doi: https://doi.org/10.15407/dopovidi2017.01.017
UDC 517.5

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Approximation of 2π-periodic functions by Taylor — Abel — Poisson operators in the integral metric

Presented by Academician of the NAS of Ukraine V.L. Makarov

We obtain direct and inverse approximation theorems of 2π-periodic functions by Taylor — Abel — Poisson operators in the integral metric.

Keywords: *direct approximation theorem, inverse approximation theorem, K-functional, linear approximation method.*

Let $L_p = L_p(\mathbb{T})$, $1 \le p \le \infty$, be the space of all functions f, given on the torus $\mathbb{T}$, with the usual norm

$$\|f\|_p := \|f\|_{L_p(\mathbb{T})} := \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(x)|^p \, \mathrm{d}x \right), & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{x \in [0,2\pi]} |f(x)|, & p = \infty. \end{cases}$$

Further, let $f \in L_1$. The Fourier coefficients of f are given by

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{e}^{-ikx} \mathrm{d}x, \qquad k \in \mathbb{Z}.$$

Here, we study approximative properties of the Taylor — Abel — Poisson operators $A_{\varrho,r}$, which are defined in the following way [1, 2]:

For $\varrho \in [0,1)$, $r \in \mathbb{N}$ and $f \in L_1$, we set

$$A_{\varrho,r}(f)(x) := \sum_{k \in \mathbb{Z}} \lambda_{|k|,r}(\varrho) \hat{f}_k \mathrm{e}^{ikx},$$

where, for $k=0,1,\ldots,r-1$, the numbers $\lambda_{k,r}(\varrho)\equiv 1$ and

$$\lambda_{k,r}(\varrho):=\sum_{j=0}^{r-1}\binom{k}{j}(1-\varrho)^j\varrho^{k-j},\ \ k=r,r+1,\ldots,\ \ \varrho\in[0,1].$$

We denote, by $f(\varrho,x)$, $0\le\varrho<1$, the Poisson integral (the Poisson operator) of f, i.e.,

$$f(\varrho,x):=\frac{1}{2\pi}\int_0^{2\pi}f(t)P(\varrho,x-t)\mathrm{d}t,$$

where $P(\varrho,t)=\dfrac{1-\varrho^2}{\left|1-\varrho\mathrm{e}^{it}\right|^2}$ is the Poisson kernel.

Leis [3] considered the transformation

$$L_{\varrho,r}(f)(x)=\sum_{k=0}^{r-1}\frac{\mathrm{d}^kf(x)}{\mathrm{d}n^k}\frac{(1-\varrho)^k}{k!},\ \ r\in\mathbb{N},$$

where

$$\frac{\mathrm{d}f(x)}{\mathrm{d}n}=-\frac{\partial f(\varrho,x)}{\partial\varrho}\bigg|_{\varrho=1}$$

is the normal derivative of the function f.

Butzer and Sunouchi [4] considered the transformation

$$B_{\varrho,r}(f)(x):=\sum_{k=0}^{r-1}(-1)^{\frac{k+1}{2}}f^{\{k\}}(x)\frac{(-\ln\varrho)^k}{k!},$$

where $f^{\{k\}}=f^{(k)}$, if $k\in 2\mathbb{Z}_+$ and $f^{\{k\}}=\tilde{f}^{(k)}$, if $(k-1)\in 2\mathbb{Z}_+$.

The relation between the operators $A_{\varrho,r}$ and the operators $L_{\varrho,r}$ and $B_{\varrho,r}$ is shown in the following relation:

$$A_{\varrho,r}(f)(x)=\sum_{k=0}^{r-1}\frac{\partial^kf(\varrho,x)}{\partial\varrho^k}\cdot\frac{(1-\varrho)^k}{k!},\tag{1}$$

which holds for any function $f\in L_1$ and for all numbers $r\in\mathbb{N}$, $\varrho\in[0,1)$, and $x\in\mathbb{T}$.

If, for a function $f\in L_1$ and for a positive integer n, there exists the function $g\in L_1$ such that

$$\hat{g}_k=0,\ \text{if}\ |k|<n\ \text{and}\ \hat{g}_k=\frac{|k|!}{(|k|-n)!}\hat{f}_k,\ \text{if}\ |k|\ge n,\ k\in\mathbb{Z},$$

then we say that, for the function f, there exists the radial derivative g of order n, for which we use the notation $f^{[n]}$. Here, we use the term “radial derivative» in view of the following fact.

