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Semilinear equations in the plane with measurable data

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We study semilinear partial differential equations in the plane, the linear part of which is written in a divergence form. The main result is given as a factorization theorem. This theorem states that every weak solution of such an equation can be represented as a composition of a weak solution of the corresponding isotropic equation in a canonical domain and a quasiconformal mapping agreed with a matrix-valued measurable coefficient appearing in the divergence part of the equation. The latter makes it possible, in particular, to remove the regularity restrictions on the boundary in the study of boundary-value problems for such semilinear equations.

Keywords: *semilinear elliptic equations, quasiconformal mappings, Beltrami equation.*

The main goal of this paper is to point out one application of quasiconformal mappings to the study of some *nonlinear* partial differential equations in the plane.

Let Ω be a domain in the complex plane $\mathbb{C}$. It is well known that the Beltrami equation

$$\omega_{\bar{z}} = \mu(z)\omega_z, \quad z \in \Omega, \tag{1}$$

where $\omega_z = \frac{1}{2}(\omega_x - i\omega_y)$, $\omega_{\bar{z}} = \frac{1}{2}(\omega_x + i\omega_y)$, $z = x + iy$, is turned out to be instrumental in the study of Riemann surfaces, Teichmüller spaces, Kleinian groups, meromorphic functions, low dimensional topology, holomorphic motion, complex dynamics, Clifford analysis, and control theory.

As known, a K-quasiconformal mapping $\omega : \Omega \to \mathbb{C}$, $K \geqslant 1$, is just a homeomorphic $W^{1,2}_{\text{loc}}(\Omega)$ solution to the Beltrami equation when the measurable coefficient μ satisfies the strong ellipticity condition $|\mu(z)| \leqslant (K-1)/(K+1)$ almost everywhere in Ω. In particular, if $\mu = 0$ in a domain $\Omega \subset \mathbb{C}$, then the Beltrami equation reduces to the Cauchy—Riemann equation and a solution ω is analytic in Ω, see, e.g., [1, 2], see also [3], and the references therein.

We will deal with *semilinear* partial differential equations

$$\operatorname{div}[A(z)\nabla u] = f(u), \tag{2}$$

linear part of which contains the elliptic operator in the divergence form, where the matrix function $A(z)$ is in the class $M^{2\times 2}(\Omega)$ of 2×2 symmetric matrix functions with measurable entries $a_{jk}(z)$, $j, k = 1, 2$, the determinant 1, and the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2 \quad \text{a.e. in} \quad \Omega \tag{3}$$

for every $\xi \in \mathbb{C}$, where $1 \leqslant K < \infty$. For the case of smooth $A(z)$, see [4, 5].

We prove the following Factorization Theorem: Every weak solution $u \in C \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ of the semilinear equation (2) with arbitrary continuous $f(u)$ can be represented as $u = T \circ \omega$, where $\omega : \Omega \to G \subseteq \mathbb{C}$ is a quasiconformal mapping agreed with the matrix function A, and $T \in C \cap W^{1,2}_{\mathrm{loc}}(G)$ is a weak solution of the semilinear Poisson equation

$$\Delta T = mf(T) \quad \text{in} \quad G, \tag{4}$$

where $m(w)$, $w \in G$, is the Jacobian of the inverse mapping $\omega^{-1}(w)$. In particular, we obtain here the semilinear Gauss—Bieberbach—Rademacher equation with the weight $m(w)$ for the case $f(u) = e^u$.

1. Some definitions and preliminary remarks. Given $A \in M^{2\times 2}(\Omega)$, let us first consider the second-order elliptic homogeneous equation

$$\operatorname{div}(A(z)\nabla u) = 0 \quad \text{a.e. in} \quad \Omega. \tag{5}$$

A function u is called a weak solution to the equation if

$$\int_{\Omega} \langle A(z)\nabla u, \nabla\varphi\rangle = 0 \quad \forall \varphi \in C_0^{\infty}(\Omega). \tag{6}$$

This is meaningful at least for $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, where $W^{1,p}_{\mathrm{loc}}(\Omega)$ stands for the well-known Sobolev space. Here, we will assume a little more regularity, namely that $u \in C \cap W^{1,2}_{\mathrm{loc}}(\Omega)$.

