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# Estimation of confidence level for Value-at-Risk: statistical analysis

**Abstract**. The paper investigates the problem of estimation of the confidence level for Value-at-Risk to get the minimum VaR portfolio with a predefined level of expected return. The equation which describes the relation between the confidence level and the rate of the expected return depends on the unknown parameters of distribution of asset returns which should be estimated. The classical sample estimators for unknown parameters are used. The author has examined the properties of the estimator for the confidence level in considerable detail. Under the assumption that the asset returns are multivariate, we find the asymptotic distribution of the estimator for the confidence level. Moreover, we extend this result to the case of elliptically contoured distributed asset returns. Based on the distributional properties, the confidence interval for the confidence level for VaR is constructed and the test procedure whether the resulting portfolio is statistically different from the global minimum variance portfolio is provided. Using a simulation study, we demonstrate that our results give a good approximation even in the case of moderate sample sizes $n$=250, $n$=500 not only in the case of normally distributed asset returns, but also when asset returns follow the elliptically countered distribution. We have concluded that investors can use the results of the paper with regard to all sectors of the economy.
We used monthly asset returns of five stocks included into Dow Jones Index, namely: McDonald's, Johnson&Johnson, Procter&Gamble, AT&T, and Verizon Communications from 01 October 2010 to 01 September 2015 to give numerical illustration of our fundamental results.
**Keywords**: Portfolio Selection Problem; Value-at-Risk; Variance; Expected Return; Sample Estimator; Risk Measure
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**Оцінка рівня довіри для Value-At-Risk: статистичний аналіз**
**Анотація.** У статті досліджено проблему оцінки рівня довіри для Value-at-Risk, щоб отримати портфель з мінімальним рівнем VaR із наперед заданим рівнем очікуваної дохідності. Рівність, що описує співвідношення між рівнем довіри та рівнем очікуваної дохідності, залежить від невідомих параметрів розподілу дохідності активів, які повинні бути оцінені на практиці. У роботі використано класичні вибіркові оцінки для невідомих параметрів. Детально досліджено властивості розподілу отриманої оцінки для рівня довіри. Використовуючи властивості розподілу, побудовано інтервал довіри для рівня довіри для VaR і статистичні тести для перевірки гіпотези, чи істотно відрізняється портфель з найменшим рівнем VaR від портфеля з найменшою дисперсією. На основі методу Монте-Карло показано, що отримані результати дають хороше наближення не тільки в випадку нормально розподілених дохідностей активів, але й при дохідності активів, що мають багатовимірний еліптичний розподіл.
**Ключові слова:** інвестиційний портфель; проблема вибору структури портфеля; Value-at-Risk; дисперсія; очікувана дохідність; вибіркова оцінка; міра ризику.

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**Оценка уровня доверия для Value-At-Risk: статистический анализ**
**Аннотация.** В статье исследована проблема оценки уровня доверия для Value-At-Risk, чтобы получить портфель с минимальным уровнем VaR с заранее заданным уровнем ожидаемой доходности. Равенство, описывающее соотношение между уровнем доверия и уровнем ожидаемой доходности, зависит от неизвестных параметров распределения доходности активов, которые должны быть оценены на практике. В работе использованы классические выборочные оценки для неизвестных параметров. Подробно исследованы свойства распределения оценки для уровня доверия. На основании свойств распределения, построено интервал доверия для уровня доверия для VaR и статистические тесты для проверки гипотезы, существенно ли отличается портфель с наименьшим уровнем VaR от портфеля с наименьшей дисперсией. На основе метода Монте-Карло показано, что полученные результаты дают хорошее приближение не только в случае нормально распределенных доходностей активов, но и при доходностях активов, имеющих многомерное эллиптическое распределение.
**Ключевые слова:** инвестиционный портфель; проблема выбора структуры портфеля; Value-at-Risk; дисперсия; ожидаемая доходность; выборочная оценка; мера риска.

## 1. Introduction

To divide the wealth between assets in order to get the maximum expected return and the minimum risk is the main goal of the investor. Diversification is a basic tool for risk sharing between the financial elements used in the investment process and the consequent reduction in the overall level of risk. One of the most popular methods of application of diversification in financial activity is the use of portfolios. At the first glance, it may seem that because of risk reduction the total return of portfolio should also become smaller. But the main point in portfolio construction is that this process does not always lead to a decrease in income in spite of the fact that the total risk of the portfolio becomes smaller. This is archived by using different models of portfolio optimisation.

