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CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM WITH TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC

We determine conditions for the origin to be a center for a class of cubic differential systems having two invariant straight lines and one invariant cubic. We prove that a fine focus $O(0,0)$ is a center if and only if the first two Lyapunov quantities vanish.

Keywords: cubic differential system, invariant straight lines, focus, algebraic solution, problem of center.

Introduction

We consider the cubic differential system of the form

$$\begin{array}{l} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + \\ +mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + \\ +qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{array} \qquad (1)$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a fine focus. The aim of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that a singular point $O(0,0)$ is a center for system (1) if and only if it has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$ [19]. Also, $O(0,0)$ is a center if and only if (1) has a holomorphic integrating factor of the form $\mu = 1 + \sum \mu_j(x,y)$ in some neighborhood of $O(0,0)$ [1].

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

Quantities L_j, $j = \overline{1,\infty}$ are polynomials with respect to the coefficients of system (1) called to be the Lyapunov quantities. The origin is a fine focus of order r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ and $L_r \neq 0$. The origin is a center for (1) if and only if $L_j = 0$, $j = \overline{1,\infty}$.

By the Hilbert basis theorem, there is N such that $L_j = 0$ for all j if and only if $L_j = 0$ for all $j \leq N$. It is only necessary to find a finite number of Lyapunov quantities, though in any given case it is not known a priori how many are required.

The number N is known for quadratic systems $N = 3$ [2] and for cubic systems with only homogeneous cubic nonlinearities $N = 5$ [26]. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center has been solved only in some particular cases (see, for example, [3–15, 17, 18, 20, 21]).

In this paper we solve the problem of the center for a class of cubic differential systems (1) with two invariant straight lines and one irreducible invariant cubic. The paper is organized as follows. In Section 1 we present the known results concerning relation between integrability, invariant algebraic curves and Lyapunov quantities. In Section 2 we find twenty eight sets of conditions for the existence of two invariant straight lines and one invariant cubic. In Section 3 we obtain the center conditions for cubic system (1) with two invariant straight lines and one invariant cubic and determine the order of the fine focus $O(0,0)$.

1. Algebraic solutions and center sequences

In this paper we study the problem of the center for cubic differential system (1) assuming that the system has irreducible invariant algebraic curves: two invariant straight lines and one invariant cubic.

Definition 1. *An algebraic invariant curve of*

(1) *is the solution set in* $\mathbb{C}^2$ *of an equation* $\Phi(x,y)=0$, *where* Φ *is a polynomial in* x,y *with complex coefficients such that*

$$\frac{\partial\Phi}{\partial x}P(x,y)+\frac{\partial\Phi}{\partial y}Q(x,y)=\Phi(x,y)K(x,y),$$

for some polynomial in x,y, $K=K(x,y)$ *with complex coefficients, called the cofactor of the invariant algebraic curve* $\Phi=0$.

We say that the invariant algebraic curve $\Phi(x,y)=0$ is an *algebraic solution* of (1) if and only if $\Phi(x,y)$ is an irreducible polynomial in $\mathbb{C}[x,y]$.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x,y)=0$, $j=1,\dots,q$, then in most cases a first integral (an integrating factor) can be constructed in the Darboux form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}. \qquad (2)$$

Function (2), with $\alpha_j\in\mathbb{C}$ not all zero, is a first integral (an integrating factor) for (1) if and only if

$$\sum_{j=1}^{q}\alpha_jK_j\equiv 0 \quad \Big(\sum_{j=1}^{q}\alpha_jK_j\equiv-\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial x}\Big).$$

If system (1) has a first integral or an integrating factor of the form (2), being $\Phi_j=0$ invariant algebraic curves of (1), then system (1) is called Darboux integrable [25]. The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These years, interesting results on algebraic solutions, Lyapunov quantities and Darboux integrability have been obtained (see, for example, [6–11, 14, 16, 22, 23]).

Definition 2. *We say that* $(\Phi_k,\ k=\overline{1,M};\ L=N)$ *is a center sequence for (1) if the existence of* M *invariant irreducible algebraic curves* $\Phi_k(x,y)=0$ *and the vanishing of the Lyapunov quantities* L_ν, $\nu=\overline{1,N}$, *implies the origin* $O(0,0)$ *to be a center for (1).*

The problem of center sequences for cubic differential systems with invariant algebraic curves was considered in [5–9, 24]. In these papers, the problem of the center for cubic systems with four invariant straight lines, three invariant straight lines, two invariant straight lines and one invariant conic was completely solved. The main results of these works are summarized in the following theorem.

Theorem 1. $(a_jx+b_jy+c_j,\ j=\overline{1,4};\ \ L=2)$, $(a_jx+b_jy+c_j,\ j=\overline{1,3};\ \ L=7)$ *and* $(a_jx+b_jy+c_j,\ j=1,2,\ a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0;\ \ L=4)$ *are center sequences for the cubic system (1).*

The problem of the center for cubic system (1) having two parallel invariant straight lines and one invariant cubic was solved in [11] and for cubic system (1) having a bundle of two invariant straight lines and one invariant cubic was solved in [12], [13]. The main results of these papers are gathered in the following theorem.

Theorem 2. $(l_j=a_jx+b_jy+c_j,\ j=1,2,\ l_1||l_2,\ \Phi;\ \ L=2)$ *and* $(l_j=1+a_jx-y,\ j=1,2,\ \Phi,\ l_1\cap l_2\cap\Phi=(0,1);\ \ L=3)$, *where* $\Phi=x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3$ *is an irreducible invariant cubic, are center sequences for the cubic system (1).*

In the present paper, we shall prove that $(1+a_jx-y,\ j=1,2,\ \Phi;\ \ L=2)$, where $\Phi=x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2$ is an irreducible invariant cubic, is a center sequence for the cubic system (1).

