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NETWORK STRUCTURE ANALYSIS IN TOPOLOGICAL SPACE

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АНАЛІЗ СТРУКТУРИ МЕРЕЖІ В ТОПОЛОГІЧНОМУ ПРОСТОРІ

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Abstract. This paper focuses researches on telecommunication and/or information network engineering with respect to network topological properties study. An alternative definition of topological space and topology introduced in terms of subject-to-object interaction on the basis of A.I. Uyomov system triad approach to interpret key terms of general models theory for network graph. The direct and inverse problems of analysis/synthesis in topological space are formulated. The tensor view of topological frameworks is defined. The work aims to benefit network topology and metrics modeling through advanced methods of differential geometry and field theory.

Keywords: network graph, topological space, topology, analysis and synthesis.

Анотація. Стаття присвячена дослідженню топологічних властивостей телекомунікаційних та/або інформаційних мереж. В роботі представлено альтернативне визначення топологічного простору і топології в термінах взаємодії суб’єкта з об’єктом на базі системної тріади А.І. Уйомова для інтерпретації ключових термінів загальної теорії моделей на графі мережі. Сформульовано пряму та зворотну задачі аналізу/синтезу в топологічному просторі. Визначено тензорний вигляд топологічних структур. Робота спрямована на імплементацію сучасних методів диференційної геометрії і теорії поля для моделювання топологічних і метричних властивостей мереж.

Ключові слова: граф мережі, топологічний простір, топологія, аналіз та синтез.

Аннотация. Статья посвящена вопросам исследования топологических свойств телекоммуникационных и/или информационных сетей. В работе представлено альтернативное определение топологического пространства и топологии в терминах взаимодействия субъекта с объектом на базе системной триады А.И. Уёмова для интерпретации ключевых терминов общей теории моделей на графе сети. Сформулировано прямую и обратную задачу анализа/синтеза в топологическом пространстве. Определён тензорный вид топологических структур. Работа направлена на имплементацию современных методов дифференциальной геометрии и теории поля для моделирования топологических и метрических свойств сетей.

Ключевые слова: граф сети, топологическое пространство, топология, анализ и синтез.

Introduction

The modeling of complex objects in terms of their topological features is an effective instrument for researchers, engineers and scientists. According to Wolfram MathWorld, “*Topology is the mathematical study of the properties that are preserved through deformations, twisting, and stretching of objects. Tearing, however, is not allowed…*”. The math topology largely deals with topological spaces (TS) where important characteristics include continuity, connectedness and compactness [1]. The core idea of topological view on the objects is analyzing relationships between distinct parts of the objects regardless specific details of these attitudes (just seeing whether relation exists or not). There are various disciplines handling this subject, e.g. "general topology", "combinatorial topology" that now evolved to "algebraic topology"[2–3].

In telecoms term "topology" is widely used in less formal sense as arrangement of various network components (nodes, links, and peripherals) if so called "physical" topology considered, or information flow relationships among networks and sub-networks with respect to "logical topology". In fact, other types of various "topologies" may be defined as network models on different layers of the open system interconnection model OSI. The telecoms specific topological model is network graph in different forms, e.g. depicted graph and matrix graph defined by the coupled links (edges) mutually connecting pairs of network nodes (vertices).

However, key terms of math topology are not yet explicitly mapped on network engineering (e.g. unlike general topology, network graph link may be directed and multiproduct, network graphs are typically unaware of being "open topological space" that is character to classic topology). Therefore, more researches still needed to implement classic topology methods in neoteric network models. In particular, holistic approach to object relationships analysis of A.I. Uyomov is perspective [4]. *This work aims to advance network objects study in terms of topological properties manifested due to relations between object and subject*.