**2. Brief Literature Review**

The first model of portfolio optimisation was described by Markowitz in 1952 (Markowitz, 1952). In his work, Markowitz proposed to construct an optimal portfolio by minimising portfolio risk for a fixed level of expected return or, equivalently, by maximizing portfolio expected return for a given level of risk. To measure the portfolio risk Markowitz took the portfolio variance. All optimal, according to Markowitz, portfolios lie on the parabola in mean variance space and the hyperbola in mean standard deviation space (Merton, 1972). This set is known as the efficient frontier. The portfolios which belong to this frontier are called efficient by Markowitz. According to Markowitz, the portfolio is efficient if and only if its expected return cannot be increased without increasing its variance or its variance cannot be decreased without decreasing its expected return. It means that an investor who uses Markowitz's method for portfolio construction should choose a suitable portfolio only among the efficient portfolios.

The theory developed by Markowitz has one serious drawback. Portfolio variance is not a good risk measure. It may happen that high returns increase the variance. The better risk measures are based on either positive values of losses or negative values of returns. Such measures are known as downside risk measures (Krokhmal et al., 2011). The example of such measures is quantile-based measures. The most popular of each is the so-called Value-at-Risk (VaR). The VaR is nowadays recommended as a standard tool for banking supervision (Basel committee on banking supervision, 2001). The VaR shows the maximal level of loses with the probability $\alpha$. The quantity $\alpha$ is known as a confidence level. The values for $\alpha$ which are usually chosen are 0.9; 0.95; 0.99; 0.999.

The VaR framework for portfolio construction was considered in (Alexander and Baptista, 2002; Bodnar et al., 2002; Rockafellar et al., 2006a, 2006b; Kilianova and Pflug, 2009). For example, a theoretical background for the portfolio VaR minimisation is presented in (Alexander and Baptista, 2002). It is shown that the minimum VaR portfolio is efficient by Markowitz. In (Bodnar et al., 2002), the problem of parameter uncertainty for the minimum VaR portfolio is taken into account and the exact and asymptotic distributions of the characteristics of the minimum VaR portfolio are found. In spite of the fact that VaR minimization is very popular for portfolio construction it can happen that the expected return of the minimum VaR portfolio is smaller than the desirable level. This problem is actual for private investors because they are free in choosing the confidence level $\alpha$. It is a well-known fact that the minimum VaR portfolio expected return is inversely proportional to the confidence level. However, a smaller confidence level makes the portfolio more risky. This problem is on a par with the problem of the Sharpe ratio maximisation when the VaR is chosen as the portfolio risk measure (Bodnar and Zabolotskyy, 2013). It is shown that a portfolio with the maximum Sharpe ratio can be constructed as the minimum VaR portfolio. But it is also proved that such a portfolio is very risky. For another risk measures such investigation is provided in (Rockafellar et al., 2006a, 2006b). In (Zabolotskyy and Bilyi, 2014), the relation between the expected return of the minimum VaR portfolio and the VaR confidence level is considered. Unfortunately, the investor has no possibility to use this finding in the portfolio construction process because it depends on unknown parameters of the asset return process, namely the mean vector and the covariance matrix. It means that investor should first somehow estimate these parameters. The parameter estimators are in general case random values, which is why the estimator of confidence level is also a random value.

**3. Purpose**

The purpose of the paper is to provide a statistical analysis of the estimator of the confidence level for VaR for which the expected return of minimum VaR portfolio is equal to some predefined level.

**4. Theoretical background**

Let $P_t$ be a price of an asset at time point $t$. By $X_t$ =100ln($P_t$ /$P_{t-1}$), we denote the return of the asset at time point $t$. We assume that the investor has already chosen the assets which she/he would like to include in the portfolio. Let the number of assets in the portfolio be equal to $k$. We denote the vector of asset returns at time point $t$ by $\boldsymbol{X}_t=(X_{t1}, \ldots, X_{tk})'$. We assume that the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\Sigma$ of vector $\boldsymbol{X}_t$ do not depend on time. In other words, we suppose that $\boldsymbol{X}_t$ follows the weakly stationary process.

