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Optimality of the minimum VaR portfolio using CVaR as a risk proxy in the context of transition to Basel III: methodology and empirical study

Abstract. The transition to the new standards in risk management announced by the Basel Committee (Basel III) leads to a change in the instrument of portfolio risk calculation. Such a transition, in particular, may lead to a loss of optimality of already formed portfolios and consequently to the necessity of portfolio restructurization. It should be noted that the process of portfolio restructurization is often quite costly not only in terms of financial costs but also in terms of time consuming. Therefore, an actual problem is the construction of tools that confirm the necessity of portfolio restructurization and, consequently, the expediency of investing resources in this process. Different statistical tests are often used to solve this problem. We are interested in tests for significance of the differences between the main characteristics of optimal portfolios obtained under different risk measures, in our case VaR and CVaR.
The paper suggests a method for testing the minimum VaR portfolio for optimality in the case when CVaR is used as a measure for risk calculation. Sample estimators of two differences between the expected returns of the minimum VaR and the minimum CVaR portfolios and between the corresponding coefficients of investor risk aversion are considered. The asymptotic distributions of these estimates are provided.
For empirical research, we select the daily returns of assets from the Dow Jones Industrial Average (DJIA) list that contains information on the prices of assets of 30 companies for the period from 01.September 2017 to 31. August 2018 (a total of 252 observations). We provide the Kolmogorov-Smirnov test about the normality of distribution of all the 30 asset returns, and for our analysis we choose only those assets for which the null hypothesis cannot be rejected at the 5% level of significance. We got 10 assets: the Coca-Cola Company; the Walt Disney Company; the Boeing Company; Johnson & Johnson; the Goldman Sachs Group; Apple Inc.; the Home Depot Inc.; Verizon Communication Inc.; UnitedHealth Group; DowDuPont Inc.
Using simulation studies based on empirical data, we show that empirical distributions of the sample estimator of the difference between the expected returns of the minimum VaR and the minimum CVaR portfolios even for a small number of assets in portfolio (k=5) are significantly asymmetric and biased, and their convergence rate to the asymptotic distribution is rather slow. Instead, the properties of the sample estimator of the difference between the corresponding coefficients of investor risk aversion are significantly better. Moreover, an adjusted estimator for this difference is constructed. It is shown that for this estimator the convergence rate of empirical variances to the asymptotic one is slightly slower than for sample estimator while the empirical biases are close to zero. This fact justifies the possibility of using this estimator in practice.
Keywords: Value-at-Risk (VaR); Conditional Value-at-Risk (CVaR); Basel III; Optimal Portfolio; Portfolio Expected Return; Investor Risk Aversion; Dow Jones Industrial Average (DJIA)
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Оптимальність портфеля фінансових активів з найменшим рівнем VaR за використання міри CVaR при обчисленні ризику в контексті переходу до критеріїв Базель III: методологія та емпіричне дослідження

Анотація. У роботі запропоновано метод тестування портфеля фінансових активів з найменшим рівнем VaR на оптимальність за умови, що основною мірою для обчислення ризиків є CVaR. Розглянуто вибіркові оцінки двох різниць між очікуваними дохідностями портфелів з найменшим рівнем VaR та CVaR та коефіцієнтами, що описують ставлення інвестора до ризику, що відповідають цим портфелям. Знайдено асимптотичні розподіли цих оцінок. На основі емпіричних даних показано, що емпіричним розподілам вибіркової оцінки різниці між очікуваними дохідностями портфелів з найменшим рівнем VaR та CVaR навіть при невеликій кількості активів у портфелі (k = 5) притаманні істотні асиметрія та зміщення, а збіжність їх до асимптотичного розподілу є доволі повільною. Натомість властивості вибіркової оцінки різниці між коефіцієнтами, що описують ставлення інвестора до ризику, що відповідають портфелям з найменшим рівнем VaR та CVaR, є значно кращими. Крім того, в роботі запропоновано виправлену оцінку для цієї різниці, для якої збіжність емпіричних дисперсій до асимптотичної дещо сповільнилася; натомість емпіричні зміщення є близькими до нуля, що обґрунтовує доцільність використання цієї оцінки на практиці.
Ключові слова: міра ризику; Value-at-Risk; умовне Value-at-Risk; оптимальний портфель; очікувана дохідність портфеля; вибіркова оцінка; коефіцієнт, що описує ставлення інвестора до ризику.

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Оптимальность портфеля финансовых активов с наименьшим уровнем VaR при использование меры CVaR для исчисления риска в контексте перехода на критерии Базель III: методология и эмпирическое исследование

Аннотация. В работе предложен метод тестирования портфеля финансовых активов с наименьшим уровнем VaR на оптимальность при условии, что основной мерой для вычисления рисков является CVaR. Рассмотрены выборочные оценки двух разниц между ожидаемыми доходностями портфелей с наименьшим уровнем VaR и CVaR и коэффициентами, описывающими отношение инвестора к риску, соответствующие этим портфелям. Найдены асимптотические распределения этих оценок. На основе эмпирических данных показано, что эмпирическим распределениям выборочной оценки разницы между ожидаемыми доходностями портфелей с наименьшим уровнем VaR и CVaR даже при небольшом количестве активов в портфеле (k = 5) присущи существенные асимметрия и смещения, а сходимость их к асимптотическому распределению есть довольно медленной. Зато свойства выборочной оценки разницы между коэффициентами, описывающими отношение инвестора к риску, соответствующим портфелям с наименьшим уровнем VaR и CVaR, значительно лучше. Кроме того, в работе предложена исправленная оценка для разницы, для которой сходимость эмпирических дисперсий к асимптотической несколько замедлилась. Вместе с тем эмпирическое смещение близко к нулю, что обосновывает целесообразность использования этой оценки на практике.
Ключевые слова: мера риска; Value-at-Risk; условное Value-at-Risk; оптимальный портфель; ожидаемая доходность портфеля; выборочная оценка; коэффициент, описывающий отношение инвестора к риску.