2. Conditions for the existence of an invariant cubic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a real singular point (x_0,y_0). By rotating the system of coordinates ($x\to x\cos\varphi-y\sin\varphi, y\to x\sin\varphi+y\cos\varphi$) and rescaling the axes of coordinates ($x\to\alpha x$, $y\to\alpha y$), we obtain $l_1\cap l_2=(0,1)$. In this case the invariant straight lines can be written as

$$l_j\equiv 1+a_jx-y=0,\ a_j\in\mathbb{C},\ j=1,2;\ a_2-a_1\neq 0. \qquad (3)$$

According to [10] the straight lines (3) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$k=(a-1)(a_1+a_2)+g,\ l=-b,$$
$$r=-f-1,\ s=(1-a)a_1a_2,$$

$$m=(a_1+a_2)(c-a_1-a_2)+a_1a_2-$$
$$-a+d+2,\ q=(a_1+a_2-c)a_1a_2-g,$$
$$p=(f+2)(a_1+a_2)+b-c,$$
$$n=-(f+2)a_1a_2-(d+1).$$

In this case the cubic system (1) looks:

$$\begin{gathered}\dot{x}=y+ax^2+cxy+fy^2+[(a-1)\times\\ \times(a_1+a_2)+g]x^3+\\ +[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]\times\\ \times x^2y+[(f+2)(a_1+a_2)+b-c]\times\\ \times xy^2-(f+1)y^3\equiv P(x,y),\\ \dot{y}=-x-gx^2-dxy-by^2+\\ +(a-1)a_1a_2x^3+[g+a_1a_2(c-\\ -a_1-a_2)]x^2y+[(f+2)a_1a_2+d+1]\times\\ \times xy^2+by^3\equiv Q(x,y).\end{gathered}\tag{4}$$

In this section for cubic system (1) with two invariant straight lines (4) we find conditions for the existence of one irreducible invariant cubic curve

$$\Phi(x,y)\equiv x^2+y^2+a_{30}x^3+a_{21}x^2y+$$
$$+a_{12}xy^2+a_{03}y^3=0,\tag{5}$$

where $(a_{30},a_{21},a_{12},a_{03})\neq 0$ and $a_{ij}\in\mathbb{R}$.

By Definition 1, the cubic curve (5) is an invariant cubic curve for system (1) if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01}\in\mathbb{R}$ such that

$$P(x,y)\frac{\partial\Phi}{\partial x}+Q(x,y)\frac{\partial\Phi}{\partial y}\equiv\Phi(x,y)\times$$
$$\times(c_{20}x^2+c_{11}xy+c_{02}y^2+c_{10}x+c_{01}y).\tag{6}$$

Identifying the coefficients of the monomials x^iy^j in (6), we reduce this identity to a system of fifteen equations $\{F_{ij}=0\}$ for the unknowns $a_{30},a_{21},a_{12},a_{03},c_{20},c_{11},c_{02},\ c_{10},c_{01}$. When $i+j=3$, we find that $c_{10}=2a-a_{21}$, $c_{01}=a_{12}-2b$, $d=(3a_{21}-3a_{03}-2a+2f)/2$, $g=(3a_{30}-3a_{12}+2b+2c)/2$ and $a_{30},a_{21},a_{12},a_{03}$ are the solutions of the following systems of algebraic equations:

$$\begin{aligned}F_{50}&\equiv 9a_{12}a_{30}-2a_{30}(3(a_1+a_2)(a-1)+3b+\\&+3c-c_{20})+2a_{21}a_1a_2(1-a)-9a_{30}^2=0,\\ F_{41}&\equiv 9a_{03}a_{30}+9a_{12}a_{21}+4a_{12}a_1a_2(1-a)-\\&-18a_{21}a_{30}+2a_{21}(c_{20}+(a_1+a_2)(a_1a_2-\\&-2a+2)-ca_1a_2-3b-3c)+\\&+2a_{30}(c_{11}+6a-3f-6+3(a_1+a_2)\times\\&\times(a_1+a_2-c)-3a_1a_2)=0,\\ F_{32}&\equiv 9a_{03}a_{21}+6a_{03}a_1a_2(1-a)+9a_{12}^2-\\&-9a_{21}^2-9a_{12}a_{30}+2a_{12}(c_{20}+(a_1+a_2)\times\\&\times(1-a)+2a_1a_2(a_1+a_2-c)-3b-3c)+\\&+2a_{21}(c_{11}+5a-3f-5+2(a_1+a_2)\times\\&\times(a_1+a_2-c)-(f+4)a_1a_2)+2a_{30}\times\\&\times(c_{02}-3b+3c-3(f+2)(a_1+a_2))=0,\\ F_{23}&\equiv 2a_{03}(c_{20}+3a_1a_2(a_1+a_2-c)-3b-\\&-3c)-9a_{12}a_{21}+6(f+1)a_{30}+2a_{12}\times\\&\times(c_{11}+4a-3f-4+(a_1+a_2)\times\\&\times(a_1+a_2-c)-(2f+5)a_1a_2)+9a_{03}\times\\&\times(2a_{12}-a_{30})+2a_{21}(c_{02}-2(f+2)\times\\&\times(a_1+a_2)-3b+2c)=0,\\ F_{14}&\equiv a_{03}(9a_{03}-9a_{21}+2(c_{11}+3(a-f-1)-\\&-3(f+2)a_1a_2))+2a_{12}(c_{02}-3b+c-\\&-(f+2)(a_1+a_2))+4(f+1)a_{21}=0,\\ F_{05}&\equiv a_{03}(c_{02}-3b)+(f+1)a_{12}=0,\end{aligned}\tag{7}$$

$$\begin{aligned}F_{40}&\equiv 3a_{12}(a_{21}-2)-a_{30}(a_{21}-2a-6)-\\&-2(b+c)a_{21}+2(2b+2c+2(a_1+a_2)\times\\&\times(a-1)-c_{20})=0,\\ F_{31}&\equiv 2a_{30}(2b+3c-4a_{12})+a_{21}(2a-2f+\\&+6-a_{21})+a_{12}(6a_{12}-4b-4c)+3a_{03}\times\\&\times(a_{21}-2)+4a(a_1a_2-2)+4(a_1+a_2)\times\\&\times(c-a_1-a_2)-2c_{11}+4f+8=0,\\ F_{22}&\equiv a_{03}(15a_{12}-9a_{30}-6b-6c)+2a_{12}\times\\&\times(a-2f-3-3a_{21})+2a_{21}(b+2c)+\\&+2(4b-c_{02}-c_{20}+2(f+2)(a_1+a_2)+\\&+3(f+1)a_{30}-2a_1a_2(a_1+a_2-c))=0,\\ F_{13}&\equiv a_{03}(2a-6f-6-7a_{21}+9a_{03})+\\&+2a_{12}(c-a_{12})+2(2f+3)a_{21}-\\&-2(2a+c_{11}-(2f+4)a_1a_2)=0,\\ F_{04}&\equiv a_{12}(a_{03}-f)+b(a_{03}-2)+c_{02}=0.\end{aligned}\tag{8}$$

The conditions for the existence of an invariant cubic for system (4) will be found studying the consistency of the system of equations {(7), (8)} and assuming that $a_{03}=0$. In this case the invariant cubic curve (5) looks as

$$\Phi(x,y)\equiv x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2=0.\tag{9}$$

Then from the equation $F_{05}=0$ of (7), we can see that either $a_{12}=0$ or $f=-1$.