Let $\mathbf{w}=(w_1; \ldots; w_k)'$ be the vector of portfolio weights and $\boldsymbol{i}'\mathbf{w}=1$, where $\boldsymbol{i}$ denotes the $k$-dimensional vector. The weight of the $i$-th asset in the portfolio is denoted by $w_i$. The return of the portfolio at time point $t$ can be expressed as

$$X_{\mathbf{w}}(t) = \mathbf{w}'\mathbf{X}_t = \sum_{i=1}^{k} X_{ti} w_i .$$

The expected return of portfolio with the weight vector $\mathbf{w}$ is the mean of $X_{\mathrm{w}}(t)$, i.e. $R_{\mathrm{w}}=\mathrm{E}(X_{\mathrm{w}}(t))=\mathbf{w}'\boldsymbol{\mu}$. The variance of the portfolio is equal to $V_{\mathrm{w}}=Var(X_{\mathrm{w}}(t))=\mathbf{w}'\Sigma\mathbf{w}$. The VaR at the confidence level $\alpha$ ($VaR_\alpha$) of the portfolio with the weight vector $\mathbf{w}$ is equal to the rate of return such that $P\{X_w<-VaR_\alpha\}=1-\alpha$.

The optimisation problem of the portfolio VaR minimisation (Alexander and Baptista, 2002) is given by

$$VaR_\alpha \rightarrow \min \text{ subject to } \mathbf{i}'\mathbf{w}=1, \tag{1}$$

It should be noted that the short sales are allowed, i.e. we do not use the condition $0 \le w_i \le 1$ in (1). If we additionally assume that the vector of asset returns $\boldsymbol{X}_t$ is multivariate normally distributed, then the problem (1) can be rewritten in the following form:

$$z_\alpha\sqrt{V_{\mathbf{w}}} - R_{\mathbf{w}} = z_\alpha\sqrt{\mathbf{w}'\Sigma\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu} \rightarrow \min \text{ subject to } \mathbf{i}'\mathbf{w}=1, \tag{2}$$

where $z_\alpha=-\phi^{-1}(1-\alpha)$ is the $\alpha$-quantile of the standard normal distribution. If we additionally denote

$$\mathbf{w}_{GMV}=\Sigma^{-1}\mathbf{i}/\mathbf{i}'\Sigma^{-1}\mathbf{i},\ R_{GMV}=\boldsymbol{\mu}'\Sigma^{-1}\mathbf{i}/\mathbf{i}'\Sigma^{-1}\mathbf{i},$$
$$V_{GMV}=1/\mathbf{i}'\Sigma^{-1}\mathbf{i},\ \mathbf{R}=\Sigma^{-1}-\Sigma^{-1}\mathbf{i}\,\mathbf{i}'\Sigma^{-1}/\mathbf{i}'\Sigma^{-1}\mathbf{i},\ s=\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}$$

then the solution to problem (2) can be written as follows:

$$\mathbf{w}_{VaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{z_\alpha^2 - s}}\mathbf{R}\boldsymbol{\mu} . \tag{3}$$

It may happen that the expected return $R_{VaR}$ of the minimum VaR portfolio with the weights $\mathbf{w}_{VaR}$ at confidence level $\alpha$ is smaller than the desirable value. In this situation, the investor can decrease the confidence level to get larger expected return. In (Zabolotskyy and Bilyy, 2014) it is shown that if the investor is interested in the minimum VaR portfolio with the expected return equal to $R_0$ then she/he should choose the confidence level equal to

$$\alpha_0 = \Phi\left(\sqrt{\frac{s^2 V_{GMV}}{(R_{GMV}-R_0)^2} + s}\right). \tag{4}$$

The expression (4) cannot be used in practice because it depends on the unknown parameters of the asset returns process $\boldsymbol{\mu}$ and $\Sigma$. These quantities have to be estimated. We make use of the sample estimators. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be independent realisations of the vector of asset returns. The sample estimators are expressed as

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j,\ \hat{\Sigma} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{X}_j-\hat{\boldsymbol{\mu}})(\mathbf{X}_j-\hat{\boldsymbol{\mu}})'. \tag{5}$$

Replacing the unknown parameters $\boldsymbol{\mu}$ and $\Sigma$ in (4) by their estimators (5) we get an estimator of $\alpha_0$

$$\hat{\alpha}_0 = \Phi\left(\sqrt{\frac{\hat{s}^2 \hat{V}_{GMV}}{(\hat{R}_{GMV}-R_0)^2} + \hat{s}}\right). \tag{6}$$

Our aim is to investigate the distributional properties of the sample estimator of $\alpha_0$ (6). These properties are heavily dependent on the distributional properties of the vector of asset returns $\boldsymbol{X}_t$.

