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On analogs of some group-theoretic concepts and results for Leibniz algebras

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An algebra L over a field F is said to be a Leibniz algebra (more precisely a left Leibniz algebra) if it satisfies the Leibniz identity: $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ for all $a, b, c \in L$. Leibniz algebras are generalizations of Lie algebras. We consider some classes of generalized nilpotent Leibniz algebras (hypercentral, locally nilpotent algebras, and algebras with the idealizer condition) and show their some basic properties.

Keywords: *Leibniz algebra, Lie algebra, ideal, ascendant subalgebra, left center, right center, center of a Leibniz algebra, idealizer condition, locally nilpotent Leibniz algebra, hypercentral Leibniz algebra, Leibniz algebra with the idealizer condition.*

The concept of nilpotency arises in many algebraic disciplines and plays a key role there. One of the sources of its origin is triangular matrices. The ring theoretical concept of a commutator of two triangular matrices led to the zero-triangular matrices, the nilpotency in associative rings, the lower central series, and the concept of nilpotency in Lie algebras. The concept of a group-theoretic commutator of two nonsingular triangular matrices led to unitriangular matrices and to the concept of the lower central series in a group of matrices. At the first stage, this commonality of origin brought some parallelism in approaches. However, then the specificity of each theory introduces its own modifications. Nevertheless, it turned out that, in many cases, the same approaches led to comparable results in groups and Lie algebras. This parallelism runs through the book [1] and was noted in many articles devoted to Lie algebras, in particular, in work [2]. One of the interesting generalizations of Lie algebras is Leibniz algebras. Therefore, the following question naturally arises: Which of the group-theoretic concepts and results have analogs in Leibniz algebras?

An algebra L over a field F is said to be a *Leibniz algebra* (more precisely, a *left Leibniz algebra*) if it satisfies the Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]] \text{ for all } a, b, c \in L.$$

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Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a, a] = 0$ for every element $a \in L$. For this reason, we may consider Leibniz algebras as "non-anticommutative" analogs of Lie algebras.

Leibniz algebra appeared first in the works by A.M. Bloh [3–5], in which he called them the D -algebras. However, in that time, these researches were not in demand, and they have not been properly developed. Real interest in Leibniz algebras arose only after two decades. It was happened thanks to the work by J.L. Loday [6], who "rediscovered" these algebras and used the term *Leibniz algebras*, since it was Leibniz who discovered and proved the "Leibniz rule" for the differentiation of functions. The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K -theory, loop spaces, noncommutative geometry, and so on. The theory of Leibniz algebras develops quite intensively now. However, it should be noted that most of the obtained results refer to finite-dimensional Leibniz algebras, and, in a greater part of the latter, algebras over fields of characteristic zero are only considered. This also applies to nilpotent Leibniz algebras. The concept of nilpotency for the Leibniz algebras is introduced as follows.

Let L be a Leibniz algebra over a field F . If A, B are subspaces of L , then $[A, B]$ will denote a subspace generated by all elements $[a, b]$, where $a \in A, b \in B$. We note that if A is an ideal of L , then $[A, A]$ is also an ideal of L .

If M is a non-empty subset of L , then $\langle M \rangle$ denotes the subalgebra of L generated by M .

Let L be a Leibniz algebra. We define the lower central series of L

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \dots \supseteq \gamma_\alpha(L) \supseteq \gamma_{\alpha+1}(L) \supseteq \dots \supseteq \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L, \gamma_2(L) = [L, L]$ and, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals $\alpha, \gamma_\lambda(L) = \bigcap_{\mu < \lambda} \gamma_\mu(L)$ for limit ordinals λ . It is possible to show that every term of this series is an ideal of L . The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L . We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \dots, L] \dots]]$.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be a *nilpotent of the nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote by $\text{ncl}(L)$ the nilpotency class of L .

In some algebraic structures, another definition of nilpotency based on the concept of the (upper) central series is used. In fact, suppose that L is a nilpotent Leibniz algebra and $\gamma_{k+1}(L) = \langle 0 \rangle$. For each factor $\gamma_j(L)/\gamma_{j+1}(L)$, we have $[L, \gamma_j(L)] = \gamma_{j+1}(L)$ and $[\gamma_j(L), L] \leq \gamma_{j+1}(L)$, and this leads us to the following concepts.