3.1. Assume that $a_{12} = 0$ and let $a_{21} = 0$. Then $F_{05} \equiv 0, F_{14} \equiv 0$ and $f = -1$. We express c_{02}, c_{11}, c_{20} from the equations of (7) and a_1, a_{30} from $F_{04} = 0$, $F_{22} = 0$. Then reduce the equation $F_{31} = 0$ by a_2^2 from $F_{13} = 0$.

If $b^2 = 3$ and $a = 0$, then we have the following set of conditions

1) $a = 0,\ d = -1,\ f = -1,\ g = (3c - b)/3,\ b^2 = 3,\ a_1 = (3c - b - 3a_2)/3,\ 3a_2^2 + (b - 3c)a_2 - 3bc - 6 = 0$

for the existence of an invariant cubic curve $9(x^2 + y^2) - 8bx^3 = 0$.

If $b^2 = 3$ and $a \neq 0$, then we obtain the following set of conditions

2) $a = 4/3,\ c = (-7b)/9,\ d = (-7)/3,\ f = -1,\ g = -2c,\ b^2 = 3,\ 9a_1 + 9a_2 + 10b = 0,\ 9a_2^2 + 10ba_2 + 51 = 0$

for the existence of an invariant cubic curve $9(x^2 + y^2) + 8bx^3 = 0$.

Let $b^2 \neq 3$ and express c from $F_{40} = 0$. Then $F_{31} \equiv f_1 f_2 = 0$, where $f_1 = b^2(2a-3)+9(a-1)^2$ and $f_2 = (3b^2 + 7a^2 + 6a - 9)^2 + 32a^2(a - 3)^2 \neq 0$. When $f_1 = 0$ we get the following set of conditions for the existence of an invariant cubic

3) $c = b(2a - 5)/3,\ d = -a - 1,\ f = -1,\ g = [2b(5a^2 - 14a + 9)]/(6a - 9),\ b^2(2a - 3) + 9(a-1)^2 = 0,\ a_1 = (2ab - 6b - 3a_2)/3,\ 3a_2^2 + (b - 3c)a_2 + 12a + b^2 - 3bc - 9 = 0.$

The invariant cubic is $3(2a - 3)(x^2 + y^2) + 4b(a^2 - 3a + 2)x^3 = 0$.

3.2. Assume that $a_{12} = 0$ and let $a_{21} \neq 0$. Then $F_{14} = 0$ yields $f = -1$. We express c_{02}, c_{11}, c_{20} from the equations $F_{23} = 0, F_{32} = 0, F_{41} = 0$ and obtain that $F_{50} \equiv g_1 g_2 g_3 = 0$, where $g_1 = a_1 a_{21} + a_{30},\ g_2 = a_2 a_{21} + a_{30},\ g_3 = (a - 1)a_{21} + (a_1 + a_2 - c)a_{30}$.

If $g_1 = 0$, then $a_{30} = -a_1 a_{21}$ and $F_{40} \equiv (a_{21} + 1)((2a - 2 - a_{21})a_1 + 2b + 2c) = 0$.

Suppose that $a_{21} = -1$ and express a_1 from $F_{04} = 0$, then $F_{31} \equiv i_1 i_2 = 0$, where $i_1 = 2a_2 + b - 2c$ and $i_2 = 4aa_2 - 6a_2 + 3b + 6c$.

When $i_1 = 0$, then $b = 0$ and the right-hand sides of (1) have a common factor $1 + cx - y$.

When $i_1 \neq 0$, we reduce the equations $F_{22} = 0, F_{13} = 0$ by b from $i_2 = 0$. Then we calculate the resultant of the polynomials F_{22} and F_{13} with respect to a and establish that the system of equations $\{F_{22} = 0,\ F_{13} = 0\}$ is consistent if and only if $4a_2^2 + 18a + 9 = 0$. We obtain the following set of conditions for the existence of an invariant cubic

4) $a = (-b^2 - 1)/2,\ c = b(-b^2 - 5)/2,\ d = (b^2 - 4)/2,\ g = 5b(-b^2 - 3)/4,\ f = -1,\ a_1 = b(-b^2 - 3)/2,\ a_2 = (-3b)/2.$

The invariant cubic is $2(x^2 + y^2) - x^2(b^3 x + 3bx + 2y) = 0$.

Let $a_{21} + 1 \neq 0$. Then the equation $F_{40} = 0$ yields $c = (a_1 a_{21} - 2aa_1 + 2a_1 - 2b)/2$. We express a from $F_{13} = 0$ and b from $F_{04} = 0$. Calculating the resultant of F_{31} and F_{22} with respect to a_2, we obtain that $Res(F_{31}, F_{22}, a_2) = (a_1^2 + 1)h_1 h_2 h_3$, where $h_1 = a_1^2 + a_{21} + 1,\ h_2 = 3a_1^4 - 4a_1^2 a_{21} + 14a_1^2 + 27,\ h_3 = 27a_1^2 a_{21}^2 - a_{21}^3 + 15a_{21}^2 - 48a_{21} - 64$.

If $h_1 = 0$, then $a_{21} = -a_1^2 - 1$ and we get the following set of conditions

5) $b = (-2aa_1 - a_1^3 - a_1 - 2a_2)/3,\ c = (4a_2 + 5a_1 - a_1^3 - 2aa_1)/6,\ d = (-2a - 3a_1^2 - 5)/2,\ f = -1,\ g = a_1(2 - a + a_1^2),\ a = -(4a_2^2 + 9 + 21a_1^2 + a_1(a_1^2 + 1)(3a_1 + 2a_2))/(2(3a_1^2 + 2a_1 a_2 + 9)),\ F_{31} \equiv a_1^3(23a_1^2 + 5a_1 a_2 + 4a_2^2 + 54) - 18a_1^2 a_2 + 27a_1 - 4a_2^3 - 27a_2 = 0.$

The invariant cubic is $x^2 + y^2 + (a_1^2 + 1)(a_1 x - y)x^2 = 0$.

If $h_1 \neq 0$ and $h_2 = 0$, then $a_{21} = (3a_1^4 + 14a_1^2 + 27)/(4a_1^2)$ and we find the following set of conditions for the existence of an invariant cubic

6) $a = (3a_1^4 + 14a_1^2 + 27)/(8a_1^2),\ b = (-a_1^2 - 3)/a_1,\ c = a_1 - b,\ f = -1,\ d = 2a - 1,\ g = (-9a_1^4 - 34a_1^2 - 81)/(8a_1),\ a_2 = (-3b)/2,\ 5a_1^6 + 31a_1^4 + 63a_1^2 - 27 = 0.$

The invariant cubic is $4a_1^2(x^2 + y^2) - (3a_1^4 + 14a_1^2 + 27)(a_1 x - y)x^2 = 0$.