*Normally distributed asset returns*

We assume that the vector of asset returns $\boldsymbol{X}_t$ is multivariate normally distributed with the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\Sigma$. This assumption is criticised in financial literature because the returns with high frequency (daily, hourly, …) do not satisfy this assumption in practice. However, the assumption of normality is relevant to monthly return (Fama, 1976). Taking into account that our aim is to get first theoretical results about distribution of sample estimator of $\alpha_0$ (6) the assumption of normality is fully correct in our case.

To simplify our further notation we make use of the following definitions

$$\Delta=(R_{GMV}-R_0)^2;\ a=sV_{GMV}+\Delta;\ I=\sqrt{\frac{s^2V_{GMV}}{\Delta}+s} \tag{7}$$

The asymptotic distribution of sample estimator (6) is presented in Theorem 1.

**Theorem 1.** Let us form a portfolio within k assets. Let $\boldsymbol{X}_t$ be the $k$-dimensional vector of asset returns and $\boldsymbol{X}_t\sim N_k(\boldsymbol{\mu}, \Sigma)$. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be independent realisations of the vector of asset returns and $\boldsymbol{\mu}\neq\boldsymbol{\mu}\mathbf{i}$. Then, the asymptotic distribution of sample estimator of $\alpha_0$ (6) is given as follows:

$$\sqrt{n}(\hat{\alpha}_0-\alpha_0)\sim N(0,\sigma_0^2), \tag{8}$$

with

$$\sigma_0^2=\frac{\varphi^2(I)}{I^2}\cdot\frac{s}{2\Delta^3}\left(2s^3V_{GMV}^3(1+s)^2+s^3V_{GMV}^2\Delta+(2a-\Delta)^2(2+s)\Delta\right), \tag{9}$$

where $I$, $s$, $\Delta$ are given in (7) and $\varphi$ is the density of the standard normal distribution.

*Proof.* From the delta method (DasGupta, 2008; Bodnar et al., 2009) and denoted by the symbol $\xrightarrow{d}$ convergence in distribution we get

$$\sqrt{n}\left(\begin{pmatrix}\hat{R}_{GMV}\\ \hat{V}_{GMV}\\ \hat{s}\end{pmatrix}-\begin{pmatrix}R_{GMV}\\ V_{GMV}\\ s\end{pmatrix}\right)\xrightarrow{d}N\left(\begin{pmatrix}0\\0\\0\end{pmatrix};\begin{pmatrix}V_{GMV}(1+s) & 0 & 0\\ 0 & 2V_{GMV}^2 & 0\\ 0 & 0 & 2s(2+s)\end{pmatrix}\right). \tag{10}$$

Using the application of the delta method and (10) we get,

$$\sqrt{n}(\hat{\alpha}_0-\alpha_0)\sim N(0,\sigma_0^2),$$

where

$$\sigma_0^2=\left(\frac{\partial\alpha_0}{\partial R_{GMV}};\frac{\partial\alpha_0}{\partial V_{GMV}};\frac{\partial\alpha_0}{\partial s}\right)\cdot\begin{pmatrix}V_{GMV}(1+s) & 0 & 0\\ 0 & 2V_{GMV}^2 & 0\\ 0 & 0 & 2s(2+s)\end{pmatrix}\cdot\left(\frac{\partial\alpha_0}{\partial R_{GMV}};\frac{\partial\alpha_0}{\partial V_{GMV}};\frac{\partial\alpha_0}{\partial s}\right)' \tag{11}$$

This leads to:

$$\frac{\partial\alpha_0}{\partial R_{GMV}}=\frac{\varphi(I)}{I}\cdot\frac{s^2}{\Delta}\cdot\left(-\frac{V_{GMV}}{\sqrt{\Delta}}\right),\ \frac{\partial\alpha_0}{\partial V_{GMV}}=\frac{\varphi(I)}{2I}\cdot\frac{s^2}{\Delta},\ \frac{\partial\alpha_0}{\partial s}=\frac{\varphi(I)}{I}\cdot\frac{s^2}{\Delta}\cdot\left(\frac{2a-\Delta}{2s^2}\right).$$

Inserting the derivatives in (11) we get the statement of the theorem. The theorem is proved.

