УДК 517.53

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Ivan Franko National University of Lviv

# MODULO-ELLIPTIC AND MODULO-LOXODROMIC FUNCTIONS

Встановлено взаємозв'язки між $p$-локсодромними та модуль-локсодромними функціями, а також квазі-еліптичними та модуль-еліптичними функціями.

Connections between $p$-loxodromic functions and modulo-loxodromic functions and $p$-elliptic functions and modulo-elliptic functions, respectively, are established.

**Introduction.** Denote $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Let $\omega_1, \omega_2$ be complex numbers such that $Im\frac{\omega_2}{\omega_1} > 0$. A meromorphic in $\mathbb{C}$ function $g$ is called **elliptic** [1] if for every $u \in \mathbb{C}$

$$g(u+\omega_1) = g(u), \ \ g(u+\omega_2) = g(u).$$

Elliptic functions were first discovered by N. Abel as inverse functions of elliptic integrals, and their theory was developed by C. Jacobi. A more complete investigation of elliptic functions was later undertaken by K. Weierstrass, who found a simple elliptic function $\wp$, in terms of which all the others could be expressed.

Let $q, p \in \mathbb{C}^*$, $|q| < 1$.

**Definition.** [3] *A meromorphic in* $\mathbb{C}^*$ *function* $f$ *is said to be* $p$**-loxodromic of multiplicator** $q$ *if for every* $z \in \mathbb{C}^*$

$$f(qz) = pf(z).$$

In particular, if $p(z) \equiv 1$, we obtain classic loxodromic function. It was studied in the works of O. Rausenberger [9], G. Valiron [10] and Y. Hellegouarch [1]. A. Kondratyuk and his colleagues also made a great contribution to the development of this theory (see for example [2], [5], [6], [8]).

**Definition.** [7] *A meromorphic in* $\mathbb{C}^*$ *function* $f$ *is said to be* **modulo-loxodromic of multiplicator** $q$ *if for every* $z \in \mathbb{C}^*$

$$|f(qz)| = |f(z)|. \tag{1}$$

By $|\mathcal{L}|_q$ we denote the set of all modulo-loxodromic functions of multiplicator $q$.

## 1. Modulo-loxodromic functions.

Now let us consider $p$-loxodromic function of multiplicator $q$ with $p = e^{i\alpha}$. This means that the function $f$ satisfies the condition

$$f(qz) = e^{i\alpha} f(z) \tag{2}$$

for every $z \in \mathbb{C}^*$ and fixed $\alpha \in \mathbb{R}$. Denote the class of such functions by $\mathcal{L}_q^\alpha$.

**Theorem 1.** $\bigcup\limits_{\alpha\in\mathbb{R}} \mathcal{L}_q^\alpha = |\mathcal{L}|_q$. **Доведення.** Let $f \in \mathcal{L}_q^\alpha$ for some $\alpha \in \mathbb{R}$. Then the relation (10) implies

$$|f(qz)| = |e^{i\alpha} f(z)| = |f(z)|.$$

Hence, $f \in |\mathcal{L}|_q$.

Let now $f \in |\mathcal{L}|_q$. Consider the meromorphic function

$$g(z) = \frac{f(qz)}{f(z)}. \tag{3}$$

Since $f(z)$ is meromorphic in $\mathbb{C}^*$, using (11) we get that zero and pole sets of functions $f(z)$ and $f(qz)$ coincide. So, we conclude $g$ is holomorphic in $\mathbb{C}^*$ and (11) also implies

$$|g(z)| = 1$$

outside of $z = 0$, zeros and poles of $f$. Hence, all these points are removable. Applying the Liouville theorem we obtain that $g(z) \equiv c$, where $c$ is a constant. Taking into account that $|g| = 1$, we deduce that there exists $\alpha \in \mathbb{R}$, such that $c = e^{i\alpha}$ and (12) implies

$$f(qz) = e^{i\alpha} f(z),$$

that is $f \in \mathcal{L}_q^\alpha$ as well as $f \in \bigcup\limits_{\alpha\in\mathbb{R}} \mathcal{L}_q^\alpha$. This completes the proof.

