}_\beta)_{L^2(I_\alpha)\otimes L^2(I_\beta)}\end{aligned} \qquad (28)$$

is continuous. Indeed, because $\|\Omega(\lambda)\|_{H_-\otimes H_-}$, $\lambda \in \mathbb{R}^1_+$, is bounded, we have using definition (26):

$$|a(u_\beta, v_\alpha)| \le C\|u_\beta\|_{H_{\beta,+}} \cdot \|v_\alpha\|_{H_{\alpha,+}}.$$

Using chains (25) we can assert that there exist such continuous operators $R:\ H_{\beta,+} \to H_{\alpha,+}$ and $S:\ H_{\alpha,+} \to H_{\beta,-}$ that we have the representations

$$\begin{aligned}a(u_\beta, v_\alpha) = (Ru_\beta, v_\alpha)_{L^2(I_\alpha)} = (u_\beta, Sv_\alpha)_{L^2(I_\beta)}, \\ u_\beta \in H_{\beta,+},\ v_\alpha \in H_{\alpha,+}. \qquad (29)\end{aligned}$$

From (29), (28), (26) and (22) we can conclude that $\xi = Ru_\beta \in H_{\alpha,-}$ is a generalized solution, inside I_α of the equation $\Im\xi = \lambda\xi$. Namely, we have the corresponding equality (23): $\forall v_\alpha \in C^\infty_{fin}(\mathcal{Y}) \subset C^\infty_0(I)$

$$(\xi, \Im u_\alpha)_{L^2(I_\alpha)} = \lambda(\xi, v_\alpha)_{L^2(I_\alpha)}.$$

Therefore, the above-mentioned assertion gives that $Ru_\beta = \xi \in C^\infty_{fin}(\bar{I})$ and, according to (24)

$$\begin{aligned}(Ru_\beta)(x) = C_1(u_\beta)\big[\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x\big], \\ x \in \bar{I}_\alpha,\ u_\beta \in H_{\beta,+}. \qquad (30)\end{aligned}$$

Quite analogously we get the representation

$$\begin{aligned}(Sv_\alpha)(y) = C_2(v_\alpha)\big[\cos\sqrt{\lambda}y + \sin\sqrt{\lambda}y\big], \\ y \in \bar{I}_\beta,\ v_\alpha \in H_{\alpha,+}.\end{aligned}$$

Equality (29) gives that

$$\begin{aligned}C_1(u_\beta)\Big[\int_{I_\alpha} \cos\sqrt{\lambda}x v_\alpha(x)dx + \\ + \int_{I_\alpha} \sin\sqrt{\lambda}x v_\alpha(x)dx\Big] = \\ = C_2(v_\alpha)\Big[\int_{I_\beta} \cos\sqrt{\lambda}y u_\beta(y)dy + \\ + \int_{I_\beta} \sin\sqrt{\lambda}y v_\beta(y)dy\Big].\end{aligned} \qquad (31)$$

From (31) it is to conclude that some constants $r \in \mathbb{R}^1$

$$\begin{aligned}C_1(u_\beta) = \tau \int_{I_\beta} u_\beta(y)[\cos\sqrt{\lambda}y + \sin\sqrt{\lambda}y]dy; \\ u_\beta(y) \in H_{\beta+}, \qquad (32)\end{aligned}$$

substituting $C(u_\beta)$ into (30) and using (28), (29) we get:

$$\begin{aligned}(\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes u_\beta)_{L^2(I_\alpha)\otimes L^2(I_\beta)} = \\ = \tau \iint_{I_\alpha\times I_\beta} \big(\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x\big)(\cos\sqrt{\lambda}y + \\ + \sin\sqrt{\lambda}y) v_\alpha(x) u_\beta(y) dxdy, \\ u_\beta \in H_{\beta,+};\ v_\alpha \in H_{\alpha,+}.\end{aligned}$$

This equality means that $\Omega_{\alpha\beta}(\lambda)$ is a smooth function $\Omega_{\alpha\beta}(\lambda;x,y)$ and

$$\begin{gathered}\Omega_{\alpha\beta}(\lambda;x,y)=\tau_{\alpha\beta}(\lambda)\big(\cos\sqrt{\lambda}x+\\ +\sin\sqrt{\lambda}x\big)(\cos\sqrt{\lambda}y+\sin\sqrt{\lambda}y),\\ x\in\bar{I}_\alpha,\ y\in\bar{I}_\beta,\ \alpha,\beta=1,2.\end{gathered}\tag{33}$$

Let $u,v\in H_+=W^2_{2,0}(I)$. Then representations (27) and (33) give:

$$\begin{gathered}(\Omega(\lambda),v\otimes u)_{L^2\otimes L^2}=\\ =\sum_{\alpha,\beta=1}^{2}\tau_{\alpha\beta}(\lambda)\iint\limits_{I_\alpha\times I_\beta}\big(\cos\sqrt{\lambda}x+\\ +\sin\sqrt{\lambda}x\big)\big(\cos\sqrt{\lambda}y+\\ +\sin\sqrt{\lambda}y\big)v(x)u(y)dxdy=\\ =\iint\limits_{I\times I}\left(\sum_{\alpha,\beta=1}^{2}\big(\cos\sqrt{\lambda}x+\sin\sqrt{\lambda}x\big)(\cos\sqrt{\lambda}y+\right.\\ \left.+\sin\sqrt{\lambda}y)k_\alpha(x)k_\beta(y)\tau_{\alpha\beta}(\lambda)\right)v(x)u(y)dxdy.\end{gathered}\tag{34}$$