Suppose that $h_1 h_2 \neq 0$ and let $h_3 = 0$. Denote $a_{21} = 3h^2 - 1$, then

$$h_3 \equiv (3a_1 h^2 - a_1 + h^3 - 3h)\times$$
$$\times(3a_1 h^2 - a_1 - h^3 + 3h) = 0.$$

In this case we obtain the following two sets of conditions:

7) $a = (3h^2 - 1)/2,\ b = -2h,\ c = h(7a - 4)/(3a),\ g = h(13a - 4 - 3a^2)/(3a),\ f = -1,\ d = 2a - 1,\ a_1 = h(h^2 - 3)/(3h^2 - 1),\ a_2 = 3h.$

The invariant cubic is $x^2 + y^2 + (3hx - h^3x + 3h^2y - y)x^2 = 0.$

8) $a = (3h^2 - 1)/2,\ b = 2h,\ c = -h(7a - 4)/(3a),\ g = -h(13a - 4 - 3a^2)/(3a),\ f = -1,\ d = 2a - 1,\ a_1 = -h(h^2 - 3)/(3h^2 - 1),\ a_2 = -3h.$

The invariant cubic is $x^2 + y^2 + (h^3x - 3hx + 3h^2y - y)x^2 = 0.$

The case $g_2 = 0$ is symmetric to $g_1 = 0$ and we get the conditions 4) – 8).

Assume that $g_1g_2 \neq 0$ and let $g_3 = 0$. Then $a = (a_{21} + ca_{30} - a_2a_{30} - a_1a_{30})/a_{21}$. We express a_1 from $F_{04} = 0$ and reduce the equations of (8) by a_2^2 from $F_{13} = 0$. Consider the equation $F_{40} - F_{22} = 0$ and suppose that $ba_{30} - 3a_{21}^2 + b^2a_{21} = 0$. In this case we get the following set of conditions for the existence of an invariant cubic

9) $a = (b^2 + 4)/4,\ c = (-3b)/2,\ d = (b^2 - 4)/2,\ f = -1,\ g = b(3b^2 - 4)/8,\ a_1 = -a_2 - 2b,\ 4a_2^2 + 8ba_2 + 5b^2 + 4 = 0.$

The invariant cubic is $4(x^2 + y^2) + b^2x^2(bx + 2y) = 0.$

Suppose that $ba_{30} - 3a_{21}^2 + b^2a_{21} \neq 0$ and express c from the equation $F_{40} - F_{22} = 0$. If $a_{30} = 0$, then we have the following set of conditions

10) $a = 1,\ b = 2c,\ d = 10,\ f = -1,\ g = 3c,\ a_1 = 3\sqrt{3},\ a_2 = -3\sqrt{3}.$

The invariant cubic is $x^2 + y^2 + 8x^2y = 0.$

Let $a_{30} \neq 0$ and express a_{30} from $F_{31} = 0$. In this case we get the following set of conditions for the existence of an invariant cubic

11) $a = (2a_{21} + ba_{30})/(2a_{21}),\ c = [-b(a_{21}^2 + a_{21}(b^2 - 32) + 4b^2)]/[2(3a_{21}^2 - b^2a_{21} - 4b^2)],\ d = (3a_{21} - 2a - 2)/2,\ f = -1,\ g = (3a_{30} + 2b + 2c)/2,\ a_{30} = [(3a_{21} - b^2)(a_{21} - 8)a_{21}^2]/[b(3a_{21}^2 - b^2a_{21} - 4b^2)],\ a_1 = (2c - b - 2a_2)/2,\ \ a_{21}(2a_2^2 + (b - 2c)a_2 + b^2 - 2bc + 2) - 7a_{21}^2 + 4ba_{30} = 0,\ F_{40} \equiv 81a_{21}^4 - 30b^2a_{21}^3 + b^2(b^2 + 24)a_{21}^2 + 8b^4a_{21} + 16b^4 = 0.$

The invariant cubic is $x^2 + y^2 + x^2(a_{30}x + a_{21}y) = 0.$

3.3. Assume that $a_{12} \neq 0$ and let $f = -1$. We express c_{02}, c_{11}, c_{20} from the equations $F_{14} = 0, F_{23} = 0, F_{32} = 0$ of (7) and calculate the resultant of the polynomials F_{41} and F_{50} with respect to a. We obtain that $Res(F_{41}, F_{50}, a) = j_1j_2j_3j_4$, where $j_1 = 4a_{12}a_{30} - a_{21}^2$, $j_2 = a_1 + a_2 - c$, $j_3 = a_1^2a_{12} + a_1a_{21} + a_{30}$, $j_4 = a_2^2a_{12} + a_2a_{21} + a_{30}$.

3.3.1. Suppose that $j_1 = 0$. Then $a_{30} = a_{21}^2/(4a_{12})$ and $F_{41} \equiv h_1h_2h_3 = 0$, where $h_1 = 2a_1a_{12} + a_{21}$, $h_2 = 2a_2a_{12} + a_{21}$, $h_3 = 2(a - 1)a_{12} + (a_1 + a_2 - c)a_{21}$.

Let $a_1 = (-a_{21})/(2a_{12})$, then $h_1 \equiv 0$ and $F_{50} \equiv 0$. We express b from $F_{04} = 0$ and reduce the equations $F_{31} = 0$ and $F_{22} = 0$ by a_2^2 from $F_{13} = 0$. Suppose that $a_{21} + 1 \neq 0$ and express c from $F_{40} = 0$. Then $F_{22} - F_{31} \equiv (a_{21} - 2a)(a_{12}^2 - 2a_{21} - 4) = 0$.

If $a_{21} = 2a$ and $a = 4$, then we have the following two sets of conditions for the existence of an invariant cubic:

12) $a = 4,\ b = 7 - g,\ c = 2g - 7,\ d = 7,\ f = -1,\ g^2 - 14g + 46 = 0,\ a_1 = -1,\ a_2 = 3g - 17.$

The invariant cubic is $x^2 + y^2 + 4x(x + y)^2 = 0.$

13) $a = 4,\ b = -7 - g,\ c = 2g + 7,\ d = 7,\ f = -1,\ g^2 + 14g + 46 = 0,\ a_1 = 1,\ a_2 = 3g + 17.$

The invariant cubic is $x^2 + y^2 - 4x(x - y)^2 = 0.$

If $a_{21} = 2a$ and $a \neq 4$, then we find the following set of conditions

14) $b = [3(a^2 - a_{12}^2)]/[2a_{12}(a - 4)],\ c = (4a^2 - 2aa_{12}^2 - 4a + 5a_{12}^2)/[a_{12}(4 - a)],\ d = 2a - 1,\ g = (a_{12}^2 - 3a^3 + 17a^2 - aa_{12}^2 - 8a)/[2a_{12}(4 - a)],\ a_1 = (-a)/a_{12},\ a_2 = (a_{12}^2 - 9a^2 + 2aa_{12}^2)/[2a_{12}(a - 4)],\ F_{13} \equiv 27a_{12}^4 - 2a_{12}^2(4a^3 - 3a^2 + 48a + 32) + 27a^4 = 0.$

The invariant cubic is $a_{12}(x^2+y^2)+x(ax+a_{12}y)^2=0$.

