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ON SOME GENERALIZATIONS OF P-LOXODROMIC FUNCTIONS

Розглянуто функціональне рівняння $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\backslash\{0\}$, $q \in \mathbb{C}\backslash\{0\}$, $|q| < 1$. При певних фіксованих елементарних функціях $p(z)$ знайдено його мероморфні та голоморфні розв'язки. Ці розв'язки є деякими узагальненнями p-локсодромних функцій.

The functional equation of the form $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\backslash\{0\}$, $q \in \mathbb{C}\backslash\{0\}$, $|q| < 1$ is considered. For certain fixed elementary functions $p(z)$, meromorphic as well as holomorphic solutions of this equation are found. These solutions are some generalizations of p-loxodromic functions.

Introduction.

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For $z \in \mathbb{C}^*$ consider the equation of the form

$$f(qz) = p(z)f(z), \qquad (1)$$

where $p(z)$ is some function, $q \in \mathbb{C}^*$, $|q| < 1$.

If $p(z) \equiv const$, then meromorphic solution of this equation is p-loxodromic function [4]. In particular, if $p(z) \equiv 1$, we have classic loxodromic function. The class of loxodromic functions is denoted by $\mathcal{L}_q$. It was studied in the works of O. Rausenberger [12], G. Valiron [15] and Y. Hellegouarch [2]. Such functions have many applications, and not only theoretical. A particularly practical one can be found in [13]. Various generalizations and properties of such functions were considered recently by A. Kondratyuk and his students in [3], [5-8], [10-11].

The aim of this article is to obtain holomorphic and meromorphic solutions of the equation (1), where $p(z)$ are some elementary functions. These solutions will be certain generalizations of p-loxodromic functions.

We will consider two cases $p(z) = \dfrac{1}{z^m}$ and $p(z) = \dfrac{1}{(1-z)^m}$, where $m \in \mathbb{Z}$. In fact, we consider only the case $m \neq 0$, because in the case $m = 0$ we obtain classic loxodromic functions. In Section 1 we describe meromorphic solutions of equation (1) for such $p(z)$. Section 2 deals with holomorphic solutions of this equation for the same $p(z)$.

1. Meromorphic generalizations.

Let us consider functional equation

$$f(qz) = \frac{1}{z^m} f(z), \quad z \in \mathbb{C}^*, \quad m \in \mathbb{Z}. \qquad (2)$$

Our task now is to find its meromorphic in $\mathbb{C}^*$ solutions.

Definition. *The function*

$$P(z) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)\left(1 - \frac{q^n}{z}\right)$$

is called the Schottky-Klein prime function.

It was introduced by Schottky [14] and Klein [9] for the study of conformal mappings of double-connected domains (see also [1]). This function is holomorphic in $\mathbb{C}^*$ and has zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. It is easily shown that the Schottky-Klein prime function has the following properties

$$P(qz) = -z^{-1}P(z), \qquad (3)$$

$$P\left(\frac{z}{q}\right) = -\frac{z}{q}P(z). \qquad (4)$$

Theorem 1. *The meromorphic in $\mathbb{C}^*$ function of the form $f(z) = P^m((-1)^m z)g(z)$, where $g \in \mathcal{L}_q$, satisfies (2).*

Доведення. Applying equality (3), we have

$$f(qz) = P^m(q(-1)^m z)g(qz) \overset{(3)}{=}$$

$$= \left(-\frac{1}{(-1)^m z}P((-1)^m z)\right)^m g(z) = \frac{1}{z^m}f(z).$$

Theorem 2. *Every meromorphic in $\mathbb{C}^*$ solution of (2) can be represented in the form $f(z) = P^m((-1)^m z)g(z)$, where $g \in \mathcal{L}_q$.*

Доведення. Let f be a meromorphic solution of (2). Consider the function $g(z) = \dfrac{f(z)}{P^m((-1)^m z)}$. Since f is meromorphic and P is holomorphic, it follows that g is meromorphic. Taking into account (2) and (3), we get

$$g(qz) = \frac{f(qz)}{P^m((-1)^m qz)} =$$

$$= \frac{\frac{1}{z^m} f(z)}{\frac{1}{((-1)^m z)^m} P^m((-1)^m z)} = g(z).$$

So, we can conclude that $g(qz) = g(z)$ for every $z \neq q^n, n \in \mathbb{Z}$. It is sufficient to make a conclusion that g is loxodromic. This completes the proof.