1. Introduction

Every financial institution planning its own activity faces the problem of financial risk estimation. From the theory and practice of finance, it is well known that investments in one asset are rather risky and the risk estimation process for each asset takes a lot of time. Therefore, financial asset portfolios are often used in practice.

2. Brief Literature Review

Markowitz's approach to portfolio construction (Markowitz, 1952) is not the only method of choosing an optimal portfolio structure. For example W. Sharpe (1994) described the method of portfolio constructing based on the Sharpe ratio maximization. It is easy to show that the resulting portfolio lies on the efficient frontier. The main disadvantage of this method is that mathematical expectation for the sample estimator of the portfolio weights with the maximum Sharpe ratio does not exist. In addition, it is shown (Schmid & Zabolotskyy, 2008) that it is impossible to construct an unbiased estimator for the given portfolio. This fact causes certain warnings concerning practical use of this portfolio. Y. Okhrin and W. Schmid (2006) considered a method for portfolio constructing based on the portfolio expected utility function maximization. The problem of determining the investor's risk aversion is one of the main drawbacks of this method. In spite of this, maximum expected utility optimal portfolios are widely used.

Most works on optimal portfolio construction use portfolio variance as a risk proxy. However, such a choice of risk measure is not optimal, since variance has several important disadvantages. In recent years, a risk measure Value-at-Risk (VaR) became very popular for calculating portfolio risk. The advantage of VaR over variance is that VaR is a quantile-based risk measure (Krokhmal et al., 2011) and therefore it takes into account only the positive values of the loss function (negative values of asset returns), so the probability of high profits does not affect the risk of loss. G. Alexander and M. Baptista (2002) suggested using VaR as a risk proxy in the portfolio theory.

At the end of the last century, P. Artzner et al. (1999) formulated four main properties of coherence, which risk measures should satisfy. These properties are monotonicity, sub-additivity, homogeneity and translational invariance. VaR is not sub-additive in the general case (Pflug, 2000). As a result, it may happen that the total risk of two assets can be greater than the sum of the risks of these assets. This fact leads to a contradiction with the basic rule of portfolio theory: the use of diversification never leads to higher risks. Another drawback of VaR is that it is not convex in the case of discrete distributed asset returns and, consequently, it can have many local extremes (Kyshakevych, 2012).

Obviously, the Artzner's axioms do not describe a single risk measure. There are several coherent risk measures, but one of the most famous is the so-called conditional VaR ($CVaR$), which is a generalization of VaR. G. Pflug (2000) proved that $CVaR$ satisfies all conditions of coherence. G. Alexander and M. Baptista (2004) showed that the minimum $CVaR$ portfolio lies on the efficient frontier or, in other words, is efficient by Markowitz. The expected return of this portfolio lies between the expected return of the global minimum variance portfolio and the expected return of the minimum VaR portfolio under the assumption that asset returns are independent and normally distributed. The main drawback of $CVaR$ is that it is not always possible to calculate its value. For example, assuming that the return of some financial asset follows the Cauchy distribution, we get that $CVaR$ cannot be definite for such an asset. It is obvious that VaR is free from such a disadvantage.

It should be pointed out that VaR has some advantages over $CVaR$ in terms of practical application in spite of all the above-mentioned disadvantages. In particular, calculating

CVaR is more laborious procedure (Chatterjee, 2014, Sarykalin et al., 2008) than calculating *VaR*. Moreover, *VaR* is more robust than *CVaR*, and *CVaR* evaluation procedure requires much more data and is much more sensitive to estimation error than *VaR*. In addition, the result of a *CVaR* calculation is reliable only if a correct model is used to describe the distribution tails.

The transition to Basel III in risk management leads to a change in the instrument of portfolio risk calculation. Such a transition, in particular, may lead to a loss of optimality of already formed portfolios and consequently to the necessity of portfolio restructurization. It should be noted that the process of portfolio restructurization is often quite costly not only in terms of financial costs but also in terms of time consuming. Therefore, an actual problem is the construction of tools that confirm the necessity of portfolio restructurization and, consequently, the expediency of investing resources in this process. Different statistical tests are often used to solve this problem. We are interested in tests for significance of the differences between the main characteristics of optimal portfolios obtained under different risk measures in our case *VaR* and *CVaR*.

3. Purpose

The purpose of the paper is a probabilistic analysis of the estimators of differences between the portfolio characteristics with structures derived under risk measures *VaR* and *CVaR* and construction of statistical tests for testing the significance of values of these differences based on this analysis.

4. Theoretical background

An important step before portfolio construction is the choice of a risk measure to calculate portfolio risks. The most popular risk measures are variance, *VaR* and *CVaR*. Let us consider them in more detail.

Let P_t be the price of some financial asset at the time point t. We define the return of this asset as follows: $X_t = 100\ln(P_t / P_{t-1})$. The main properties of log returns can be found in (Fan & Yao, 2015).

By the mid-1990s, the variance was the basis for the risk calculation. Nowadays, it is considered that better measures for practical use are measures that calculate the risk based on the corresponding quantiles of the loss function (Krokhmal et al., 2011). The most popular and most commonly used measures are Value-at-Risk (*VaR*) and its extension to a coherent measure - Conditional Value-at-Risk (*CVaR*).

Let us include k assets in the portfolio. Denoted by $\mathrm{X}_t = (X_{1t}, X_{2t}, \dots, X_{kt})'$, the k-dimensional vector of asset returns is included in the portfolio. The fraction of i-th asset in a portfolio is denoted by w_i and the portfolio - the vector of fractions $\mathrm{w} = (w_1, w_2, \dots, w_k)'$. We assume that the vector X_t follows a k-dimensional normal distribution with the mean vector $\mathrm{E}(\mathrm{X}_t) = \boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \mathrm{D}(\mathrm{X}_t)$. The main characteristics of the portfolio can be calculated as follows: expected return $R_{\mathbf{w}} = \mathrm{E}(X_{\mathbf{w}t}) = \boldsymbol{\mu}'\mathbf{w}$, variance $V_{\mathbf{w}} = \mathrm{D}(X_{\mathbf{w}t}) = \mathbf{w}'\Sigma\mathbf{w}$, where $X_{\mathbf{w}t}$ - portfolio return at time point t. Note that the assumption of normality of the distribution of the asset returns vector X_t is one of the main in the classical portfolio theory. Despite criticism of this assumption in recent decades, it is often used not only in practice, but also in theoretical works. This is because the normal distribution has attractive theoretical properties: consistency with the classical portfolio theory and with the assumptions of CAPM; the equivalence of the rules of decision-making in one and many periodic cases (Markowitz, 1991). In addition, for low-frequency returns, for example, monthly and annual asset returns, the assumption of normality of their distribution, are consistent with practical observation (Fama, 1976). Moreover, the calculating methods for *VaR* in Basel II or *CVaR* in Basel III are based on the assumption that asset returns are normally distributed.