Let A, B be the ideals of L such that $A \leq B$. The factor B/A is called *central* (in L) if $[L, B], [B, L] \leq A$.

The center $\zeta(L)$ of a Leibniz algebra L is defined in the following way:

$$\zeta(L) = \{ x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L \}.$$

Clearly, $\zeta(L)$ is an ideal of L . In particular, we can consider the factor-algebra $L/\zeta(L)$. Starting from the center, we can define the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \dots \leq \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \dots \leq \zeta_\gamma(L) = \zeta_\infty(L)$$

of Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L , and, recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α , and $\zeta_\lambda(L) = \bigcup_{\mu < \lambda} \zeta_\mu(L)$ for limit ordinals λ .

By definition, each term of this series is an ideal of L . The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L . A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter. Denote, by $zl(L)$, the length of the upper central series of L . In work [7], the connection between the lower and upper central series in nilpotent Leibniz algebras has been considered. It was proved that, in this case, the lengths of the lower and upper central series coincide. Moreover, they are the least among the lengths of all other central series.

The concepts of upper and lower central series introduced here immediately lead to the following classes of Leibniz algebras.

A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be *hypocentral* if it coincides with the lower hypercenter.

In the case of finite dimensional algebras, these two concepts coincide, but in general, these two classes are very different. Thus, for finitely generated hypercentral Leibniz algebras, we have

Theorem A. *Let L be a finitely generated Leibniz algebra over a field F . If L is hypercentral, then L is nilpotent. Moreover, L has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group-theoretic result proved by A.I. Mal'cev [8].

At the same time, a finitely generated hypocentral Leibniz algebra can have infinite dimension. Thus, a cyclic Leibniz algebra $\langle a \rangle$, where an element a has infinite depth, is hypocentral and has infinite dimension (see [9]).

A Leibniz algebra L is said to be *locally nilpotent* if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras.

We obtained the following characterization of hypercentral Leibniz algebras.

Theorem B. *Let L be a Leibniz algebra over a field F . Then L is hypercentral if and only if, for each element $a \in L$ and every countable subset $\{x_n \mid n \in \mathbb{N}\}$ of elements of L , there exists a positive integer k such that all commutators $[x_p, \dots, x_p, a, x_{j+p}, \dots, x_k]$ are zeros for all $j, 0 \leq j \leq k$.*

Corollary. *Let L be a Leibniz algebra over a field F . Then L is hypercentral if and only if every subalgebra of L having finite or countable dimension is hypercentral.*

These results are analogs of the results proved for the groups by S.N. Chernikov.

Let L be a Leibniz algebra. If A, B are nilpotent ideals of L , then their sum $A + B$ is a nilpotent ideal of L [10, Lemma 1.5]. In this connection, the following question arises: whether an analogous assertion is valid for locally nilpotent ideals. As it was shown by B. Hartley [11], this assertion takes place for Lie algebras. Our next result gives a positive answer to this question.

Theorem C. *Let L be a Leibniz algebra over a field F , and let A, B be locally nilpotent ideals of L . Then $A + B$ is locally nilpotent.*

Corollary 1. *Let L be a Leibniz algebra over a field F and \mathcal{S} be a family of locally nilpotent ideals of L . Then a subalgebra generated by \mathcal{S} is locally nilpotent.*

Corollary 2. *Let L be a Leibniz algebra over a field F . Then L has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over field F . The greatest locally nilpotent ideal of L is called the *locally nilpotent radical* of L and will be denoted by $\text{Ln}(L)$.

These results are an analog of the results in groups proved by K.A. Hirsch [12] and B.I. Plotkin [13] (see also survey [14]).

The subalgebra $\text{Nil}(L)$ generated by all nilpotent ideals of L is called the *nil-radical* of L . Clearly, $\text{Nil}(L)$ is an ideal of L . If $L = \text{Nil}(L)$, then L is called a Leibniz *nil-algebra*. Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. Every Leibniz nil-algebra is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil-algebra, which is not hypercentral (see, e.g., [1, Ch. 6]).