Suppose that $a_{21}\neq 2a$ and let $a_{21}=(a_{12}^2-4)/2$. If $5a_{12}^2-108=0$, then $F_{22}\equiv(15a-64)(5a_{12}a_2-234)=0$ and we get the following two sets of conditions:

15) $a=64/15$, $b=(504-25a_{12}a_2)/(75a_{12})$, $c=2(25a_{12}a_2+897)/(75a_{12})$, $f=-1$, $d=119/15$, $g=(25a_{12}a_2+2046)/(75a_{12})$, $5a_{12}^2-108=0$, $a_1=(-22)/(5a_{12})$, $75a_2^2-145a_{12}a_2-819=0$.

The invariant cubic is $16a_{12}(x^2+y^2)+x(a_{12}^2x-4x+4a_{12}y)^2=0$.

16) $b=6(15a-101)/(25a_{12})$, $c=2(45a+497)/(25a_{12})$, $d=(61-5a)/5$, $f=-1$, $g=4(45a+76)/(25a_{12})$, $5a_{12}^2-108=0$, $a_1=(-22)/(5a_{12})$, $a_2=234/(5a_{12})$.

The invariant cubic is $16a_{12}(x^2+y^2)+x(a_{12}^2x-4x+4a_{12}y)^2=0$.

If $5a_{12}^2-108\neq 0$, then we obtain the following set of conditions

17) $b=(4ca_{12}-4a_2a_{12}-3a_{12}^2-4)/(4a_{12})$, $d=(3a_{12}^2-4a-16)/4$, $g=(3a_{12}^4-96a_{12}^2-32a_{12}a_2+64ca_{12}+16)/(32a_{12})$, $n=(4aa_{12}+a_{12}^2a_2-4a_2-3a_{12}^3+12a_{12})/(4a_{12})$, $c=[a_{12}^2(16a^2+104a-a_{12}^4-10a_{12}^2-64)+96a-160]/[16a_{12}(6a-4-a_{12}^2)]$, $a_2=[4aa_{12}^2(5a_{12}^2-4)-(5a_{12}^2-12)(a_{12}^2+4)(a_{12}^2-6)]/[32a_{12}(6a-4-a_{12}^2)]$, $F_{13}\equiv 3a_{12}^8-4a_{12}^6(8a+1)+4a_{12}^4(28a^2+8a-13)+32a_{12}^2(1-4a^3-2a^2+4a)-64=0$.

The invariant cubic is $16a_{12}(x^2+y^2)+x(a_{12}^2x-4x+4a_{12}y)^2=0$.

Let $a_{21}=-1$. In this case $F_{40}\equiv(a-1)(4a_{12}a_2-1)=0$. If $a=1$, then the system (8) is not consistent. Assume that $a\neq 1$ and let $a_2=1/(4a_{12})$. The case $a=(-1)/2$ is contained in 14). If $a\neq(-1)/2$ and $a_{12}^2=2$, then we get the following set of conditions

18) $a=(-3)/4$, $b=1/(2a_{12})$, $c=13/(4a_{12})$, $d=(-7)/4$, $f=-1$, $g=9/(8a_{12})$, $a_{12}^2=2$, $a_1=1/(2a_{12})$, $a_2=1/(4a_{12})$.

The invariant cubic is $4a_{12}(x^2+y^2)+x(x^2-4a_{12}xy+8y^2)=0$.

The case $h_2=0$ is symmetric to $h_1=0$ if we replace a_2 with a_1 and we obtain the sets of conditions 12) – 18).

Assume that $h_1h_2\neq 0$ and let $h_3=0$. We express a_1 from $F_{04}=0$ and reduce the equations of (8) by a_2^2 from $F_{13}=0$. Then $h_3=0$ yields $a=(a_{12}a_{21}+2a_{12}+ba_{21}]/(2a_{12})$. Denote $\Delta_1=a_{12}a_{21}-2a_{12}-3ba_{21}$ and $\Delta_2=4a_{12}^2(a_{21}+16)-3a_{21}^3$.

Let $\Delta_1\neq 0$ and express c from $F_{22}=0$. If $a_{21}(a_{21}+4)=0$, then the system (8) is not consistent. If $a_{21}=8$, then $a_{12}=\pm 4$ and we get the following two sets of conditions:

19) $a=b+5$, $c=b+12$, $d=6-b$, $f=-1$, $g=2(b+6)$, $a_1=8-a_2$, $a_2^2-8a_2=11$.

The invariant cubic is $x^2+y^2+4x(x+y)^2=0$.

20) $a=5-b$, $c=b-12$, $d=6+b$, $f=-1$, $g=2(b-6)$, $a_1=-a_2-8$, $a_2^2+8a_2=11$.

The invariant cubic is $x^2+y^2-4x(x-y)^2=0$.

Suppose that $a_{21}(a_{21}+4)(a_{21}-8)\Delta_1\neq 0$ and let $\Delta_2=0$. Then the system (8) is not consistent.

Suppose that $a_{21}(a_{21}+4)(a_{21}-8)\Delta_1\Delta_2\neq 0$ and reduce the equations $\{F_{40}=0,\ F_{31}=0\}$ by b^2 from $H\equiv a_{21}F_{40}+a_{12}F_{31}=0$. Then $F_{31}\equiv e_1e_2=0$, where

$$e_1=44a_{12}^2a_{21}-64a_{12}^2+ +16ba_{12}a_{21}^2-128ba_{12}a_{21}-9a_{21}^3,$$
$$e_2=432a_{12}^4-16a_{12}^2a_{21}^3+24a_{12}^2a_{21}^2- -768a_{12}^2a_{21}-1024a_{12}^2+27a_{21}^4.$$

If $e_1=0$, then express b and obtain that $F_{31}\equiv F_{40}\equiv 0$ and $H\equiv\Delta_1\Delta_2^2\neq 0$.