An important role in portfolio theory plays unconditional with respect to the portfolio expected return minimization problem of portfolio variance:

$$V_{\mathrm{w}} = \mathrm{w}'\Sigma\mathrm{w} \to \min \text{ with respect to } \sum_{i=1}^{k} w_i = 1\,. \tag{1}$$

The solution of the problem (1) can be written in the following form:

$$\mathbf{w}_{GMV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\,, \tag{2}$$

where:

1 - k-dimensional vector of ones.

The portfolio of financial assets with the structure $\mathbf{w}_{GMV}$ is commonly used in the financial literature. It is known as the global minimum variance *(GMV)* portfolio. The characteristics of this portfolio can be calculated from:

$$R_{GMV} = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} - \text{expected return}\,,\quad V_{GMV} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} - \text{variance}\,. \tag{3}$$

The structure of the minimum *VaR* portfolio is observed from the following optimization problem (Alexander & Baptista, 2002):

$$VaR_\alpha(\mathbf{w}) = z_\alpha\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu} \to \min \text{ with respect to } \sum_{i=1}^{k} w_i = 1\,. \tag{4}$$

It should be noted that we do not impose the condition of positive portfolio weights as in the case of the *GMV* portfolio. G. Alexander and M. Baptista (2002) solved the problem (4). The minimum *VaR* portfolio structure and its characteristics using our denotation can be written:

$$\mathbf{w}_{VaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{z_\alpha^2 - s}}\mathbf{R}\boldsymbol{\mu}\,, \tag{5}$$

$$R_{VaR} = \mathbf{w}'_{VaR}\,\boldsymbol{\mu} = R_{GMV} + \frac{s}{\sqrt{z_\alpha^2 - s}}\sqrt{V_{GMV}}\,, \tag{6}$$

$$V_{VaR} = \mathbf{w}'_{VaR}\,\boldsymbol{\Sigma}\mathbf{w}_{VaR} = \frac{z_\alpha^2}{z_\alpha^2 - s}V_{GMV}\,, \tag{7}$$

$$M_{VaR} = \sqrt{z_\alpha^2 - s}\sqrt{V_{GMV}} - R_{GMV}\,, \tag{8}$$

where:

M_{VaR} denotes the *VaR* of portfolio with the structure $\mathbf{w}_{VaR}$,

$$\mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,,\quad s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}\,.$$

The necessary and sufficient condition to solve the problem (4) is:

$$s < z_\alpha^2\,. \tag{9}$$

T. Bodnar et al., 2012 analyzed the condition (9) using the data on daily stock prices from the Dow Jones list and showed that at statistically reasonable confidence levels ($\alpha \geq 0.9$) an investor has a possibility to construct the minimum *VaR* portfolio with high probability (> 0.999).

Using *CVaR* as a risk proxy, the problem of the minimum *CVaR* portfolio construction has the form (Alexander & Baptista, 2004):

$$CVaR_\alpha(\mathbf{w})\ k_\alpha\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu} \to \min \text{ with respect to } \sum_{i=1}^{k} w_i = 1\,. \tag{10}$$

G. Alexander and M. Baptista (2004) proved that $s < k_\alpha^2$ is the necessary and sufficient condition for the existence of the solution of the problem (10). It can be easily shown that we get $k_\alpha^2 \geq z_\alpha^2$ in the case of continuous distributions of asset returns. This implies that the minimum *CVaR* portfolio can be constructed when the minimum *VaR* portfolio can be constructed with the same assets.

G. Alexander and M. Baptista,(2004) presented the weights and the characteristics of the minimum *CVaR* portfolio:

$$\mathbf{w}_{CVaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{k_\alpha^2 - s}}\mathbf{R}\boldsymbol{\mu}\,. \tag{11}$$

$$R_{CVaR} = \mathbf{w}'_{CVaR}\,\boldsymbol{\mu} = R_{GMV} + \frac{s}{\sqrt{k_\alpha^2 - s}}\sqrt{V_{GMV}}\,, \tag{12}$$

$$V_{CVaR} = \mathbf{w}'_{CVaR}\,\boldsymbol{\Sigma}\mathbf{w}_{CVaR} = \frac{k_\alpha^2}{k_\alpha^2 - s}V_{GMV}\,, \tag{13}$$

$$M_{CVaR} = \sqrt{k_\alpha^2 - s}\sqrt{V_{GMV}} - R_{GMV}\,. \tag{14}$$

Since $k_\alpha^2 \geq z_\alpha^2$ for every confidence level α satisfies (9), we get $R_{GMV} \leq R_{CVaR} \leq R_{VaR}$. Moreover, the inequalities are strict for all acceptable values of $\alpha < 1$. Instead, all the portfolios with minimal risk coincide for $\alpha = 1$.

In 2019, Basel III should be finally implemented. The important difference between Basel II and Basel III is the proposed risk measures. In Basel II recommendations, VaR is a risk proxy instead of Basel III – $CVaR$. For continuous distributions at the same level of confidence, the value of $CVaR$ is greater than the value of VaR. The question arises: how the expected returns of the minimum VaR and the minimum $CVaR$ portfolios differ at the same confidence level. From (6) and (12), we obtain:

$$\begin{aligned}\Delta &= R_{VaR} - R_{CVaR} = \\ &= R_{GMV} + \frac{s}{\sqrt{z_\alpha^2 - s}}\sqrt{V_{GMV}} - R_{GMV} + \frac{s}{\sqrt{k_\alpha^2 - s}}\sqrt{V_{GMV}} = \\ &= s\sqrt{V_{GMV}}\left(\frac{1}{\sqrt{z_\alpha^2 - s}} - \frac{1}{\sqrt{k_\alpha^2 - s}}\right)\end{aligned} \tag{15}$$

Note that the equality (15) remains correct under the assumption that vector of asset returns X_t follows the k-dimensional conditional normal distribution with the parameters $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$.