The arbitrariness of the functions $u,v\in W^2_2(I)$ in (34) shows that $\Omega(\lambda)$ is an ordinary kernel $\Omega(\lambda;x,y)$, and regard that $\Omega(\lambda;-x,-y)=\Omega(\lambda;x,y)$ we get

$$\begin{gathered}\Omega(\lambda;x,y)=\sum_{\alpha,\beta=1}^{2}(\cos\sqrt{\lambda}x\cos\sqrt{\lambda}y+\\ +\sin\sqrt{\lambda}x\sin\sqrt{\lambda}y)\tau_{\alpha\beta}(\lambda),\ x,y\in I.\end{gathered}\tag{35}$$

Note that the matrix $\tau(\lambda)=(\tau_{\alpha\beta}(\lambda))^2_{\alpha,\beta=1}$ is nonnegative definite for every $\lambda\in\mathbb{R}^1$. Indeed, from (34) and (21) we can conclude:

$$\sum_{\alpha,\beta=1}^{2}\tau_{\alpha\beta}(\lambda)C_\alpha C_\beta=(\Omega(\lambda),u\otimes u)_{L^2\otimes L^2}\geq 0,$$

where

$$\begin{gathered}C_\alpha=\int\limits_{I_\alpha}(\cos\sqrt{\lambda}x+\sin\sqrt{\lambda}x)u(x)dx,\\ u\in H_+,\quad \alpha=1,2.\end{gathered}$$

The nonnegativeness of $\tau(\lambda)$ gives:

$$\begin{gathered}\tau_{11}(\lambda)\geq 0,\ \tau_{22}(\lambda)\geq 0,\\ |\tau_{12}(\lambda)|^2\leq\tau_{11}(\lambda)\tau_{22}(\lambda),\quad \lambda\in\mathbb{R}^1_+.\end{gathered}\tag{36}$$

Using the measure ρ from Proposition 1 we introduce the matrix-valued nonnegative Borel measure $d\sigma(\lambda)$ on $\mathbb{R}^1_+$:

$$\begin{gathered}d\sigma(\lambda)=\tau(\lambda)d\rho(\lambda):=(\tau_{\alpha\beta}(\lambda)d\rho(\lambda))^2_{\alpha,\beta=1}=\\ =(d\sigma_{\alpha\beta}(\lambda))^2_{\alpha,\beta=1}.\end{gathered}\tag{37}$$

After substituting representation (35) into (20) we get (4).

The convergence of integrals (4) follows from (36), (37). The inverse statement is evident: every integral (4) has form (2) with continuous $k_{\alpha\beta}(t)$ and is a bounded positive definite kernel, because $\forall f\in C^\infty_{fin}(I)$

$$\begin{gathered}\int\limits_I\int\limits_I K(x,y)f(y)f(x)dxdy=\\ =\iint\limits_{I\times I}\left(\int\limits_{\mathbb{R}^1_+}(\cos\sqrt{\lambda}x\cos\sqrt{\lambda}y+\right.\\ +\sin\sqrt{\lambda}x\sin\sqrt{\lambda}y)\times\\ \left.\times\sum_{\alpha,\beta=1}^{2}k_\alpha(x)k_\beta(y)d\sigma_{\alpha\beta}(\lambda)\right)f(x)f(y)dxdy=\\ =\int\limits_{\mathbb{R}^1_+}\left(\sum_{\alpha,\beta=1}^{2}\int\limits_{I_\alpha}\cos\sqrt{\lambda}xf(x)dx\times\right.\\ \left.\times\int\limits_{I_\beta}\cos\sqrt{\lambda}yf(y)dy\right)d\rho(\lambda)+\\ +\int\limits_{\mathbb{R}^1_+}\left(\sum_{\alpha,\beta=1}^{2}\int\limits_{I_\alpha}\sin\sqrt{\lambda}xf(x)dx\times\right.\\ \left.\times\int\limits_{I_\beta}\sin\sqrt{\lambda}yf(y)dy\right)d\rho(\lambda)=\\ =\int\limits_{\mathbb{R}^1_+}\left|\int\limits_I\cos\sqrt{\lambda}xf(x)dx\right|^2d\rho(\lambda)+\\ +\int\limits_{\mathbb{R}^1_+}\left|\int\limits_I\sin\sqrt{\lambda}xf(x)dx\right|^2d\rho(\lambda)\geq 0.\end{gathered}$$

Theorem is proved.

Remark. The proof of this theorem shows that it holds true for a more general situation, namely, if the functions $I_{\alpha\beta} \ni t \to k_{\alpha\beta}(t) \in C^1$ are only measurable. In this case, the corresponding integral (4) is, as before continuous. It is proved that the difference between K and this integral is a positive definite kernel (defined for almost all x, y).

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