Let $A \in M^{2\times 2}(\Omega)$ and $u \in C \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution to (5). Then there exists $v \in C \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ called the *stream function* of u, such that

$$\nabla v = HA\nabla u \quad \text{a.e. in} \quad \Omega, \quad \text{where } H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Setting $\omega(z) = u(z) + iv(z)$ we see that ω satisfies the Beltrami equation

$$\omega_{\bar z}(z) = \mu(z)\omega_z(z) \quad \text{a.e. in} \quad \Omega, \tag{8}$$

where the complex dilatation $\mu(z)$ is given by

$$\mu(z) = \frac{a_{22}(z) - a_{11}(z) - 2ia_{12}(z)}{\det(I + A(z))}, \tag{9}$$

see, e.g., Theorem 16.1.6 in [6]. The condition of ellipticity (3) now is written as

$$|\mu(z)| \leqslant \frac{K-1}{K+1} \quad \text{a.e. in} \quad \Omega. \tag{10}$$

Thus, given any $A \in M^{2\times 2}(\Omega)$, one produces by (9) the complex dilatation $\mu(z)$ for which, in turn, by the Measurable Riemann mapping theorem, see, e.g., Theorem V.B.3 in [1] and Theorem V.1.3 in [2], the Beltrami equation (8) generates, as its solution, a quasiconfomal homeomorphism ω. We say that the matrix function A generates the corresponding quasiconformal mapping ω, or that A and ω *are agreed*.

Note also the useful fact that the quasiconformal mappings ω admit a change of variables in integrals, because homeomorphisms of the class $W^{1,2}_{\mathrm{loc}}$ are absolute continuous with respect to the area measure, see, e.g., Theorem III.6.1, Lemmas III.2.1 and III.3.3 in [2].

We complete this section with the following very important result on the composition operators in Sobolev spaces, see, e.g., [7—9].

Proposition 1. *Let* $\omega : \Omega \to \mathbb{C}$ *be a quasiconformal homeomorphism and let* $\varphi : G \to \mathbb{C}$ *belong to the class* $W^{1,2}_{\mathrm{loc}}(\Omega)$. *Then the composition function* $\varphi \circ \omega \in W^{1,2}_{\mathrm{loc}}(G)$.

The study of the superposition operators on Sobolev spaces stems from the classical article [10], see also, e.g., [11—13] for the detailed history and bibliography.

2. The basic identity. It is well known that every positive definite quadratic form

$$ds^2 = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2, \tag{11}$$

defined in a plane domain Ω, can be reduced, by means of a suitable quasiconformal change of variables, to the canonical form

$$ds^2 = \Lambda(du^2 + dv^2), \quad \Lambda \neq 0 \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{12}$$

provided that $ac - b^2 \geqslant \Delta_0 > 0$, $a > 0$, a.e. in Ω, see, e.g., [14, pp. 10-12]. This key result can be extended to every linear divergent operator of the form $\mathrm{div}[A(z)\nabla u(z)]$, $z = x + iy$, with an arbitrary matrix function $A \in M^{2\times 2}(\Omega)$.

Namely, we have already seen by direct computation that if the function T and the entries of A are sufficiently smooth, then

$$\mathrm{div}\,[A(z)\nabla(T(\omega(z)))] = J_\omega(z)\Delta T(\omega(z)), \quad z \in \Omega, \tag{13}$$

see [4, 5]. Here, $J_w(z)$ stands for the Jacobian of the mapping $w(z)$, i.e., $J_\omega(z) = \det D_\omega(z)$, where $D_w(z)$ is the Jacobian matrix of the mapping ω at the point $z \in \Omega$. Equality (14) below can be viewed as a weak counterpart to equality (13).

Proposition 2. *Let* Ω *be a domain in* $\mathbb{C}$, $A \in M^{2\times 2}(\Omega)$ *and* $\omega : \Omega \to G$ *be a quasiconformal mapping agreed with* A. *Then the equality*

$$\int_\Omega \langle A(z)\nabla(T(\omega(z))), \nabla\varphi(z)\rangle dm_z = \int_\Omega \langle D^{-1}_\omega(z)\nabla T(\omega(z)), \nabla\varphi(z)\rangle J_\omega(z)dm_z \tag{14}$$

holds for every $T \in W^{1,2}_{\mathrm{loc}}(G)$ *and for all* $\varphi \in W^{1,2}_0(\Omega)$.

Proof. Assuming that $T \in W^{1,2}_{\mathrm{loc}}(G)$ and that $\omega : \Omega \to G$ is a quasiconformal mapping agreed with $A(z)$, we see, by Proposition 1, that $u := T \circ \omega \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Since

$$\nabla u(z) = D^t_\omega(z)\nabla T(\omega(z)), \tag{15}$$

where $D^t_\omega(z)$ stands for the transpose matrix to $D_\omega(z)$ and ω satisfies the Beltrami equation (8), that can be written in the matrix form as

$$A(z)D^t_\omega(z) = D^{-1}_\omega(z)J_\omega(z)\,, \tag{16}$$

we arrive successively at the required equality (14):

$$\int\limits_\Omega \langle A(z)\nabla(T(\omega(z))), \nabla\varphi(z)\rangle dm_z = \int\limits_\Omega \langle A(z)D^t_\omega(z)\nabla T(\omega(z)), \nabla\varphi(z)\rangle dm_z = \\ = \int\limits_\Omega \langle D^{-1}_\omega(z)\nabla T(\omega(z)), \nabla\varphi(z)\rangle J_\omega(z) dm_z. \tag{17}$$