If $e_1\neq 0$ and $e_2=0$, then we have the following set of conditions

21) $a=(a_{12}a_{21}+2a_{12}+ba_{21})/(2a_{12})$, $c=[4a_{12}^2(2a_{21}-7)+12ba_{12}(2-3a_{21})+9a_{21}^2-12b^2a_{21}]/[4(a_{12}a_{21}-2a_{12}-3ba_{21})]$, $d=(2a_{12}a_{21}-4a_{12}-ba_{21})/(2a_{12})$, $f=-1$, $g=(3a_{21}^2-12a_{12}^2+8ba_{12}+8ca_{12})/(8a_{12})$,

$a_1 = c - b - a_{12} - a_2$, $2a_{12}a_2^2 + 2a_{12}a_2(a_{12} + b - c) + 4ba_{12}^2 + a_{12}(2b^2 - 2bc + 2) + a_{21}(4b - 3a_{12}) = 0$, $H \equiv b^2(4a_{12}^2a_{21} + 64a_{12}^2 - 3a_{21}^3) + ba_{12}(4a_{12}^2a_{21} - 32a_{12}^2 + a_{21}^3 - 8a_{21}^2) + a_{12}^2(12a_{12}^2 - a_{21}^2 - 16a_{21}) = 0$, $e_2 \equiv 432a_{12}^4 - 16a_{12}^2a_{21}^3 + 24a_{12}^2a_{21}^2 - 768a_{12}^2a_{21} - 1024a_{12}^2 + 27a_{21}^4 = 0$.

The invariant cubic is $4a_{12}(x^2 + y^2) + x(a_{21}x + 2a_{12}y)^2 = 0$.

Let $\Delta_1 = 0$. Then $b = [a_{12}(a_{21} - 2)]/(3a_{21})$ and the system of equations $\{F_{22} = 0, F_{31} = 0\}$ is consistent if and only if $a_{12} = \pm 4$, $a_{21} = 8$. In this case we obtain the sets of conditions 19) ($b = 1$) and 20) ($b = -1$).

3.3.2. Assume that $j_1 \neq 0$ and let $j_2 = 0$. Then $c = a_1 + a_2$ and $F_{41} \equiv i_1i_2 = 0$, where $i_1 = a - 1$, $i_2 = 2a_1a_2a_{12}^2 + (a_1 + a_2)a_{12}a_{21} - 2a_{12}a_{30} + a_{21}^2$.

Let $i_1 = 0$. Then $F_{50} \equiv 0, F_{41} \equiv 0$ and $F_{04} = 0$ yields $a_{12} = -b$. We obtain that

$$F_{40} \equiv (2a_1 + 2a_2 + a_{30} + 5b)(a_{21} + 1) = 0.$$

Suppose that $a_{21} = -1$. Then $F_{22} = 0$ implies $a_{30} = -(2a_1 + 2a_2 + 7b)/3$ and $F_{31} \equiv (2a_1 + a_2 + 3b)(a_1 + 2a_2 + 3b) = 0$. In this case the system (8) is not consistent.

Suppose that $a_{21} \neq -1$. Then $F_{40} = 0$ yields $a_{30} = -2a_1 - 2a_2 - 5b$ and $F_{22} \equiv (a_1 + a_2 + 2b)(a_{21} + 4) = 0$. If $a_{21} = -4$, then the system (8) is not consistent.

If $a_{21} \neq -4$ and $a_1 = -a_2 - 2b$, then $F_{13} = 0$ implies $a_{21} = [2(a_2 + b)^2 + 2]/7$. In this case we get the following set of conditions for the existence of an invariant cubic

22) $a = 1$, $c = -2b$, $d = 10$, $f = -1$, $g = -b$, $a_1 = -a_2 - 2b$, $a_2^2 + 2ba_2 + b^2 - 27 = 0$.

The invariant cubic is $x^2 + y^2 - x(bx^2 - 8xy + by^2) = 0$.

Let $i_1 \neq 0$ and $i_2 = 0$. Then $a_{30} = (2a_1a_2a_{12}^2 + (a_1 + a_2)a_{12}a_{21} + a_{21}^2)/(2a_{12})$ and the equations $F_{50} = 0$, $F_{04} = 0$ yield $a_{21} = b(a_1 + a_2)$, $a_{12} = -b$. We express a from $F_{13} = 0$ and reduce the equations $F_{40} = 0$ and $F_{31} = 0$ by b^3 from $F_{22} = 0$. In this case the equation $G \equiv F_{40} + a_2F_{31} = 0$ becomes

$$G \equiv -2(b(a_1 - a_2) + a_2^2 + 1)(2a_1 + 5b)(a_2^2 + 1) = 0.$$

If $a_1 = (-5b)/2$, then $a_2 = (-46)/(11b)$ and $b^2 = 4/11$. We find the following set of conditions for the existence of an invariant cubic

23) $a = (-61)/11$, $c = -14b$, $d = (-34)/11$, $f = -1$, $g = (-299b)/11$, $b^2 = 4/11$, $a_1 = (-5b)/2$, $a_2 = (-23b)/2$.

The invariant cubic is $2(x^2 + y^2) - x(6b^2x + 2x + by)(5bx + 2y) = 0$.

If $a_1 \neq (-5b)/2$, then $G = 0$ implies $a_1 = (ba_2 - a_2^2 - 1)/b$ and $F_{40} = 0$ yields $a_2 = (-5b)/2$. In this case $b^2 = 4/11$ and we obtain the set of conditions 23).

3.3.3. Assume that $j_1j_2 \neq 0$ and let $j_3 = 0$. Then $a_{30} = -a_1(a_1a_{12} + a_{21})$ and $F_{41} \equiv r_1r_2 = 0$, where $r_1 = a_{12}(a_1 + a_2) + a_{21}$, $r_2 = (a - 1)a_{12} + (a_1 + a_2 - c)(a_1a_{12} + a_{21})$.

Suppose that $r_1 = 0$. Then $a_{21} = -(a_1 + a_2)a_{12}$. We express a_1 from $F_{04} = 0$ and reduce the equations $\{F_{40} = 0, F_{31} = 0, F_{22} = 0\}$ by a_2^2 from $F_{13} = 0$.

Denote $\Delta_3 = a_{12} - 3b + c$ and let $\Delta_3 \neq 0$. If $\Delta_3 = 0$ the system (8) is not consistent. We express a from $F_{22} = 0$ and calculate the resultant of the polynomials F_{40} and F_{31} with respect to c. We obtain that $Res(F_{40}, F_{31}, c) = 1048576ba_{12}s_1s_2 \cdots s_9$, where $s_1 = a_{12} - 2b$, $s_2 = a_{12} - b$, $s_3 = 3a_{12}^2 - 8ba_{12} - 4$, $s_4 = a_{12}^2 + (3a_{12} - 4b)^2 + 8$, $s_5 = (a_{12} - 4b)^2 + 4$, $s_6 = 9a_{12}^2 + 4$, $s_7 = 5a_{12}^2 + 4$, $s_8 = a_{12}^2 + 4$, $s_9 = b^2 + 1$ and $s_4s_5 \cdots s_9 \neq 0$.