Note the following important properties of locally nilpotent Leibniz algebras.

Theorem D. *Let L be a locally nilpotent Leibniz algebra over a field F .*

(i) *If $A, B, A \leq B$ are the ideals of L such that the factor B/A is L -chief, then B/A is central in L (that is, $B/A \leq \zeta(L/A)$). In particular, $\dim_F(B/A) = 1$.*

(ii) *If A is a maximal subalgebra of L , then A is an ideal of L .*

Let L be a Leibniz algebra over a field F , and let H be a subalgebra of L . The *idealizer* of H is defined by the following rule:

$$\mathbf{I}_L(H) = \{x \in L \mid [h, x], [x, h] \in H \text{ for all } h \in H\}.$$

It is possible to prove that the idealizer of H is a subalgebra of L . If L is a hypercentral (in particular, nilpotent) Leibniz algebra, then $H \neq \mathbf{I}_L(H)$. This leads us to the following class of Leibniz algebras.

Let L be a Leibniz algebra over a field F . We say that L *satisfies the idealizer condition* if $\mathbf{I}_L(A) \neq A$ for every proper subalgebra A of L .

A subalgebra A is called *ascendant* in L if there is an ascending chain of subalgebras

$$A = A_0 \leq A_1 \leq \dots \leq A_\alpha \leq A_{\alpha+1} \leq \dots \leq A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant. The last our result is the following

Theorem E. *Let L be a Leibniz algebra over a field F . If L satisfies the idealizer condition, then L is locally nilpotent.*

This result is analogous to the result proved for groups by B.I. Plotkin [15].

Again, it should be noted that Leibniz algebras with the idealizer condition will form a subclass of the class of locally nilpotent Leibniz algebras, since this is already the case for Lie algebras (see, e.g., [1, Ch. 6]).

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ПРО АНАЛОГИ ДЕЯКИХ ТЕОРЕТИКО-ГРУПОВИХ ПОНЯТЬ ТА РЕЗУЛЬТАТІВ ДЛЯ АЛГЕБР ЛЕЙБНИЦА

Алгебра L над полем F називається алгеброю Лейбніца (точніше лівою алгеброю Лейбніца), якщо вона задовольняє таку тотожність Лейбніца: $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ для всіх $a, b, c \in L$. Алгебри Лейбніца являють собою узагальнення алгебр Лі. В роботі розглянуто деякі класи узагальнено нільпотентних алгебр Лейбніца (гіперцентральні, локально нільпотентні алгебри та алгебри з ідеалізаторною умовою) та показано деякі їх базові властивості.

Ключові слова: алгебра Лейбніца, алгебра Лі, ідеал, зростаюча підалгебра, лівий центр, правий центр, центр алгебри Лейбніца, ідеалізаторна умова, локально нільпотентна алгебра Лейбніца, гіперцентральна алгебра Лейбніца, алгебра Лейбніца з ідеалізаторною умовою.

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ОБ АНАЛОГАХ НЕКОТОРЫХ ТЕОРЕТИКО-ГРУППОВЫХ ПОНЯТИЙ И РЕЗУЛЬТАТОВ ДЛЯ АЛГЕБР ЛЕЙБНИЦА

Алгебра L над полем F называется алгеброй Лейбница (точнее левой алгеброй Лейбница), если она удовлетворяет следующему тождеству Лейбница: $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ для всех $a, b, c \in L$. Алгебры Лейбница представляют собой обобщение алгебр Ли. В работе рассмотрены некоторые классы обобщенно нильпотентных алгебр Лейбница (гиперцентральные, локально нильпотентные алгебры и алгебры с идеализаторным условием) и показаны некоторые их базовые свойства.

Ключевые слова: алгебра Лейбница, алгебра Ли, идеал, возрастающая подалгебра, левый центр, правый центр, центр алгебры Лейбница, идеализаторное условие, локально нильпотентная алгебра Лейбница, гиперцентральная алгебра Лейбница, алгебра Лейбница с идеализаторным условием.