The universal method for the efficient frontier construction is the method of maximization of the portfolio expected utility. In the portfolio theory, the portfolio with expected quadratic utility is determined as follows:

$$U(\mathbf{w}) = R_{\mathbf{w}} - \frac{\beta}{2}\sqrt{V_{\mathbf{w}}}\,,$$

where β denotes the coefficient of investor's risk aversion. We get the weights of the optimal portfolio with the maximum expected quadratic utility from the following optimization problem:

$$U(\mathbf{w}) \to \max \text{ with respect to } \sum_{i=1}^{k} w_i = 1.$$

They can be written as follows:

$$\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}\boldsymbol{\Sigma}^{-1}\mathbf{i}} + \beta^{-1}\mathbf{R}\boldsymbol{\mu}\,. \tag{16}$$

Y. Okhrin and W. Schmid (2006) pointed out that by changing values of investor's risk aversion coefficient from 0 to +∞ we can get an arbitrary portfolio from Markowitz's efficient frontier. G. Alexander and M. Baptista (2004) proved that the minimum VaR and the minimum $CVaR$ portfolios are efficient by Markowitz. It means that it is possible for investors who construct their portfolio by its risk minimization to define the risk aversion coefficient (Das et al, 2010, Alexander & Baptista, 2011). We are interested only in cases when VaR and $CVaR$ are used as risk measures. Equating the expressions for the minimum VaR and the minimum $CVaR$ portfolio weights (5) and (11) with the weights of the maximum expected utility portfolio (16) and solving the equation with respect to corresponding coefficients of investor's risk aversion we get the following:

- for investors who constructs their portfolio by VaR minimization, the risk aversion coefficient is equal to

$$\beta_{VaR} = \frac{\sqrt{z_\alpha^2 - s}}{\sqrt{V_{GMV}}}\,;$$

- for investors who construct their portfolio by $CVaR$ minimization, the risk aversion coefficient is equal to:

$$\beta_{CVaR} = \frac{\sqrt{k_\alpha^2 - s}}{\sqrt{V_{GMV}}}\,.$$

Hence, the difference between the presented above coefficients of investor's risk aversion is equal to

$$\Delta_{ra} = \frac{1}{\sqrt{V_{GMV}}}\left(\sqrt{k_\alpha^2 - s} - \sqrt{z_\alpha^2 - s}\right). \tag{17}$$

We have shown that changing the risk proxy from VaR to $CVaR$ with the same confidence level we get different minimum risk portfolios. However, the Basel II and Basel III recommendations suggest different confidence levels for risk measures. For VaR the recommended level is 99% while for $CVaR$ it is 99.9%. It should be noted that such a high confidence level for $CVaR$ causes a negative reaction from practitioners. In a general case, we agree with such a reaction because we could not find any reasonable explanation of the choice of the confidence level for $CVaR$. From investors' point of view the new confidence level for $CVaR$ should satisfy the following: the minimum $CVaR$ portfolio with a new confidence level should be equivalent to the minimum VaR portfolio with α = 99%. We will find a relation between confidence levels for VaR and $CVaR$ under which the corresponding portfolios with the minimal risk are equivalent. Since portfolios with the minimum VaR and the minimum $CVaR$ are efficient by Markowitz, the necessary and sufficient condition for these portfolios to be equivalent is the coincidence of their expected returns. Consequently, by equating (6) and (12) we get that for a certain confidence level for VaR the confidence level for $CVaR$ should satisfy the following equation:

$$\frac{e^{-z_{\alpha_{CVaR}}^2/2}}{\sqrt{2\pi}(1 - \alpha_{CVaR})} = z_{\alpha_{VaR}}\,. \tag{18}$$

The equation (18) could not be solved analytically with respect to the confidence level for $CVaR$. We use the values for confidence level α_{VaR} which are commonly used in practice, namely {0.9, 0.95, 0.99, 0.999} and solve (18) with respect to α_{CVaR}. The results are presented in Table 1.

The results in Table 1 show us that under our assumption of equivalence of portfolios with the minimal risk we need to reduce the confidence level for $CVaR$, that is, increasing accuracy in risk treatment decreases its confidence. For example, to get equivalent the minimum VaR and the minimum $CVaR$ portfolios for α_{VaR} = 99% we should choose α_{CVaR} equal to 97.5%. The natural question arises: for which values of α_{VaR} the value of α_{CVaR} still statistically reasonable i. e. greater than 90%. Putting α_{CVaR} = 0.9 and solving the equation (18) with respect to α_{VaR} we get α_{VaR} = 0.960355. Consequently, if an investor constructs his/her portfolio by minimizing its VaR at a confidence level less than 0.96, then there is no equivalent minimum $CVaR$ portfolio at a confidence level greater than 90%.

We are not able to use the previous results concerning parameters Δ and Δ_{ra} in practice because they depend on unknown parameters of distribution of asset returns vector X_t – $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Firstly, we need to estimate these parameters. The most used method of the parameter estimation is the

Tab. 1: **Relation between confidence levels for VaR and $CVaR$ under which the corresponding portfolios with the minimal risk are equivalent**

Confidence level for VaR α_{VaR}	Confidence level for $CVaR$ α_{CVaR}
0.9	0.7543511
0.95	0.8745023
0.99	0.9742017
0.999	0.9973862

Source: Developed by the authors in program R historical method. Let us have a sample of previous values of vectors of asset returns $X_1, X_2,\dots X_n$. Based on this sample, we construct the following estimators of unknown parameters of distribution of X_t :

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}X_i,\ \hat{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})(X_i - \hat{\mu})'. \quad (19)$$