1. Analysis and synthesis tasks for topological frameworks

In this section we distinguish two marginal concepts of system approach: analysis and synthesis. The common paradigm of system analysis is studying relationships among different parts of an object [5]. Following [4] we study object properties through the subject-to-object relations where *subject* is a principal party of object's model formalism (e.g. subjective view point S on the object). The subject and object entities we treat as *own open neighborhoods* (S) and (O) of abstract points S and O, where $\varepsilon_S^0=(S)$, $\varepsilon_S^1=(\varepsilon_S^0,...)=((S),...)$ are *null-order* and *first-order open neighborhoods* (ONH) of S. The ONH hypothesis is declared in various forms (e.g. "empty set $\varnothing$ is open") as a key thesis of general model theory to get insight of *continuity*, *connectivity* and *convergence* properties among distinguished object things [1, 2]. Herewith, we postulate *local and wide* subject's open neighborhoods ε_L and ε_W : $\varepsilon_L,\varepsilon_W\in\varepsilon_S^1=((S),\varepsilon_L,\varepsilon_W)$, Fig. 1,a. By default ε_L and ε_W consider being empty, e.g. ε_L is empty local area network, ε_W is empty wide area network and (S) is gateway interface. As shown in Fig. 1,b, (S) is encapsulated into the so called "*universe*" $U:(S)\subset U=(\varepsilon_L\bigcup(S)\bigcup\varepsilon_W)$. We endow (S) with front and back faces like two-dimensional torus, Fig. 1,a. The unions $S_L=\varepsilon_L\bigcup(S)$ and $S_W=(S)\bigcup\varepsilon_W$ we respectively call *first-order simplex topological spaces* (where only first-order neighborhoods of point S defined). The union $CS=S_L\bigcup S_W$ we call *first-order complex topological space* (CTS).

Introduce local and wide objects $(L),(W)$ as ONHs of abstract points L and W, Fig. 1,b:

$$((L)\subset((L)\bigcup(S))\subset((L)\bigcup(S)\bigcup(W)))\subset U\ . \tag{1}$$

Define local and wide *simplex topological space-structures (simplexes)*:

$$\begin{cases} SL=((L)\bigcup(S))\subset U; \\ SW=((W)\bigcup(S))\subset U\ . \end{cases} \tag{2}$$

Let *complex topological space-structure (complex)*:

$$C=SL\bigcup SW\ . \tag{3}$$

The category "space" (SP) we treat as Subject's *operator* (method, procedure etc.) applied to the Object for study its properties through the "Subject-to-Object" interaction resulted in Object's image reflection in Subject's space; this image we call "*space-structure*" SPS. The SPS accusation is *direct problem* (Object *analysis*). The *inverse problem* is Object *model synthesis*, e.g. constructing "*inverse operator*" SP^{-1} for SPS. The explication of abstract categories "space", "space-structure" and "model" are "*topological space*" TS, "*topological space-structure*" and "*topological model*" TM (these terms are discussed in more detail within the section 2 of this paper):

$$\begin{cases} TS \times O = TSS; \\ TS^{-1} \times TSS = TM \to O; \\ TS \times TS^{-1} = TS^{-1} \times TS \to I. \end{cases} \quad (4)$$

Onwards, we discuss simplex topological frameworks particular for local object (L) understood as *open local area network graph* (G) with not more than N *open poles* for outside communication where any point (vertex) has not more than $N+1$ ports for edge links. The graph considers be simplest (non-directed, non-weighted). Any edge of the graph (G) presumably means the adjacent point's square conductivity in *hop-units* (similar to electrical circuit conductivity $g^2 = \frac{1}{R^2}$ as inverse function to resistance R).

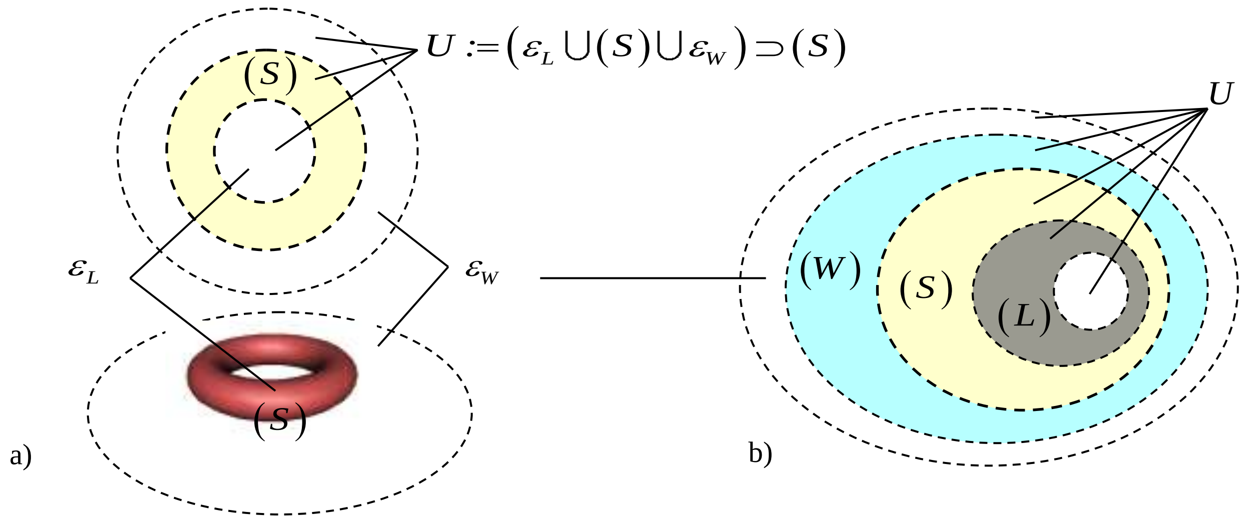

Figure 1 – Complex topological space (a) and space-structure (b) with trivial topology

2. Network graph presentation in topological space

To apply generic topological framework to network graph analysis consider classic definition of topological space and topology after [6].