Let $b = 0$. If $a_{12} = c$, then the invariant cubic is reducible. If $a_{12} \neq c$ and $a_{12}^2 = 4/3$, then $c^2 - 12 = 0$. We get the following set of conditions

24) $a = (-7)/3$, $b = 0$, $d = (-8)/3$, $f = -1$, $g = c$, $c^2 - 12 = 0$, $a_1 = (2c - 3a_2)/3$, $3a_2^2 - 2ca_2 + 3 = 0$.

The invariant cubic is $3(x^2 + y^2) + x(cx^2 - 8xy + cy^2) = 0$.

Let $s_1 = 0$ and $b \neq 0$. Then $a_{12} = 2b$ and $c = (3b^2 + 1)/(2b)$. In this case the right hand side of (1) have a common linear factor.

Let $s_2 = 0$ and $bs_1 \neq 0$. Then $a_{12} = b$ and $c = 0$. We find the following set of conditions

25) $a = b^2 + 1$, $c = 0$, $d = 2(b^2 - 1)$, $f = -1$, $g = b(3b^2 + 1)$, $a_{1,2} = -b \pm i\sqrt{b^2 + 1}$

for the existence of an invariant cubic $x^2+y^2+(2b^3+b)x^3+2b^2x^2y+bxy^2=0$.

Let $s_3=0$ and $bs_1s_2\neq 0$. Then $b=(3a_{12}^2-4)/(8a_{12})$ and $c=(15a_{12}^4+32a_{12}^2+16)/(16a_{12}^3)$. In this case we get the following set of conditions

26) $a=(7a_{12}^4-48a_{12}^2-48)/(32a_{12}^2)$, $b=(3a_{12}^2-4)/(8a_{12})$, $c=(15a_{12}^4+32a_{12}^2+16)/(16a_{12}^3)$, $d=(7a_{12}^2-52)/16$, $f=-1$, $g=(9a_{12}^6-132a_{12}^4+432a_{12}^2+320)/(128a_{12}^3)$, $a_1=(4-a_{12}^2)/(4a_{12})$, $a_2=(16-3a_{12}^4+24a_{12}^2)/(16a_{12}^3)$.

The invariant cubic curve is

$$64a_{12}^3(x^2+y^2)+x(3a_{12}^4x-24a_{12}^2x-\\-16x+16a_{12}^3y)(a_{12}^2x-4x+4a_{12}y)=0.$$

Assume that $r_1\neq 0$ and let $r_2=0$. Then $a=[a_{12}-(a_1+a_2-c)(a_1a_{12}+a_{21})]/a_{12}$. Denote $\Delta_4=a_{12}(a_1-a_2+a_{12})+3a_{21}$, $\Delta_5=a_{12}(a_1^2-a_1a_{12}-1)+2a_1a_{21}$ and suppose that $\Delta_4\Delta_5\neq 0$. We express b from $F_{04}=0$, c from $F_{13}=0$ and reduce the equations $\{F_{40}=0, F_{31}=0\}$ by a_2^2 from $F_{22}=0$. Then express a_2 from $F_{40}=0$ and obtain that $F_{31}=u_1u_2u_3u_4\Delta_4\Delta_5$, where $u_1=3a_1a_{12}+2+2a_{21}$, $u_2=4a_1a_{12}-a_{12}^2+4+4a_{21}$, $u_3=a_1^2+2a_1a_{12}+1+a_{21}$, $u_4=(a_1a_{12}+a_{21})^2+a_{12}^2\neq 0$.

If $u_1=0$, then the system (8) is not consistent. If $u_1\neq 0$ and $u_2=0$, then $a_{21}=(a_{12}^2-4a_1a_{12}-4)/4$ and $F_{22}=0$ yields $a_1=(a_{12}^4-72a_{12}^2-432)/(16a_{12}^3)$. In this case we obtain the following set of conditions

27) $a=(3a_{12}^4-16a_{12}^2+144)/(32a_{12}^2)$, $b=(-5a_{12}^2-36)/(8a_{12})$, $c=(35a_{12}^4-432)/(16a_{12}^3)$, $d=(3a_{12}^4+76a_{12}^2+576)/(16a_{12}^2)$, $f=-1$, $g=(236a_{12}^4-3a_{12}^6-144a_{12}^2-8640)/(128a_{12}^3)$, $a_1=(a_{12}^4-72a_{12}^2-432)/(16a_{12}^3)$, $a_2=(7a_{12}^2+36)/(4a_{12})$

for the existence of an invariant cubic

$$64a_{12}^3(x^2+y^2)-x(a_{12}^4x-72a_{12}^2x-\\-432x-16a_{12}^3y)(a_{12}^2x-4x+4a_{12}y)=0.$$

If $u_1u_2\neq 0$ and $u_3=0$, then $a_{21}=-a_1^2-2a_1a_{12}-1$ and $F_{22}=0$ yields $a_{12}=(-7a_1^4-18a_1^2-27)/(8a_1^3)$. In this case we get the following set of conditions

28) $a=(3a_1^6-31a_1^4+81a_1^2+243)/[8a_1^2(a_1^2+9)]$, $b=(7a_1^4+18a_1^2+27)/[2a_1(a_1^2+9)]$, $c=[(a_1^4-18a_1^2-27)(5a_1^2+9)]/[4a_1^3(a_1^2+9)]$, $d=[2a(a_1^2+9)+26a_1^2+18]/(a_1^2+9)$, $f=-1$, $g=(3a_1^8+94a_1^6-288a_1^4-1134a_1^2-243)/[16a_1^3(a_1^2+9)]$, $a_2=-(19a_1^4+54a_1^2+27)/(8a_1^3)$

for the existence of an invariant cubic

$$8a_1^3(x^2+y^2)+x(a_1^5x-10a_1^3x-27a_1x+\\+7a_1^4y+18a_1^2y+27y)(a_1x-y)=0.$$

Let $\Delta_4=0$. Then $a_{21}=a_{12}(a_2-a_1-a_{12})/3$ and the equations $F_{13}=0$ yields $a_2=(a_{12}^2+3a_1a_{12}+6)/(a_{12}+6a_1)$. In this case the right-hand sides of (1) have a common factor.

Assume that $\Delta_4\neq 0$ and let $\Delta_5=0$. Then $a_{21}=a_{12}(1+a_1a_{12}-a_1^2)/(2a_1)$. If $a_{12}=-2a_1$, then the right-hand sides of (1) have a common factor. If $a_{12}\neq -2a_1$, then express c from $F_{22}=0$ and the system of equations (8) is not consistent.

3.3.4. Assume that $j_1j_2j_3\neq 0$ and let $j_4=0$. The case $j_4=0$ is equivalent with $j_3=0$ if we take into consideration the symmetry $F_{ij}(a_1,a_2)=F_{ij}(a_2,a_1)$ in the algebraic system of equations $\{(7), (8)\}$.