Now, consider functional equation of the form

$$f(qz) = \frac{1}{(1-z)^m} f(z), \quad z \in \mathbb{C}^*, \quad m \in \mathbb{Z}. \tag{5}$$

Let us find meromorphic in $\mathbb{C}^*$ solutions of (5).

Define the entire function with the zero sequence $\{q^{-n}\}$, $n \in \mathbb{N} \cup \{0\}$, $0 < |q| < 1$,

$$H(z) = \prod_{n=0}^{\infty} (1 - q^n z).$$

Theorem 3. *Let $g \in \mathcal{L}_q$. The meromorphic in $\mathbb{C}^*$ function $f(z) = H^m(z)g(z)$ satisfies equation (5).*

Доведення. The proof is straightforward. At first, let us consider $H(qz)$, we have

$$H(qz) = \prod_{n=0}^{\infty} \left(1 - q^{n+1} z\right) = \prod_{k=1}^{\infty} \left(1 - q^k z\right) =$$

$$= \frac{1}{1-z} \prod_{n=0}^{\infty} (1 - q^n z) = \frac{1}{1-z} H(z). \tag{6}$$

Since g is loxodromic, we obtain

$$(1-z)^m f(qz) = (1-z)^m g(qz) H^m(qz) =$$

$$= (1-z)^m g(z) H^m(qz) \overset{(6)}{=}$$

$$= (1-z)^m g(z) \frac{1}{(1-z)^m} H^m(z) = f(z).$$

Theorem 4. *Every meromorphic in $\mathbb{C}^*$ solution of (5) can be represented in the form $f(z) = H^m(z)g(z)$, where $g \in \mathcal{L}_q$.* **Доведення.** The proof is analogous to the proof of Theorem 2. Let f be a solution of equation (5). Consider the function $g = \dfrac{f}{H^m}$. Since f is meromorphic and H is holomorphic this implies that g is meromorphic. Using (5) and (6), we get

$$g(qz) = \frac{f(qz)}{H^m(qz)} = \frac{\frac{1}{(1-z)^m} f(z)}{\frac{1}{(1-z)^m} H^m(z)} = g(z).$$

Therefore, for all $z \neq q^{-n}$, $n \in \mathbb{N} \cup \{0\}$ we obtain that $g(qz) = g(z)$, i. e. g is loxodromic. The proof is finished.

2. Holomorphic generalizations

We also are interested in finding holomorphic in $\mathbb{C}^*$ solutions of equations (2) and (5).

Theorem 5. *If m is a positive integer, then holomorphic in $\mathbb{C}^*$ function $f(z) = C \prod_{j=1}^{m} P\left(\frac{z}{c_j}\right)$, where $c_1, c_2, \ldots c_m$ are nonzero complex numbers, not necessarily distinct, such that $\prod_{j=1}^{m} c_j = (-1)^m$, C is a constant, satisfies (2).*

Доведення. Using formula (3), we obtain

$$f(qz) = C \prod_{j=1}^{m} P\left(\frac{qz}{c_j}\right) =$$

$$= C \frac{\prod_{j=1}^{m} c_j}{(-z)^m} \prod_{j=1}^{m} P\left(\frac{z}{c_j}\right) = \frac{1}{z^m} f(z).$$