The straightforward use of the result of Theorem 1 is impossible because the variance $\sigma_0^2$ depends on the unknown parameters of the asset return distribution $\boldsymbol{\mu}$ and $\Sigma$. The investor can estimate these parameters using the sample estimators (5). Then, the estimator of $\sigma_0^2$ will have the following form:

$$\hat{\sigma}_0^2=\frac{\varphi^2(\hat{I})}{\hat{I}^2}\cdot\frac{\hat{s}}{2\hat{\Delta}^3}\left(2\hat{s}^3\hat{V}_{GMV}^3(1+\hat{s})^2+\hat{s}^3\hat{V}_{GMV}^2\hat{\Delta}+(2\hat{a}-\hat{\Delta})^2(2+\hat{s})\hat{\Delta}\right). \tag{12}$$

The correctness of the estimator (12) is provided by the next theorem.

**Theorem 2.** Under conditions of Theorem 1, by $n\to\infty$ the estimator (12) almost surely converges to its true value of $\sigma_0^2$

$$\hat{\sigma}_0^2=\frac{\varphi^2(\hat{I})}{\hat{I}^2}\cdot\frac{\hat{s}}{2\hat{\Delta}^3}\left(2\hat{s}^3\hat{V}_{GMV}^3(1+\hat{s})^2+\hat{s}^3\hat{V}_{GMV}^2\hat{\Delta}+(2\hat{a}-\hat{\Delta})^2(2+\hat{s})\hat{\Delta}\right)\xrightarrow{a.s.}\sigma_0^2.$$

*Proof.* The theorem follows directly from the proof of Theorem 1 and the continuous mapping theorem (DasGupta, 2008).

The results of Theorems 1 and 2 make it possible to construct $(1-\beta)$ - confidence interval for $\alpha_0$

$$\left[\hat{\alpha}_0-\frac{\hat{\sigma}_0}{\sqrt{n}}z_{1-\beta/2};\hat{\alpha}_0+\frac{\hat{\sigma}_0}{\sqrt{n}}z_{1-\beta/2}\right]. \tag{13}$$

The confidence interval (13) allows us to test the hypothesis whether the confidence level $\alpha_0$ significantly differs from 1 or, in other words, whether the minimum VaR portfolio with the confidence level $\alpha_0$ significantly differs from the global minimum variance portfolio. To test this, it is enough to check whether the interval (13) contains 1. If it is so, then with the confidence level $(1-\beta)$ minimum VaR portfolio and the global minimum variance portfolio are statistically identical.

*Elliptically contoured distributed asset returns*

We extend the obtained results to the case of elliptically contoured distributed asset returns. We assume that the vector of asset returns $\boldsymbol{X}_t$ is elliptically contoured distributed with $\mathrm{E}(\boldsymbol{X}_t)=\boldsymbol{\mu}$ and $Var(\boldsymbol{X}_t)=\Sigma$.

The $k$-dimensional vector Y is elliptically contoured distributed if its characteristics function is given by

$$\mathrm{E}(\exp(i\mathbf{x'Y}))=\exp(i\boldsymbol{\mu}'\mathbf{x})\psi(\mathbf{x'Dx})\ \text{for}\ \mathbf{x}\in\mathbb{R}^k \tag{14}$$

where $\mathbf{D}=\boldsymbol{\Sigma}/\gamma^2$, $\gamma=(-\psi'(0)/2)^{1/2}$ and the function $\psi$ are known as the characteristic generator of the elliptical distribution. This class of distributions contains the multivariate normal distribution, the multivariate $t$-distribution, the multivariate Laplace distribution, the multivariate symmetric stable distribution among others. The distributions from this class are often used in financial literature (Gupta et al., 2013, DasGupta, 2008, Chamberlain, 1983, Berk, 1997).

Under this assumption, we get:

$$\alpha_0=F\left(\gamma\cdot\sqrt{\frac{s^2V_{GMV}}{(R_{GMV}-R_0)^2}+s}\right), \tag{15}$$

where $F$ denotes the univariate marginal distribution function of the elements in $\boldsymbol{X}_t$ and $\gamma=(-\psi'(0)/2)^{1/2}$.