## 2. Modulo-elliptic functions

**Definition 1.** *A meromorphic in* $\mathbb{C}$ *function g is called* **quasi-elliptic**, *if there exist* $\omega_1, \omega_2 \in \mathbb{C}^*$, $Im\frac{\omega_2}{\omega_1} > 0$, *and* $p \in \mathbb{C}^*$, $q \in \mathbb{C}^*$, *such that for every* $u \in \mathbb{C}$

$$g(u+\omega_1) = pg(u), \;\; g(u+\omega_2) = qg(u).$$

Consider the case $p = e^{i\alpha}$, $q = e^{i\beta}$. It means that function $g$ satisfies two conditions for every $u \in \mathbb{C}$

$$g(u+\omega_1) = e^{i\alpha}g(u), \;\; g(u+\omega_2) = e^{i\beta}g(u). \quad (4)$$

Denote the class of such functions by $\mathcal{QE}_{\alpha\beta}$.

The following definition was introduced by A. Kondratyuk [4].

**Definition.** *A meromorphic in* $\mathbb{C}$ *function f is said to be* **modulo-elliptic**, *if there exist* $\omega_1, \omega_2 \in \mathbb{C}^*$, *such that* $Im\frac{\omega_2}{\omega_1} > 0$ *and for every* $u \in \mathbb{C}$

$$|f(u+\omega_1)| = |f(u)|, \;\; |f(u+\omega_2)| = |f(u)|.$$

The class of modulo-elliptic functions is denoted by $|\mathcal{E}|$.

**Theorem 2.** $\bigcup\limits_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} \mathcal{QE}_{\alpha\beta} = |\mathcal{E}|$. **Доведення.**

Let $f \in \mathcal{QE}_{\alpha\beta}$. Then, using (4) we have

$$|f(u+\omega_1)| = |e^{i\alpha}f(u)| = |f(u)|,$$

$$|f(u+\omega_2)| = |e^{i\beta}f(u)| = |f(u)|.$$

Hence, $f \in |\mathcal{E}|$.

Let now $f \in |\mathcal{E}|$. In other words, for every $u \in \mathbb{C}$ the following equalities are valid

$$|f(u+\omega_1)| = |f(u)|, \;\; |f(u+\omega_2)| = |f(u)|.$$

Consider the first of these equalities

$$|f(u+\omega_1)| = |f(u)|, \;\; u \in \mathbb{C}. \quad (5)$$

If $f(u) \neq 0$ and $f(u) \neq \infty$ we can divide (5) by $|f(u)|$ and we obtain

$$\left|\frac{f(u+\omega_1)}{f(u)}\right| = 1. \quad (6)$$

The function $g(u) = \dfrac{f(u+\omega_1)}{f(u)}$ is meromorphic in $\mathbb{C}$. From (6) it follows that function $g$ is holomorphic and bounded in $\mathbb{C}$ without a set of zeros and poles of $f$. Since $g$ is bounded, these points are removable, and relation (6) implies

$$\forall u \in \mathbb{C}: \;\; |g(u)| = 1.$$

By the Liouville theorem $g$ is constant and the last equality implies there exists $\alpha \in \mathbb{R}$, such that $g(u) = e^{i\alpha}$. It means

$$\forall u \in \mathbb{C}: \;\; f(u+\omega_1) = e^{i\alpha}f(u).$$

Similarly as above, we obtain that there exists $\beta \in \mathbb{R}$ and

$$\forall u \in \mathbb{C}: \;\; f(u+\omega_2) = e^{i\beta}f(u).$$

So, we can conclude that $f \in \mathcal{QE}_{\alpha\beta}$ as well as $\bigcup\limits_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} \mathcal{QE}_{\alpha\beta}$.

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