Theorem 6. *If m is a positive integer, then every holomorphic in $\mathbb{C}^*$ solution of (2) can be represented in the form $f(z) = C \prod_{j=1}^{m} P\left(\frac{z}{c_j}\right)$, where $c_1, c_2, \ldots c_m$ are* *nonzero complex numbers, not necessarily distinct, such that* $\prod_{j=1}^{m} c_j = (-1)^m$ *and* C *is a constant.*

Доведення. Let m be an even positive integer. Suppose, function f is a holomorphic in $\mathbb{C}^*$ solution of (2). Therefore, by Theorem 2,

$$f(z) = P^m((-1)^m z)g(z), \tag{7}$$

where $g \in \mathcal{L}_q$. Since functions f and P are holomorphic in $\mathbb{C}^*$, then g is either holomorphic in $\mathbb{C}^*$ or has the poles only in the points $\{(-1)^m q^n\}$, $n \in \mathbb{Z}$ and multiplicity of each pole is $l_n \leq m$, $l_n \in \mathbb{N}$.

If g is holomorphic, then $g(z) \equiv const$ due to the fact that the only holomorphic loxodromic function is constant [2, p. 93]. So

$$f(z) = CP^m((-1)^m z) = C\prod_{j=1}^{m} P\left(\frac{z}{c_j}\right),$$

where $c_1 = c_2 = \cdots = c_m = (-1)^m$.

In the second case we use the loxodromic function's representation by Schottky-Klein prime functions (see [2], [15] for more details). Namely, let $a_1, a_2, ..., a_l$ and $b_1, b_2 ..., b_l$ be the zeros and the poles of function g in the annulus $A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \leq R\}$, $R > 0$, respectively, and $\partial A_q(R)$ contains neither zeros nor poles of $g \in \mathcal{L}_q$. It is known [2, p. 93] that each loxodromic function has equal numbers of zeros and poles (counted according to their multiplicities) in every such annulus $A_q(R)$. Then

$$g(z) = Kz^p \frac{P\left(\frac{z}{a_1}\right) P\left(\frac{z}{a_2}\right) \cdot ... \cdot P\left(\frac{z}{a_l}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot ... \cdot P\left(\frac{z}{b_l}\right)}, \tag{8}$$

where $\frac{a_1 a_2 \ldots a_l}{b_1 b_2 \ldots b_l} = q^{-p}$, $p \in \mathbb{Z}$ and K is a constant. Using 4 we obtain for $p > 0$,

$$z^p P\left(\frac{z}{a_1}\right) = \frac{z}{a_1} P\left(\frac{z}{a_1}\right) z^{p-1} a_1 =$$

$$= -qP\left(\frac{z}{qa_1}\right) z^{p-1} a_1 =$$

$$= \frac{z}{qa_1} P\left(\frac{z}{qa_1}\right) z^{p-2} a_1^2 (-q^2) =$$

$$= -qP\left(\frac{z}{q^2 a_1}\right) z^{p-2} a_1^2 (-q^2) =$$

$$= (-1)^2 z^{p-2} a_1^2 q q^2 P\left(\frac{z}{q^2 a_1}\right) =$$

$$= \cdots = (-1)^p z^{p-p} a_1^p q q^2 q^3 \ldots q^p P\left(\frac{z}{q^p a_1}\right) =$$

$$= (-1)^p a_1^p q^{\frac{p(p+1)}{2}} P\left(\frac{z}{q^p a_1}\right).$$

In the same way, for $p < 0$, applying formula (3), we get

$$z^p P\left(\frac{z}{a_1}\right) = \left(\frac{1}{z}\right)^{-p} P\left(\frac{z}{a_1}\right) =$$

$$= \frac{a_1}{z} P\left(\frac{z}{a_1}\right) \left(\frac{1}{z}\right)^{-p-1} \frac{1}{a_1} =$$

$$= (-1)P\left(\frac{qz}{a_1}\right) \left(\frac{1}{z}\right)^{-p-1} \frac{1}{a_1} =$$