<u>Definition 1</u>.

A class Ç of objects («points») is a topological space if and only if it can be expressed as a union of a family J of point sets which contains: 1) the intersection of every pair of its sets; 2) the union of the sets in every subfamily. J is topology for the space Ç, and the elements of J are called open sets relative to the topology J. A family B of open sets is a base for the topology J if and only if every set of J is the union of the sets in B. A given space may admit more than one topology; every space admits the indiscrete (trivial) topology comprising only Ç and the empty set, and the discrete topology comprising all the subsets of.

Definition 1 relies on: eight prior terms (class, point, object, point set, family, subfamily, element of family, set in subfamily; two prior operations (union of sets, intersection of sets); one rule to prove whether class Ç is topological space or not (if and only if a class *can be expressed* as a union of a family of point sets"). It seems there is excessive number of prior terms in definition 1, as some of them look synonyms: class, set and family; subfamily and "subset" (not mentioned in definition); object, element of set and element of family.

To adapt category of topological space to engineering application we introduce an equivalent definition relied on the E. Zermelo choice axiom [7]. Following that a "set" is defined if a choice function CHF to determine what things are included into the set. Now, synonyms class, point, object, point set, family etc. can be eliminated due to the "set of concrete things" (e.g. the set of open neighborhoods of point S). Therefore, the "subset" is another set. We also avoid terms "empty set" $\varnothing$ and "open set" in TS definition; instead, subject and object neighborhoods suppose open entities.

Definition 2. *The topological space is the union of subject open neighborhoods.*

Definition 3. *The object's presentation in topological space is topology.*

Definition 2 does not explicitly declares the properties of point sets intersections and unions (given in classic TS definition 1), as these properties are transparent with definition 2. Truly, any two subject's ONH always have intersection with (S); therefore, any couple ONH intersection belongs to the union of ONH. Again, any partial union of ONH belongs to their common union. Also, one may see that no topology defined for topological space itself in definition 2 (e.g. TS has no particular topology until a concrete object is reflected in this TS in view of topological space-structure TSS). Thus, topology as cognitive category is determined due to subject-object interaction in definition 3. For the same reason, topology term here is not referred to as solely object property regardless subject's view point.

Definitions 2 and 3 are explicated for open network graph G with vertices A, B, C, D, E and F where A and B are open poles of the graph related to subject view point S, Fig. 2,a. The topological space TS is formed by the union of four nested subject's ONHs $\varepsilon_S^0 = (S) \subset \varepsilon_S^1 \subset \varepsilon_S^2 \subset \varepsilon_S^3$:

$$TS = \bigcup\left(\varepsilon_S^0, \varepsilon_S^1, \varepsilon_S^2, \varepsilon_S^3\right). \tag{5}$$

It is easy to verify neighborhoods intersection and unification properties: $\varepsilon_S^0 \cap \varepsilon_S^1 = (S) \subset TS$, $\varepsilon_S^1 \cap \varepsilon_S^2 = \varepsilon_S^1 \subset TS$,… ; $\varepsilon_S^0 \cup \varepsilon_S^1 = \varepsilon_S^1 \subset TS$, $\varepsilon_S^0 \cup \varepsilon_S^1 \cup \varepsilon_S^2 = \varepsilon_S^2 \subset TS$, … . No particular topology determined through these four neighborhoods but their nested structure. The subject-object related topology T towards network graph G appears in different forms regarding the applied mechanism of subject's ONH design.