**Theorem 3.** Let us form another portfolio within $k$ assets. Let $\boldsymbol{X}_t$ be the $k$-dimensional vector of asset returns and $\boldsymbol{X}_t$ is an elliptically contoured distributed asset return with $\mathrm{E}(\boldsymbol{X}_t)=\boldsymbol{\mu}$ and $Var(\boldsymbol{X}_t)=\Sigma$ and the characteristic function given in (14). Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be independent realisations of the vector of asset returns and $\boldsymbol{\mu}\neq\boldsymbol{\mu}\mathrm{i}$. Then, the asymptotic distribution of the sample estimator of $\alpha_0$ is given as follows:

$$\sqrt{n}(\hat{\alpha}_0-\alpha_0)\sim N(0,\sigma_0^2), \tag{16}$$

with

$$\sigma_0^2=\frac{\gamma^2\cdot f^2(\sqrt{\gamma}I)}{I^2}\cdot\frac{s}{2\Delta^3}\left(2s^3V_{GMV}^3(1+s\lambda)^2+s^3V_{GMV}^2\Delta\lambda+(2a-\Delta)^2(2+s\lambda)\Delta\right), \tag{17}$$

where $I$, $s$, $\Delta$ are given in (7), $\lambda=\psi''(0)/(\psi'(0))^2$ and f is the univariate marginal density function of the elements in $\boldsymbol{X}_t$.

*Proof.* The proof is similar to the proof of Theorem 1 taking into account the fact that in the case of elliptically contoured distributed asset returns we have

$$\sqrt{n}\left(\begin{pmatrix}\hat{R}_{GMV}\\ \hat{V}_{GMV}\\ \hat{s}\end{pmatrix}-\begin{pmatrix}R_{GMV}\\ V_{GMV}\\ s\end{pmatrix}\right)\xrightarrow{d}N\left(\begin{pmatrix}0\\0\\0\end{pmatrix};\begin{pmatrix}V_{GMV}(1+s\lambda) & 0 & 0\\ 0 & 2V_{GMV}^2\lambda & 0\\ 0 & 0 & 2s(2+s\lambda)\end{pmatrix}\right). \tag{18}$$

*Numerical illustration*

In this section we analyse the finite sample properties of $\hat{\alpha}_0$. We choose monthly asset returns of five ($k$=5) stocks included into Dow Jones Index (McDonald's, Johnson&Johnson, Procter&Gamble, AT&T, Verizon Communications) from 01 October 2010 to 01 September 2015. Estimating from the data the mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$ and using these estimators we can calculate the expected returns of the global minimum variance portfolio and the minimum VaR portfolio with the confidence level $\alpha$=0,95: $R_{GMV}$=0,387 and $R_{VaR}$=0,428. We construct 12 confidence intervals (for confidence levels $(1-\beta)$ from {0,9; 0,95; 0,99} and sample sizes n from {120; 250; 500; 1000}) for the confidence level for VaR to get the expected return $R_0$ equal to 0,428. The results are presented in Table 1. We observe that minimum $\text{VaR}_{0,95}$ portfolio is statistically identical for all sample sizes and for all confidence levels if compared to the global minimum variance portfolio.

sample density of $\sqrt{n}(\hat{\alpha}_0 - \alpha_0)$ can be well approximated by a normal distribution and the resulting approximation performs very well for moderate sample sizes in spite of the fact that the convergence to the limit distribution is a little bit slower in the case of t-distributed asset returns.

**5. Conclusion**

The minimum VaR portfolios are very popular not only in financial literature but also from a practical perspective. They give a simple interpretation of the optimal portfolio and its risk. The dependency of these portfolios on the confidence level gives additional information about the portfolio risk. A higher confidence level implies more reliable results. However, increasing the confidence level we decrease the expected return of the minimum VaR portfolio. It may happen that the level of the expected return is smaller than the desirable value. By choosing the smaller confidence level the investor can increase the expected return of the minimum VaR portfolio.

Tab. 1: **Confidence intervals for $\alpha_0$ with $R_0$=0,428**

| | n=120 | | | n=250 | | | n=500 | | | n=1000 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.9 | 0.95 | 0.99 | 0.9 | 0.95 | 0.99 | 0.9 | 0.95 | 0.99 | 0.9 | 0.95 | 0.99 |
| Lower bound | −0.77 | −1.10 | −1.75 | −0.24 | −0.47 | −0.92 | 0.11 | −0.05 | −0.37 | 0.35 | 0.24 | 0.02 |
| Upper bound | 2.67 | 3.00 | 3.65 | 2.14 | 2.37 | 2.82 | 1.79 | 1.95 | 2.27 | 1.55 | 1.66 | 1.88 |

Source: Own research

The largest value for the confidence level for VaR for which the minimum VaR portfolio significantly differs from the global minimum variance portfolio with a probability of 0.95 can be found by equating the upper bound of the 0.95-confidence interval (13) to 1 (for different values of $n$). The results are presented in Table 2. We observe that the portfolios are very risky in all the cases.