$$= (-1)\frac{a_1}{qz} P\left(\frac{qz}{a_1}\right) \left(\frac{1}{z}\right)^{-p-2} \frac{1}{a_1^2} q =$$

$$= (-1)^2 P\left(\frac{q^2 z}{a_1}\right) \left(\frac{1}{z}\right)^{-p-2} \frac{1}{a_1^2} q = \cdots =$$

$$= (-1)^p P\left(\frac{q^{-p} z}{a_1}\right) \left(\frac{1}{z}\right)^{-p-(-p)} \frac{1}{a_1^{-p}} q q^2 \ldots$$

$$\ldots q^{-p-1} = (-1)^p a_1^p q^{\frac{p(p+1)}{2}} P\left(\frac{z}{q^p a_1}\right).$$

The case $p = 0$ is trivial.

Then we can rewrite (8) in the following way

$$g(z) = C\frac{P\left(\frac{z}{q^p a_1}\right) P\left(\frac{z}{a_2}\right) \cdot ... \cdot P\left(\frac{z}{a_l}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot ... \cdot P\left(\frac{z}{b_l}\right)}, \tag{9}$$

where $C = (-a_1)^p q^{\frac{p(p+1)}{2}} K$. Let us denote $q^p a_1 = c_1$, $a_2 = c_2, \ldots, a_l = c_l$. Accordingly to this notation, we can rewrite (9) as

$$g(z) = C\frac{P\left(\frac{z}{c_1}\right) P\left(\frac{z}{c_2}\right) \cdot ... \cdot P\left(\frac{z}{c_l}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot ... \cdot P\left(\frac{z}{b_l}\right)}, \tag{10}$$

where C is a constant.

Obviously, every annulus $A_q(R)$ contains only one point from the sequence $\{(-1)^m q^n\}$, $n \in \mathbb{Z}$. It is convenient to choose such annulus $A_q(R)$, which contains the pole $b_1 = b_2 = \dots = b_l = (-1)^m q^0 = (-1)^m$. Note that $l = l_0$, where l_0 is the multiplicity of the pole at $z = (-1)^m$. Since $\prod_{j=1}^{l} b_j = (-1)^{ml}$ it follows $q^p a_1 a_2 \dots a_l = (-1)^{ml}$. In other words, $\prod_{j=1}^{l} c_j = (-1)^{ml}$. Thus, expression (10) can be rewritten in the form

$$g(z) = C\frac{\prod_{j=1}^{l} P\left(\frac{z}{c_j}\right)}{\prod_{j=1}^{l} P((-1)^m z)} = C\frac{\prod_{j=1}^{l} P\left(\frac{z}{c_j}\right)}{P^l((-1)^m z)}. \tag{11}$$

Using (7), we also can write g in the form

$$g(z) = \frac{f(z)}{P^m((-1)^m z)}. \tag{12}$$

Equating the right hand sides of formulas (11) and (12), we see that

$$f(z) = CP^{m-l}((-1)^m z)\prod_{j=1}^{l} P\left(\frac{z}{c_j}\right).$$

Note that $(-1)^{m^2} = (-1)^m$. So in the case $l = m$ Theorem 6 is proved. If $l < m$, then set $c_{l+1} = c_{l+2} = \dots = c_m = (-1)^m$ to get

$$f(z) = C\prod_{j=1}^{m} P\left(\frac{z}{c_j}\right)$$

and again use the property $(-1)^{m^2} = (-1)^m$ to obtain $\prod_{j=1}^{m} c_j = (-1)^m$.