Inserting the estimators (19) instead of unknown parameters μ and Σ in expressions for Δ and Δ_{ra} (15), (17) we get the sample estimators of these parameters. Let us denote them by using a symbol $\hat{}$, i. e.:

$$\hat{\Delta} = \hat{s}\sqrt{\hat{V}_{GMV}}\left(\frac{1}{\sqrt{z_\alpha^2 - \hat{s}}} - \frac{1}{\sqrt{k_\alpha^2 - \hat{s}}}\right) \text{and } \hat{\Delta}_{ra} = \frac{1}{\sqrt{\hat{V}_{GMV}}}\left(\sqrt{k_\alpha^2 - \hat{s}} - \sqrt{z_\alpha^2 - \hat{s}}\right). \quad (20)$$

It is clear that we should interpret the estimators (20) as random values (not as constants) because the estimators of the parameters of distribution of the vector X_t - μ and Σ are random values. Therefore, to get the maximum information about the estimators of differences Δ and Δ_{ra}, we should investigate their distributions.

Let us denote $\theta = (\mu', \text{vech}(\Sigma)')'$ - a vector of unknown parameters and $\hat{\theta} = (\hat{\mu}', \text{vech}(\hat{\Sigma})')'$ - a sample estimator of θ. The operator vech is definite for an arbitrary square symmetric matrix $A = (a_{ij})$ of dimension $k \times k$ and transforms it onto $k(k+1)/2$ - dimension vector by the rule vech $(A) = (a_{11},\dots,a_{k1},\dots,a_{ii},\dots,a_{ki},\dots,a_{kk})'$. The main properties of matrix operators can be found in (Harville, 2008).

Since parameters Δ and Δ_{ra} can be treated as functions of θ, i. e. $\Delta = f(\theta)$ and $\Delta_{ra} = g(\theta)$, then from the delta-method (Brockwell and Davis, 2006) we get:

$$\sqrt{n}\left(f(\hat{\theta}) - f(\theta)\right) \xrightarrow{d} N(0, G_1'\Omega G_1),\ \sqrt{n}\left(g(\hat{\theta}) - g(\theta)\right) \xrightarrow{d} N(0, G_2'\Omega G_2),$$

where the vectors G_1 and G_2 of dimension $1\times k(k+3)/2$ formed from partial derivatives of functions f and g respectively by the vector of parameters θ, i. e. $G_1 = (\partial\Delta/\partial\mu,\ \partial\Delta/\partial\text{vech}(\Sigma))'$ and $G_2 = (\partial\Delta_{ra}/\partial\mu,\ \partial\Delta_{ra}/\partial\text{vech}(\Sigma))'$ and the matrix Ω can be found in (Brockwell & Davis, 2006).

On the other hand, we can treat the parameters Δ and Δ_{ra} as functions of parameters of the efficient frontier R_{GMV}, V_{GMV}, s, i. e. $\Delta = f_1(R_{GMV}, V_{GMV}, s)$ and $\Delta_{ra} = f_2(R_{GMV}, V_{GMV}, s)$. We get:

$$G_1'\Omega G_1 =$$
$$= (\partial f_1/\partial R_{GMV})^2(\partial R_{GMV}/\partial\theta)'\Omega(\partial R_{GMV}/\partial\theta) + (\partial f_1/\partial V_{GMV})^2(\partial V_{GMV}/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta) +$$
$$+ (\partial f_1/\partial s)^2(\partial s/\partial\theta)'\Omega(\partial s/\partial\theta) + 2(\partial f_1/\partial R_{GMV})(\partial f_1/\partial V_{GMV})(\partial R_{GMV}/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta) +$$
$$+ 2(\partial f_1/\partial R_{GMV})(\partial f_1/\partial s)(\partial R_{GMV}/\partial\theta)'\Omega(\partial s/\partial\theta) + 2(\partial f_1/\partial V_{GMV})(\partial f_1/\partial s)(\partial s/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta).$$

Analogical equality we get also for $G_2'\Omega G_2$.

From (Bodnar & Schmid, 2009) we get:

$$(\partial R_{GMV}/\partial\theta)'\Omega(\partial R_{GMV}/\partial\theta) = V_{GMV}(1+s),\ (\partial V_{GMV}/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta) = 2V_{GMV}^2,$$

$$(\partial s/\partial\theta)'\Omega(\partial s/\partial\theta) = 2s(2+s),$$

$$(\partial R_{GMV}/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta) = (\partial R_{GMV}/\partial\theta)'\Omega(\partial s/\partial\theta) = (\partial s/\partial\theta)'\Omega(\partial V_{GMV}/\partial\theta) = 0$$

Taking into account that:

$$(\partial f_1/\partial R_{GMV}) = (\partial f_2/\partial R_{GMV}) = 0,$$

$$(\partial f_1/\partial V_{GMV}) = \frac{s}{2\sqrt{V_{GMV}}}\frac{a_{ks}}{b_{ks}},\ (\partial f_2/\partial V_{GMV}) = -\frac{a_{ks}}{2V_{GMV}^{3/2}},$$

$$(\partial f_1/\partial s) = \sqrt{V_{GMV}}\left(\frac{a_{ks}}{b_{ks}} - s\frac{c_{ks}}{2b_{ks}^3}\right),\ (\partial f_2/\partial V_{GMV}) = -\frac{a_{ks}}{2b_{ks}\sqrt{V_{GMV}}},$$

where:

$$a_{ks} = \sqrt{k_\alpha^2 - s} - \sqrt{z_\alpha^2 - s},\ b_{ks} = \sqrt{k_\alpha^2 - s}\sqrt{z_\alpha^2 - s},\ c_{ks} = \left(\sqrt{k_\alpha^2 - s}\right)^3 - \left(\sqrt{z_\alpha^2 - s}\right)^3, \quad (21)$$

we are able to find asymptotic distributions of sample estimators of parameters Δ and Δ_{ra}. The next theorem summarizes the previous findings.