3. The main result. Let Ω be a bounded domain in $\mathbb{C}$ and let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. In this section, we study a model semilinear equation

$$\operatorname{div}[A(z)\nabla u(z)] = f(u(z)), \quad z\in\Omega\,, \tag{18}$$

as well as its Laplace counterp3art:

$$\Delta T(w) = J(w)f(T(w)), \quad w\in G = \omega(\Omega)\,, \tag{19}$$

where $\omega:\Omega\to G$ is a quasiconformal mapping agreed with $A(z)$ and $J(w)$ stands for the Jacobian of the inverse mapping $\omega^{-1}:G\to\Omega$.

We say that a function $u\in C\cap W^{1,2}_{\rm loc}(\Omega)$ is a *weak solution to Eq.* (18) if

$$\int\limits_\Omega \langle A(z)\nabla u(z), \nabla\varphi(z)\rangle dm_z + \int\limits_\Omega f(u(z))\varphi(z) dm_z = 0 \quad \forall\varphi\in C\cap W^{1,2}_0(\Omega). \tag{20}$$

We also say that a function $T\in C\cap W^{1,2}_{\rm loc}(G)$ is a *weak solution to Eq.* (19) if

$$\int\limits_G \langle \nabla T(w), \nabla\psi(w)\rangle dm_w + \int\limits_G J(w)f(T(w))\psi(w) dm_w = 0 \quad \forall\psi\in C\cap W^{1,2}_0(G). \tag{21}$$

Since $J(w)$ is the Jacobian of the mapping $\omega^{-1}(w)$ it is easy to verify, by performing the change of a variable by the formula $w=\omega(z)$ that the second integral in (21) is well-defined. Here, we again made use of the fact from Proposition 1 that the composed mapping $u(z) = T(\omega(z))$ is in $C\cap W^{1,2}_{\rm loc}(\Omega)$ if $T\in C\cap W^{1,2}_{\rm loc}(G)$ and ω is quasiconformal.

Theorem 1. *Let Ω be a domain in $\mathbb{C}$, $A\in M^{2\times2}(\Omega)$ and let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Then every weak solution u of the semilinear equation*

$$\operatorname{div}[A(z)\nabla u(z)] = f(u(z)), \quad z\in\Omega\,, \tag{22}$$

can be represented as the composition

$$u(z) = T(\omega(z))\,, \tag{23}$$

where $\omega:\Omega\to G$ is a quasiconformal mapping agreed with A and T is a weak solution to the equation

$$\Delta T(w) = J(w)f(T(w)), \quad w\in G. \tag{24}$$

Corollary 1. *If $f(u) \geqslant 0$, then the function T in Theorem 1 is subharmonic.*

Proof. Let u be a weak solution of the semilinear equation (22) and $T = u \circ \omega^{-1}$. Then $T \in C \cap W^{1,2}_{\mathrm{loc}}(G)$ by Proposition 1, and we have that

$$\int_{\Omega} \langle A(z)\nabla(T(\omega(z))), \nabla\varphi(z)\rangle dm_z + \int_{\Omega} f(T(\omega(z)))\varphi(z)\, dm_z = 0 \qquad (25)$$

for all $\varphi \in C \cap W_0^{1,2}(\Omega)$. Next, by Proposition 2,

$$\int_{\Omega} \langle A(z)\nabla(T(\omega(z))), \nabla\varphi(z)\rangle dm_z = \int_{\Omega} \langle D_{\omega}^{-1}(z)\nabla T(\omega(z)), \nabla\varphi(z)\rangle J_{\omega}(z)\, dm_z, \qquad (26)$$

and, therefore,

$$\int_{\Omega} \langle J_{\omega}(z) D_{\omega}^{-1}(z)\nabla T(\omega(z)), \nabla\varphi(z)\rangle dm_z + \int_{\Omega} f(T(\omega(z)))\varphi(z)\, dm_z = 0 \qquad (27)$$

for all $\varphi \in C \cap W_0^{1,2}(\Omega)$.