We further compare the asymptotic densities of the sample estimator of $\alpha_0$ (6) from Theorem 1 and Theorem 3 with the exact ones. For the presentation of the results of Theorem 3 we choose the multivariate $t$-distribution with 5 degrees of freedom. The results of the simulation study are based on $10^5$ independent repetitions and are presented in Figure 1. For the desired level of the expected return we choose $R_0=2R_{VaR;0,95}$. This finding illustrates that the finite

Tab. 2: **The largest values for the confidence level $\alpha_0$ for which the minimum VaR portfolio significantly differs from the minimum variance portfolio with a probability of 0.95**

| | n=120 | n=250 | n=500 | n=1000 |
|---|---|---|---|---|
| Confidence level α | 0.64 | 0.66 | 0.69 | 0.71 |

Source: Own research

The equality which describes the relation between confidence level and the rate of the expected return of the minimum VaR portfolio was given in (Zabolotskyy and Bilyi, 2014). This equality does not have a direct practical application because it depends on the unknown parameters of distribution of asset returns process.

In the present paper, the authors have conducted a statistical analysis of the sample estimator of the confidence level for VaR, for which the minimum VaR portfolio has a desirable rate of return. Under the assumption that the asset returns are multivariate normally distributed, we find the asymptotic distribution of the estimator for confidence level. Moreover, we extend this result to the case of elliptically contoured distributed asset returns. We construct the confidence interval and provide the test theory for the confidence level. Using historical data of monthly asset returns of five stocks included into Dow Jones index, we compare the asymptotic distributions with the exact ones for different sample sizes. We have shown that the exact distribution of the estimator of confidence level for VaR in both cases can be well approximated by a normal distribution and the resulting approximation performs very well for moderate sample sizes. We have concluded that the investor can put the results of the paper into practice with regard to all sectors of the economy.

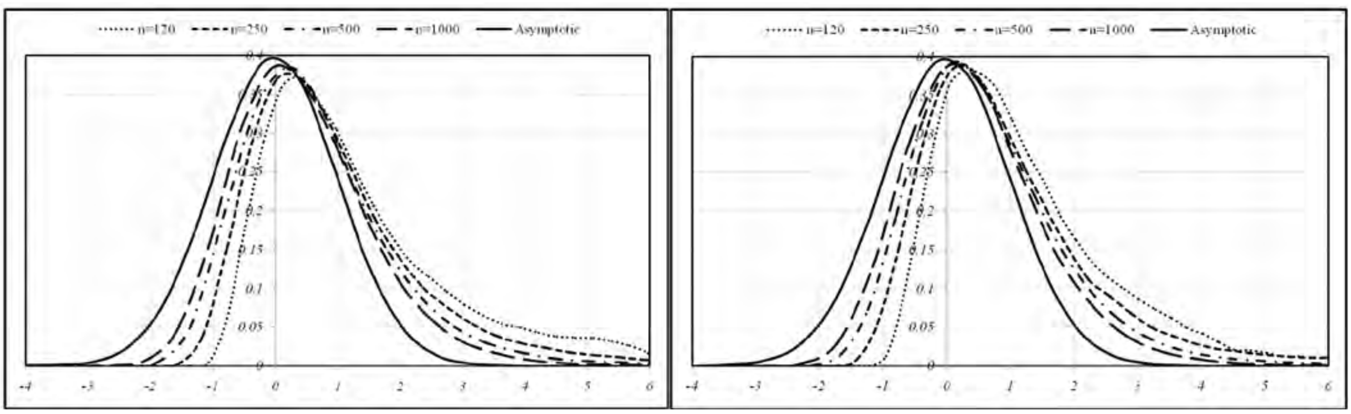


Fig. 1: **Exact and asymptotic densities of $\sqrt{n}(\hat{\alpha}_0 - \alpha_0)$ for $n$ from {120, 250, 500, 1000} in the case of normally distributed asset returns (left) and in the case of elliptically contoured distributed asset returns (right)**

Source: Own research

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