If the function $f^{[r]}\in L_1$, then its Poisson integral can be presented as

$$f^{[r]}(\varrho,x)=(f(\varrho,\cdot))^{[r]}(x)=\varrho^r\frac{\partial^rf(\varrho,x)}{\partial\varrho^r},\ \ \varrho\in[0,1),\ \ x\in\mathbb{T}.$$

Hence, by virtue of the theorem of limit values of a Poisson integral (see, for example, [5, p. 27]), for almost all $x \in \mathbb{T}$, we have $f^{[r]}(x) = \lim\limits_{\varrho \to 1-} f^{[r]}(\varrho, x)$.

In the space L_p, the K-functional of a function f (see, for example, [6, Ch. 6]) generated by the radial derivative of order n is the following quantity:

$$K_n(\delta, f)_p := \inf\left\{\|f-h\|_p + \delta^n \|h^{[n]}\|_p : h^{[n]} \in L_p\right\}, \qquad \delta > 0.$$

Theorem 1. *Assume that* $f \in L_p$, $1 \leqslant p \leqslant \infty$ $n, r \in \mathbb{N}$, $n \leqslant r$, *and* $0 < \alpha < n$. *If*

$$K_n(f^{[r-n]})_p = O(\delta^{\alpha}), \quad \delta \to 0+, \tag{2}$$

then

$$\|f - A_{\varrho,r}(f)\|_p = O((1-\varrho)^{r-n+\alpha}), \quad \varrho \to 1-. \tag{3}$$

Theorem 2. *Assume that* $f \in L_p$, $1 \leqslant p \leqslant \infty$, $n, r \in \mathbb{N}$, $n \leqslant r$, *and* $0 < \alpha < n$. *If relation (3) holds, then* $f^{[r-n]} \in L_p$, *and relation (2) also holds.*

Note that the relation $\|f - A_{\varrho,r}(f)\|_p = o((1-\varrho)^r)$, $\varrho \to 1-$, holds only in the trivial case where

$$f = \sum_{|k| \leqslant r-1} \hat{f}_k e^{ikx}.$$

In such case, the theorems are easily true. This fact is related to the so-called saturation property of the approximation method generated by the operator $A_{\varrho,r}$. In particular, in [1], it was shown that the operator $A_{\varrho,r}$ generates the linear approximation method of holomorphic functions, which is saturated in the space H_p with the saturation order $(1-\varrho)^r$ and the saturation class $H_p^{r-1}\mathrm{Lip}1$.

It is of interest to consider the case $n = 1$. In this case, by virtue of Theorem 2.4 (Ch. 6 §2 [6]), the set of all functions $f \in L_p$ satisfying the condition $K_1(\delta, f) = O(\delta^{\alpha})$, $\delta \to 0+$, $\alpha > 0$, is equivalent to the Lipschitz class

$$\mathrm{Lip}(\alpha, p) = \left\{f \in L_p : \omega(f, \delta) := \sup_{|h| \leqslant \delta} \|f(\cdot + h) - f(\cdot)\|_p = O(\delta^{\alpha}),\ \delta \to 0+\right\}.$$

Corollary 1. *Assume that* $f \in L_p$, $1 \leqslant p \leqslant \infty$, $r \in \mathbb{N}$ *and* $0 < \alpha < 1$. *The following statements are equivalent:*

1) $\|f - A_{\varrho,r}(f)\|_p = O((1-\varrho)^{r-1+\alpha})$, $\varrho \to 1-$;

2) $f^{[r-1]} \in \mathrm{Lip}(\alpha, p)$.

This work was supported in part by the FP7-People-2011-IRSES project number 295164 (EU-MLS: EU-Ukrainian Mathematicians for Life Sciences).

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Received 06.09.2016

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НАБЛИЖЕННЯ 2π-ПЕРІОДИЧНИХ ФУНКЦІЙ ОПЕРАТОРАМИ
ТЕЙЛОРА — АБЕЛЯ — ПУАССОНА В ІНТЕГРАЛЬНІЙ МЕТРИЦІ

Отримано прямі та обернені теореми наближення 2π-періодичних функцій операторами Тейлора — Абеля — Пуассона в інтегральній метриці.

Ключові слова: *пряма теорема наближення, обернена теорема наближення, K-функціонал, лінійний метод наближення.*

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ПРИБЛИЖЕНИЕ 2π-ПЕРИОДИЧЕСКИХ ФУНКЦИЙ ОПЕРАТОРАМИ
ТЕЙЛОРА — АБЕЛЯ — ПУАССОНА В ИНТЕГРАЛЬНОЙ МЕТРИКЕ

Получены прямые и обратные теоремы приближения 2π-периодических функций операторами Тейлора — Абеля — Пуассона в интегральной метрике.

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