Theorem 7. *Let m be a positive integer. The entire function $f(z) = CH^m(z)$, where C is a constant, satisfies equation (5).*

Доведення. Indeed,

$$(1-z)^m f(qz) = (1-z)^m C\left(\prod_{n=0}^{\infty}(1-q^{n+1}z)\right)^m =$$

$$= C(1-z)^m\left(\prod_{k=1}^{\infty}(1-q^k z)\right)^m =$$

$$= C\left(\prod_{n=0}^{\infty}(1-q^n z)\right)^m = f(z).$$

Theorem 8. *If m is a positive integer, then every holomorphic in $\mathbb{C}^*$ solution of (5) has the form $f(z) = CH^m(z)$, where C is a constant.*

Доведення. Assume that the function f is a holomorphic in $\mathbb{C}^*$ solution of (5). From Theorem 4 it follows that

$$f(z) = H^m(z)g(z), \tag{13}$$

where $g \in \mathcal{L}_q$. Rewrite (13), as follows

$$g(z) = \frac{f(z)}{H^m(z)}. \tag{14}$$

Functions f and H are holomorphic in $\mathbb{C}^*$. We also know that H^m has zeros of multiplicity m at the points $\{q^{-n}\}$, $n \in \mathbb{N}\cup\{0\}$.

If f has only the same zeros as H, we obtain that g does not have any zeros. So $g \in \mathcal{L}_q$ is holomorphic. Hence [2, p. 93], $g(z) \equiv const$, and theorem is proved.

Suppose that f has zeros different from $\{q^{-n}\}$, $n \in \mathbb{N}\cup\{0\}$. Then g has zeros. In this case g also should have poles [2, p. 93]. Since f is holomorphic in $\mathbb{C}^*$ solution of (5), then g has poles only at the points $\{q^{-n}\}$, $n \in \mathbb{N}\cup\{0\}$ of multiplicity $l_n \le m$.

Let us use representation (10) of $g \in \mathcal{L}_q$ in the annulus $A_q(R)$. Every annulus $A_q(R)$ contains only one point from the sequence $\{q^{-n}\}$, $n \in \mathbb{N}\cup\{0\}$. Choose such annulus $A_q(R)$, which contains the pole $b_1 = b_2 = \dots = b_l = q^0 = 1$. Note that $l = l_0$, where l_0 is the multiplicity of the pole at $z = 1$. Thus (13) takes the form

$$f(z) = C\frac{\prod_{j=1}^{l} P\left(\frac{z}{c_j}\right)}{P^l(z)}H^m(z).$$

Since P has zeros at the points $\{q^n\}$, $n \in \mathbb{Z}$, and H has zeros only at the points $\{q^{-n}\}$, $n \in \mathbb{N}\cup\{0\}$, then we obtain a contradiction. The proof is finished.

As we have seen in the proof of Theorem 8 holomorphic solutions of (5) possess the subsequent properties.

Corollary 1. *If f is a holomorphic solution of (5), then $f(z)=0$ iff $z=\{q^{-n}\}, n\in\mathbb{N}\cup\{0\}$.*

Corollary 2. *All holomorphic solutions of (5) are entire functions.*

We have considered only the case $m>0$ so far. The case $m=0$ is trivial. So it remains to consider negative m. The following theorem deals with this case.

Theorem 9. *If m is a negative integer, then equations (2) and (5) do not have any holomorphic in $\mathbb{C}^*$ solutions.*

Доведення. Consider equation (2). Let $m<0$ and to be definite, m is an even integer. Suppose that there exist a holomorphic in $\mathbb{C}^*$ solution f of (2). In this case, according to Theorem 2, it has the form $f(z)=P^m(z)g(z)$, where $g\in\mathcal{L}_q$. Hence $g(z)=f(z)P^{-m}(z)$. Obviously, g is holomorphic in $\mathbb{C}^*$. Consequently [2, p. 93], we can assert that $g(z)\equiv const$. Thus,

$$f(z)=CP^m(z),$$

where C is a constant.

But on the other hand $P^m(z)$ is not holomorphic in $\mathbb{C}^*$ in the case $m<0$. This contradicts our assumption.

Substituting H for P and using Theorem 4 we can apply similar arguments to equation (5).

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