Theorem 1. *Let us form a portfolio within k assets, denoted by X_t – k-dimensional vector of asset returns included into portfolio at time point t. Let us assume that X_t follows k-dimensional normal distribution with the parameters μ and Σ. Also, we assume that $s < z_\alpha^2$ and $k<n$. Then for $n \to \infty$:*

$$\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N(0, \sigma_1) \text{ and } \sqrt{n}(\hat{\Delta}_{ra} - \Delta_{ra}) \xrightarrow{d} N(0, \sigma_2),$$

where:

$$\sigma_1^2 = \frac{V_{GMV}s^2a_{ks}^2}{2b_{ks}^2} + V_{GMV}\left(4s + 2s^2\right)\left(\frac{a_{ks}}{b_{ks}} - \frac{s}{2}\frac{c_{ks}}{b_{ks}^3}\right)^2,$$

$$\sigma_2^2 = \frac{a_{ks}^2}{2V_{GMV}} + \frac{1}{4V_{GMV}}\left(4s + 2s^2\right)\frac{a_{ks}^2}{b_{ks}^2},$$

$$V_{GMV} = \frac{1}{1'\Sigma^{-1}1},\ s = \mu' R\mu,\ R = \Sigma^{-1} - \frac{\Sigma^{-1}11'\Sigma^{-1}}{1'\Sigma^{-1}1}.$$

a_{ks}, b_{ks}, c_{ks} - are given in (21).

In practice, we have to use the estimators of variances σ_1 and σ_2, i. e.:

$$\hat{\sigma}_1^2 = \frac{\hat{V}_{GMV}\hat{s}^2\hat{a}_{ks}^2}{2\hat{b}_{ks}^2} + \hat{V}_{GMV}\left(4\hat{s} + 2\hat{s}^2\right)\left(\frac{\hat{a}_{ks}}{\hat{b}_{ks}} - \frac{\hat{s}}{2}\frac{\hat{c}_{ks}}{\hat{b}_{ks}^3}\right)^2,$$

$$\hat{\sigma}_2^2 = \frac{\hat{a}_{ks}^2}{2\hat{V}_{GMV}} + \frac{1}{4\hat{V}_{GMV}}\left(4\hat{s} + 2\hat{s}^2\right)\frac{\hat{a}_{ks}^2}{\hat{b}_{ks}^2}.$$

Taking into account Theorem 1.14 in DasGupta (2008) and the proof of Theorem 1, we get that the previous estimators for σ_1 and σ_2 are consistent, i. e. for $n \to \infty$:

$$\hat{\sigma}_1^2 \to \sigma_1^2 \text{ and } \hat{\sigma}_2^2 \to \sigma_2^2.$$

From the result of Theorem 1, we get one- and two-sided $(1-\gamma)$ confidence intervals for values of Δ and Δ_{ra}:

- the two-sided $(1-\gamma)$ confidence interval for Δ:

$$\left[\hat{\Delta} - \frac{\hat{\sigma}_1}{\sqrt{n}}z_{1-\gamma/2}, \hat{\Delta} + \frac{\hat{\sigma}_1}{\sqrt{n}}z_{1-\gamma/2}\right];$$

- the two-sided $(1-\gamma)$ confidence interval for Δ_{ra}:

$$\left[\hat{\Delta}_{ra} - \frac{\hat{\sigma}_2}{\sqrt{n}}z_{1-\gamma/2}, \hat{\Delta}_{ra} + \frac{\hat{\sigma}_2}{\sqrt{n}}z_{1-\gamma/2}\right];$$

- the one-sided $(1-\gamma)$ confidence intervals for Δ:

$$\left[-\infty, \hat{\Delta} + \frac{\hat{\sigma}_1}{\sqrt{n}}z_{1-\gamma}\right], \left[\hat{\Delta} - \frac{\hat{\sigma}_1}{\sqrt{n}}z_{1-\gamma}, +\infty\right];$$

- the one-sided $(1-\gamma)$ confidence intervals for Δ_{ra}:

$$\left[-\infty, \hat{\Delta}_{ra} + \frac{\hat{\sigma}_2}{\sqrt{n}}z_{1-\gamma}\right], \left[\hat{\Delta}_{ra} - \frac{\hat{\sigma}_2}{\sqrt{n}}z_{1-\gamma}, +\infty\right],$$

where z_γ denotes the γ quantile of the standard normal distribution.

The constructed confidence intervals give us the possibility to use statistical tests to check whether obtained values of differences between selected characteristics of portfolios significantly differ from some desired values. In particular, if some confidence interval contains a zero value then a value obtained for the corresponding estimator does not differ significantly from zero, and therefore the corresponding portfolio remains optimal under *CVaR* as a new risk measure. This implies that there is no necessity in restructurization of this portfolio.

5. Results

As noted before, an investor should interpret the estimators of unknown parameters as random values. Accordingly, the decision-making process based on one value of some estimator is not efficient because in the case of continuously distributed asset returns the set of possible values of the estimator is infinite. It is clear that the use of additional information about a random variable that reflects some of its characteristics will lead to the improvement in the efficiency of the decision-making process. From the probability theory and mathematical statistics, it is known that the maximum information about a random variable is provided by its distribution function or density. Unfortunately, it is not always possible to obtain exact distribution or density functions, therefore it is often suggested to consider asymptotic properties of an estimator (Ling & McAleer, 2003). The convergence of the empirical distributions obtained by simulations to the asymptotic one depends on the properties of estimators of unknown parameters under specified assumptions about the behaviour of the asset returns and is not always fast. We investigate the convergence rate of empirical distributions of estimators of the parameters Δ and Δ_{ra} to the corresponding asymptotic distributions found in Theorem 1. For this purpose, we select the daily returns of assets from the Dow Jones Industrial Average (DJIA) list that contains information on the prices of assets of 30 companies for the period from 01 September 2017 to 31 August 2018 (a total of 252 observations). We apply the Kolmogorov-Smirnov test about the normality of distribution of all the 30 asset returns and choose only that asset for which the null hypothesis cannot be rejected at the 5% level of significance. We get 10 assets:

1) the Coca-Cola Company (KO, 0.210);
2) the Walt Disney Company (DIS, 0.111);
3) the Boeing Company (BA, 0.158);
4) Johnson & Johnson (JNJ, 0.179);
5) the Goldman Sachs Group (GS, 0.604);
6) Apple Inc. (AAPL, 0.151);
7) the Home Depot Inc. (HD, 0.101);
8) Verizon Communication Inc. (VZ, 0.220);
9) UnitedHealth Group (UNH, 0.343);
10) DowDuPont Inc. (DWDP, 0.758).