3. Center conditions for cubic system (1) with two invariant straight lines and one invariant cubic

In this section we derive four sets of conditions for the origin to be a center for cubic system (1) by constructing integrating factors or first integrals from invariant functions.

Theorem 3. *The following four sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $a=k=r=0$, $d=f=-1$, $g=(3c-b)/3$, $l=-b$, $m=[2(-bc-2)]/3$, $n=bc+2$, $p=(2b)/3$, $q=b$, $s=-bc-2$, $b^2=3$;

(ii) $a=(b^2+4)/4$, $c=(-3b)/2$, $d=2a-4$, $f=-1$, $g=[b(3b^2-4)]/8$, $k=(-ab)/2$, $l=-b$, $m=b^2/2$, $n=(-7b^2)/4$, $p=b/2$, $q=-b^3$, $r=0$, $s=[-b^2(5b^2+4)]/16$;

(iii) $a=1$, $c=-2b$, $d=10$, $f=-1$, $g=$

$-b,\ k=-b,\ l=-b,\ m=b^2-16,\ n=-m,\ p=q=b,\ r=s=0;$

(iv) $a=b^2+1,\ c=r=0,\ d=2(b^2-1),\ f=-1,\ g=b(3b^2+1),\ k=b(b^2+1),\ l=-b,\ m=-b^2,\ n=-4b^2,\ p=-b,\ q=b(-7b^2-3),\ s=b^2(-2b^2-1).$

Доведення. In Case (i), system (1) has a Darboux integrating factor of the form $\mu = l_1^{\alpha_1} l_2^{\alpha_2} \Phi^{\beta}$, where $l_{1,2} = (3c-b\pm\sqrt{9c^2+30bc+75})x-6y+6$, $\Phi = 9(x^2+y^2)-8bx^3$, $\alpha_1=-\alpha_2-1$, $\alpha_2=(5b+3c-\sqrt{9c^2+30bc+75}\,)/(2\sqrt{9c^2+30bc+75}\,)$, $\beta=(-4)/3$.

In Cases (ii), system (1) has a Darboux first integral of the form $l_1^{\alpha_1} l_2^{\alpha_2}\Phi^{\beta}=C$, where $l_1 = 2+(-2b+i\sqrt{b^2+4})x-2y$, $l_2=2+(-2b-i\sqrt{b^2+4})x-2y$, $\Phi=4(x^2+y^2)+b^2x^2(bx+2y)$, $\alpha_1=\alpha_2=-1$, $\beta=1$.

In Case (iii), system (1) has a Darboux first integral of the form

$$(x^2+y^2-x(bx^2-8xy+by^2))(bx-2y-1)^{-3}=C.$$

In Case (iv), system (1) has a Darboux first integral of the form

$$(x^2+y^2+(2b^3+b)x^3+2b^2x^2y+bxy^2)\times$$
$$\times(bx+2y-1)^{-1}=C.$$

Theorem 4. *Let the cubic system (1) have two invariant straight lines (3) and one invariant cubic (9). Then a singular point $O(0,0)$ is a center if and only if the first two Lyapunov quantities vanish.*

Proof. To prove the theorem, we compute the first two Lyapunov quantities L_1, L_2 in each series of conditions 1)−28) obtained in Section 2 by using the algorithm described in [9]. In the expressions for L_j we will neglect the denominators and non-zero factors.

In Case 1) the first Lyapunov quantity vanishes, then Theorem 3, (i).

In Cases 2), 3), 4), 6), 7), 8), 12), 13), 14), 15), 17), 18), 23), 24), 26), 27), 28) we have $L_1\neq 0$. Therefore the origin is a focus.

In Case 5) we calculate the resultant of F_{31} and L_1 with respect to a_2. We find that $Res(F_{31},L_1,a_2)=8192(7a_1^4+18a_1^2+27)^4(7a_1^2+4)(a_1^2+1)^2a_1\neq 0$. The origin is a focus.

In Case 9) the first Lyapunov quantity vanishes, then Theorem 3, (ii).

In Case 10) the first Lyapunov quantity is $L_1=c$. If $c=0$, then Theorem 3, (iii) ($c=0$).

In Case 11) the first Lyapunov quantity looks $L_1=81a_{21}^4-6a_{21}^3(5b^2+108)+b^2(b^2+300)a_{21}^2+4b^2(24-7b^2)a_{21}-128b^4$. We calculate the resultant of F_{40} and L_1 with respect to b taking into account that $a_{21}(a_{21}+4)\neq 0$. We find that $Res(F_4,L_1,b)=0$ if and only if $a_{21}=(-8)/5$. Let $a_{21}=(-8)/5$. Then $L_1\neq 0$. In this case the origin is a focus.

In Case 16) the first Lyapunov quantity looks $L_1=225a^2-1630a+1616$. If $L_1=0$, then the second Lyapunov quantity is $L_2\neq 0$. In this case the origin is a focus.

In Case 19) the first Lyapunov quantity is $L_1=b(b+4)$. If $b=0$, then the second Lyapunov quantity is $L_2\neq 0$. If $b=-4$, then $L_2=0$ and Theorem 3, (iii) ($b=-4$).

In Case 20) the first Lyapunov quantity is $L_1=b(b-4)$. If $b=0$, then the second Lyapunov quantity is $L_2\neq 0$. If $b=4$, then $L_2=0$ and Theorem 3, (iii) ($b=4$).

In Case 21) we reduce the first Lyapunov quantity by b^2 from $H=0$ and express b from $L_1=0$. Then $H\equiv 186624a_{12}^8-6912a_{12}^6a_{21}(2a_{21}^2-5a_{21}+56)+32a_{12}^4a_{21}^2(8a_{21}^4-40a_{21}^3+831a_{21}^2-400a_{21}+15488)-48a_{12}^2a_{21}^3(10a_{21}^4-33a_{21}^3+456a_{21}^2+2176a_{21}+12288)+81a_{21}^6(a_{21}+16)^2=0$.

We calculate the resultant of H and e_2 with respect to a_{12} taking into account that $a_{21}(a_{21}+4)(a_{21}-8)\neq 0$. We find that $Res(H,e_2,a_{12})=0$ if and only if $a_{21}^3-8a_{21}^2-16a_{21}-16=0$. Let $a_{21}^3-8a_{21}^2-16a_{21}-16=0$ and calculate the resultant of L_2 and H with respect to a_{12}. We obtain that $Res(H,L_2,a_{12})\neq 0$. Therefore the origin is a focus.

In Cases 22) and 25) we have $L_1=0$, then Theorem 3, (iii) and (iv), respectively.

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