A simple mechanism of subject's open neighborhoods (ONH) design is detecting distinguished object's partial things (vertices) captured by ONHs:

$$\varepsilon_S^0 = (S),\ \varepsilon_S^1 = ((S), A, B),\ \varepsilon_S^2 = \left(\varepsilon_S^1, C, D, E\right),\ \varepsilon_S^3 = \left(\varepsilon_S^2, F\right). \tag{6}$$

The framework (6) is topology T for topological space TS in (5) and graph G topology emerges due to direct problem solution (network graph G analysis in topological space TS): $G \times TS = T_{SG}$). The inverse problem solution (network graph G synthesis) $TS^{-1} \times T_{SG} = TM_G \to G$ is shown in Fig. 2,b. It is clear that template TS in (5) cannot sufficiently reflect graph G (Fig. 2,a); therefore, the inverse solution (graph model TM_G) is not exact copy of G but solely approaches it: $TM_G \to G$. In the train, graph edges not determined exhaustively (some link confuses are possible). Other words, the mapping $G \to T_{SG}$ for given above template is injective and the inverse problem solution is not unique (multiple variants feasible). This type of inverse task is known as *ill-posed inverse problem* [8]. The methods of this problem solution commonly referred to as *renormalization* (or *regularization*) approach [9–10]. To improve inverse reconstruction of the graph based on the direct mapping $G \times TS = T_{SG}$ some extra prior awareness towards graph links needed (e.g. prior limited number edges).

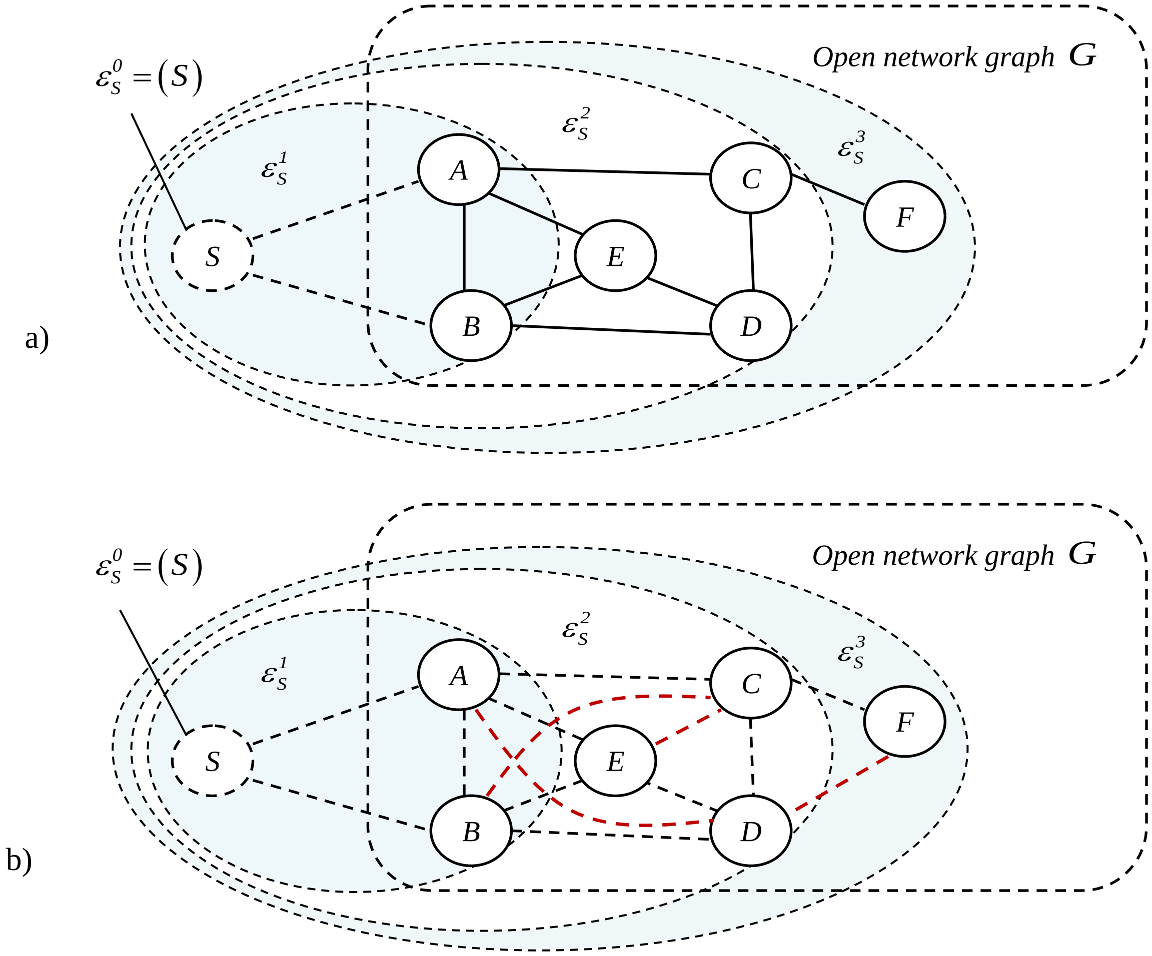

Figure 2 – Network graph presentation in topological space:
a) Analysis problem solution; b) synthesis problem solution