In brackets, we give the abbreviations of the companies' names and the p-values of the Kolmogorov-Smirnov test. Let us consider two cases: $k = 5$ (the portfolio includes 5 assets: KO, DIS, BA, JNJ, GS); $k = 10$ (the portfolio includes ten assets). Using selected asset returns as a sample from historical data, we estimate the parameters of the normal distribution according to (19) and assume that the obtained values are precise. Consequently, we obtain the precise values of the parameters Δ and Δ_{ra}:

$k = 5$: $\Delta = 0.00097598$ and $\Delta_{ra} = 0.5665$;

$k = 10$: $\Delta = 0.0029303$ and $\Delta_{ra} = 0.5986$.

Using previous values and the results of Theorem 1, we are able to construct the asymptotic densities of the estimators of parameters Δ and Δ_{ra} for $k = 5$ and $k = 10$. Using the simulation method with the number of repetitions equal to 100,000 and for different values of sample size $n = \{250, 500, 1000, 2000\}$ we provide the empirical densities of sample estimators of the unknown parameters and estimate their means and variances. The results of the simulation study are presented in Fig. 1 for for estimator of Δ and in Fig. 2 for estimator of Δ_{ra}. We observe that convergence rate of empirical distributions to asymptotic one is satisfactory

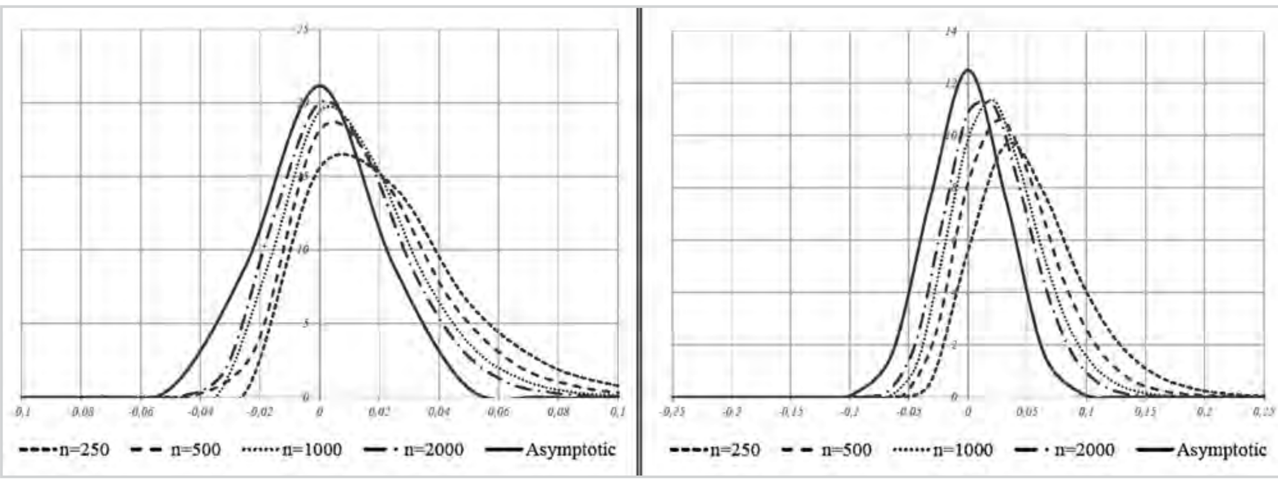

Fig. 1: **Empirical and asymptotic densities of $\sqrt{n}(\hat{\Delta} - \Delta)$ for k=5 (left) and k=10 (right) and n = {250, 500, 1000, 2000}.**
Source: Developed by the authors based on data from finance.yahoo.com in program R

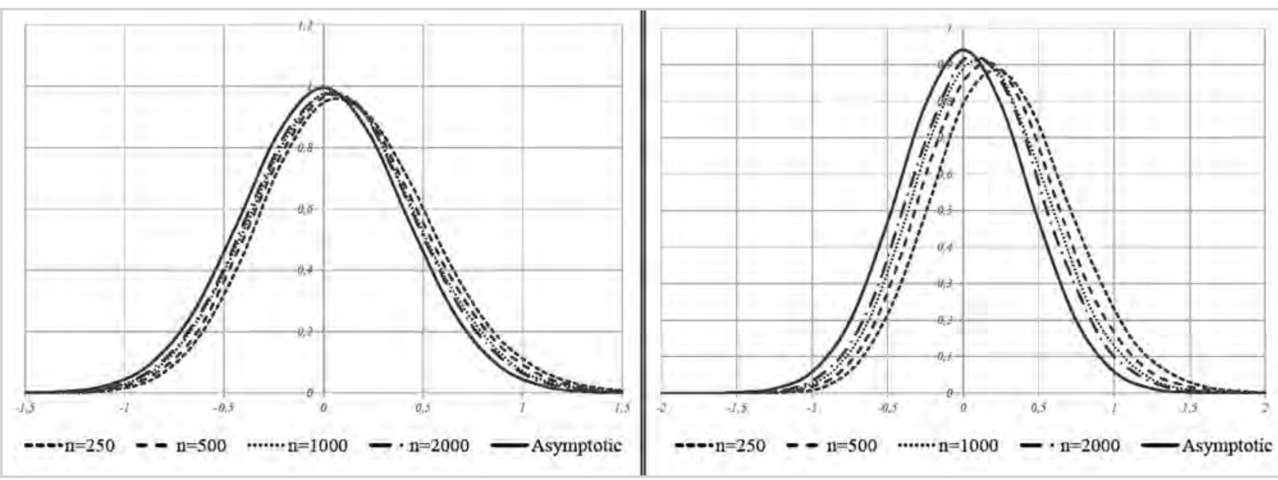

Fig. 2: **Empirical and asymptotic densities of $\sqrt{n}(\hat{\Delta}_{ra} - \Delta_{ra})$ for k=5 (left) and k=10 (right) and n = {250, 500, 1000, 2000}.**
Source: Developed by the authors based on data from finance.yahoo.com in program R