Given an arbitrary function $\psi(w) \in C \cap W_0^{1,2}(G)$, we can set $\varphi(z) = \psi(\omega(z))$ in (25) and (26), because such $\varphi \in C \cap W_0^{1,2}(\Omega)$ again by Proposition 1. Performing the change of a variable in (27) by the formula $z = \omega^{-1}(w)$, we obtain

$$\int_{G} \langle J_{\omega}(\omega^{-1}(w)) D_{\omega}^{-1}(\omega^{-1}(w))\nabla T(w), D_{\omega}^{t}(\omega^{-1}(w))\nabla\psi(w)\rangle J(w)\, dm_w +$$

$$+ \int_{G} J(w) f(T(w))\psi(w)\, dm_w = 0.$$

Since, by elementary algebraic arguments,

$$\langle J_{\omega}(\omega^{-1}(w)) D_{\omega}^{-1}(\omega^{-1}(w))\nabla T(w), D_{\omega}^{t}(\omega^{-1}(w))\nabla\psi(w)\rangle =$$

$$= J_{\omega}(\omega^{-1}(w))\langle \nabla T(w), \nabla\psi(w)\rangle,$$

and

$$J_{\omega}(\omega^{-1}(w)) = 1/J(w),$$

we see that the identity

$$\int_{G} \langle \nabla T(w), \nabla\psi(w)\rangle dm_w + \int_{G} J(w) f(T(w))\psi(w)\, dm_w = 0 \qquad (28)$$

holds for all $\psi(w) \in C \cap W_0^{1,2}(G)$. Thus, T is a weak solution to Eq. (24).

Remark 1. Inversely, since the arguments given above are invertible, we see that if T is a weak solution to Eq. (24), then the function $u(z) = T(\omega(z))$ is a weak solution to Eq. (22). Note also that, among the quasiconformal mappings $\omega: \Omega \to G$, there is a variety of the so-called volume-preserving maps, for which $J(z) \equiv 1$, $z \in \Omega$. If A generates such ω, then T is a weak solution of the quasilinear Poisson equation

$$\Delta T = f(T) \quad in \quad G \cdot \tag{29}$$

4. The final remarks. By the Measurable Riemann mapping theorem, see, e.g., Theorem V.B.3 in [1] and Theorem V.1.3 in [2], given $\mu(z)$, $z \in \Omega$, agreed with the matrix function $A \in M^{2\times 2}(\Omega)$, there exists a quasiconformal mapping $\omega : \Omega \to G$ with the complex dilatation μ. Here, if Ω is finitely connected, then G can be chosen as a circular domain whose boundary consists of circles or points, see, e.g., Theorem V.6.2 in [15]. If Ω is simply connected with a nondegenerate boundary, then we may assume that G is the unit disk $\mathbb{D}$ in $\mathbb{C}$. The latter makes it possible to remove the restrictions on the regularity of the boundary in the study of boundary-value problems for Eq. (24).

The corresponding factorization theorems can be established for other similar semilinear equations in the anisotropic case such as the nonlinear heat equation like

$$u_t - \mathrm{div}\,[A(z)\nabla u(z)] = f(u) \tag{30}$$

(the same equation describes the Brownian motion, diffusion models of the population dynamics, and many other phenomena), the nonlinear Schrödinger equation, and the nonlinear wave equation

$$u_{tt} - \mathrm{div}\,[A(z)\nabla u(z)] = f(u)\,. \tag{31}$$

Namely, one can show that every weak solution in a suitable sense for semilinear equations of such type can be factorized as the composition of a weak solution to the corresponding isotropic equation and a quasiconformal mapping agreed with the matrix function $A(z)$ as above.

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ПОЛУЛИНЕЙНЫЕ УРАВНЕНИЯ НА ПЛОСКОСТИ С ИЗМЕРИМЫМИ ДАННЫМИ

Изучены полулинейные дифференциальные уравнения в частных производных на плоскости, линейная часть которых представлена в дивергентной форме. Основной результат сформулирован в виде теоремы факторизации. Эта теорема утверждает, что любое слабое решение такого уравнения представимо в виде композиции слабого решения соответствующего изотропного уравнения в канонической области и квазиконформного отображения, согласованного с матричнозначным измеримым коэффициентом, входящим в дивергентную часть исходного уравнения. Свобода в выборе канонической области позволяет, в частности, снять некоторые ограничения на регулярность границы при исследовании краевых задач для таких полулинейных уравнений.

Ключевые слова: *полулинейные эллиптические уравнения, квазиконформные отображения, уравнение Бельтрами.*

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НАПІВЛІНІЙНІ РІВНЯННЯ НА ПЛОЩИНІ З ВИМІРНИМИ ДАНИМИ

Вивчено напівлінійні диференціальні рівняння в частинних похідних на площині, лінійна частина яких подана в дивергентній формі. Основний результат сформульований у вигляді теореми факторизації. Ця теорема стверджує, що будь-який слабкий розв'язок такого рівняння можна подати у вигляді композиції слабкого розв'язку відповідного ізотропного рівняння в канонічній області і квазіконформного відображення, узгодженого з матричнозначним вимірюваним коефіцієнтом, який входить до дивергентної частини вихідного рівняння. Свобода у виборі канонічної області дозволяє, зокрема, зняти деякі обмеження на регулярність границі при дослідженні крайових задач для таких напівлінійних рівнянь.

Ключові слова: *напівлінійні еліптичні рівняння, квазіконформні відображення, рівняння Бельтрамі.*