3. Tensor framework of topological space

Introduce differential topological neighborhood $\Delta\varepsilon_S^k$ for an arbitrary point on the open network graph G in Fig. 2,a; with respect to given subject's view point S , Fig. 3.

$$\begin{cases} \Delta\varepsilon_S^0 = \varnothing; \qquad \Delta\varepsilon_S^1 = \begin{bmatrix} S & A & B \\ A & 1 & 1 \\ B & 1 & 1 \end{bmatrix}; \\ \Delta\varepsilon_S^2 = \Delta\varepsilon_A^1 \cup \Delta\varepsilon_B^1; \\ \Delta\varepsilon_A^1 = \begin{bmatrix} A & C & E \\ C & 1 & 0 \\ E & 0 & 1 \end{bmatrix}; \qquad \Delta\varepsilon_B^1 = \begin{bmatrix} B & D & E \\ D & 1 & 1 \\ E & 1 & 1 \end{bmatrix}; \\ \Delta\varepsilon_S^3 = \Delta\varepsilon_C^1 \cup \Delta\varepsilon_D^1 \cup \Delta\varepsilon_E^1; \\ \Delta\varepsilon_C^1 = \begin{bmatrix} C & D & F \\ D & 1 & 0 \\ F & 0 & 1 \end{bmatrix}; \qquad \Delta\varepsilon_D^1 = \varnothing; \qquad \Delta\varepsilon_E^1 = \varnothing. \end{cases} \tag{7}$$

The diagonal units of matrices $\Delta\varepsilon_S^1$ and $\Delta\varepsilon_B^1$ in (7) symbolize subject's relations to open poles A and B or poles D and E for the network graph G; the non-diagonal units in $\Delta\varepsilon_S^1$ indicate non-directed linkage presence between vertices A and B (zero if no linkage, e.g. $\Delta\varepsilon_A^1$ and $\Delta\varepsilon_C^1$). According to (7) the differential framework of topological space (denote DTS) is

$$\varepsilon_S^0 = (S),\ \varepsilon_S^1 = \left((S), \varepsilon_S^0 \cup \Delta\varepsilon_S^1\right),\ \varepsilon_S^2 = \left(\varepsilon_S^1, \varepsilon_S^1 \cup \Delta\varepsilon_S^2\right),\ \varepsilon_S^3 = \left(\varepsilon_S^2, \varepsilon_S^2 \cup \Delta\varepsilon_S^3\right). \tag{8}$$

It is clear, that differential framework of topological space like given in (8) performs bijectively mapping the simple topological graph G in differential topological space-structure, or differential topology DT:

$$\begin{cases} DTS^{-1} \times DTS = I; \\ DTS \times G = G_S^{\Delta}; \\ DTS^{-1} \times G_S^{\Delta} = G. \end{cases} . \tag{9}$$

Therefore, differential topology $DT(G)$ of network graph G is isomorphic to G :

$$DT(G) = DTS \times G \leftrightarrow G. \tag{10}$$

It is quite obvious that changing subject's view point S on the graph G (e.g. varying open poles of the graph) will change the graph presentation G_S^{Δ} in differential topological space; however, the graph itself invariant to these presentations and can always be accurately reconstructed on the base of any given G_S^{Δ}. The invariance property of differential framework of topological space is character to tensor models [11].

Based in this premise we call differential framework of topological space (8) *tensor topological space*; differential topology G_S^{Δ} in (9) is called *topological tensor*. Tensor framework of topological analysis and synthesis enables evolution of general models theory towards the comprehensive study the heterogeneous network objects with directed links between the nodes, as well as tensor metric definition for telecommunication and information networks.

Conclusion

Telecommunication and information engineers primary exercises graphs theory as appropriate math model for network objects. A significant part of functional analysis fundamentals are not yet explicitly implemented in networking science disciplines because of serious issues when mapping irregular heterogeneous and multi connected topological network structures to known functional spaces of homogeneous nature. This work presents an adapted interpretation for basic categories of general model theory such as topological space, topology, topological operators and direct/inverse topological transformation problems. The core idea is that a subjective view point introduced as principal entity in topological analysis and synthesis of network graph objects wherein topology model appears as a result of subject-object interaction. Therefore, topological space itself is constructed as subject's defined coordinate system with particular topological basis. An enhanced tensor framework of topological space and topology formulated with respect to invariant topological structure of network graph obtained due to the inverse problem solution. Given approach aims to benefit network metric properties presentation in terms of advanced methods of differential geometry and field theory.

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