for the estimator of Δ_{ra} and unsatisfactory for the estimator of Δ. Even for $k=5$ and $n=2000$ empirical density of sample estimator of Δ is obviously asymmetric and significantly biased (with respect to the precise value). It is not surprising because the properties of the random variable $1/\hat{V}_{GMV}$ that has an inverse χ distribution are better than the properties of $\frac{\hat{s}}{\sqrt{z_\alpha^2-\hat{s}}}$ which distribution is unknown and the unconditional expectation does not exist. The values of asymptotic and empirical means and variances presented in Tables 2-3 confirm the above observations. In addition, we conclude that increasing the number k of assets in the portfolio decreases the convergence rate of the empirical characteristics of the investigated quantities to the asymptotic ones. This implies that increasing the number of assets in the portfolio requires a proportional increase in the size of the sample of historical values.

Tab. 2: **Empirical and asymptotic means and variances of $\sqrt{n}(\hat{\Delta}-\Delta)$ and $\sqrt{n}(\hat{\Delta}_{ra}-\Delta_{ra})$ for n = {250, 500, 1000, 2000, 3000} and k = 5**

		n=250	n=500	n=1000	n=2000	n=3000	Asympt.
$\sqrt{n}(\hat{\Delta}-\Delta)$	Mean	0.02402	0.01686	0.01194	0.00829	0.00693	0
	Variance	0.000699	0.000529	0.000442	0.000402	0.000391	0.000356
$\sqrt{n}(\hat{\Delta}_{ra}-\Delta_{ra})$	Mean	0.1226	0.0872	0.0596	0.0443	0.0333	0
	Variance	0.1693	0.1655	0.1642	0.1619	0.1612	0.1608

Source: Developed by the authors based on data from finance.yahoo.com in program R

Tab. 3: **Empirical and asymptotic means and variances of $\sqrt{n}(\hat{\Delta}-\Delta)$ and $\sqrt{n}(\hat{\Delta}_{ra}-\Delta_{ra})$ for n = {250, 500, 1000, 2000, 3000} and k = 10**

		n=250	n=500	n=1000	n=2000	n=3000	Asympt.
$\sqrt{n}(\hat{\Delta}-\Delta)$	Mean	0.05419	0.03765	0.02630	0.01846	0.01509	0
	Variance	0.001987	0.001519	0.001288	0.001171	0.001150	0.001018
$\sqrt{n}(\hat{\Delta}_{ra}-\Delta_{ra})$	Mean	0.2611	0.1836	0.1296	0.0912	0.0744	0
	Variance	0.2001	0.1899	0.1850	0.1831	0.1827	0.1803

Source: Developed by the authors based on data from finance.yahoo.com in program R

Using the described simulation algorithm, we consider the properties of the adjusted estimator of Δ_{ra}. The empirical and asymptotic densities of the random variable $\sqrt{n}(\hat{\Delta}^*_{ra}-\Delta_{ra})$ are presented in Figure 3. We present the empirical and asymptotic means and variances in Table 4. The results are expected as follows: the convergence rate of empirical variances to the asymptotic one is slightly slower than for sample estimator while the empirical means are close to zero.

Note that using the results presented in Zabolotskyy (2017), we are able to construct adjusted estimator of parameter Δ_{ra}:

$$\hat{\Delta}^*_{ra} = \sqrt{\frac{n-k-2}{(n-1)\hat{V}_{GMV}}\left(k_\alpha^2 - \frac{n-k-1}{n-1}\hat{s} + \frac{k-1}{n}\right)} - \sqrt{\frac{n-k-2}{(n-1)\hat{V}_{GMV}}\left(z_\alpha^2 - \frac{n-k-1}{n-1}\hat{s} + \frac{k-1}{n}\right)}.$$

Taking into account the results of the simulation study, we conclude that it is appropriate to use the estimator of parameter Δ_{ra} and its characteristics to compare the minimum VaR and the minimum $CVaR$ portfolios. Moreover, in the case of a sample estimator, its bias should be taken into account, and it can be omitted by using the adjusted estimator $\hat{\Delta}^*_{ra}$.

Remark 1. We have noted that the observed results are true under the following conditions $s < z_\alpha^2$ and $\hat{s} < z_\alpha^2$. It is shown by Bodnar et. al (2013) that under the condition $s < z_\alpha^2$ the inequality $\hat{s} < z_\alpha^2$ holds asymptotically with probability 1. It implies that the results presented in the paper concerning the unconditional asymptotic analysis of sample estimators of the parameters Δ and Δ_{ra} are correct.

Remark 2. All the presented results concerning the estimators of the parameters Δ and Δ_{ra} remain correct for the comparison of the minimum VaR and the minimum $CVaR$ portfolios even for different confidence levels. It can be reached by changing the corresponding quantiles in the corresponding expressions.

6. Conclusions

The paper examines the problem of decision-making on the necessity of portfolio restructurization after changing the risk proxy from VaR to $CVaR$. Taking into account the new standards in risk management announced by the Basel Committee (Basel III) which should be finally implemented in 2019 this could lead to the loss of optimality of existing portfolios and consequently the necessity of their

Tab. 4: **Empirical and asymptotic means and variances of $\sqrt{n}(\hat{\Delta}^*_{ra}-\Delta_{ra})$ for n = {250, 500, 1000, 2000, 3000} and k = 5, 10**

		n=250	n=500	n=1000	n=2000	n=3000	Asympt.
k=5	Mean	-0.0092	-0.0059	-0.0040	-0.0047	-0.0027	0
	Variance	0.16542	0.16371	0.162015	0.16142	0.16107	0.1608
k=10	Mean	-0.0113	-0.0082	-0.0075	-0.0030	-0.0012	0
	Variance	0.18945	0.18608	0.18046	0.1810	0.18097	0.18025

Source: Developed by the authors based on data from finance.yahoo.com in program R

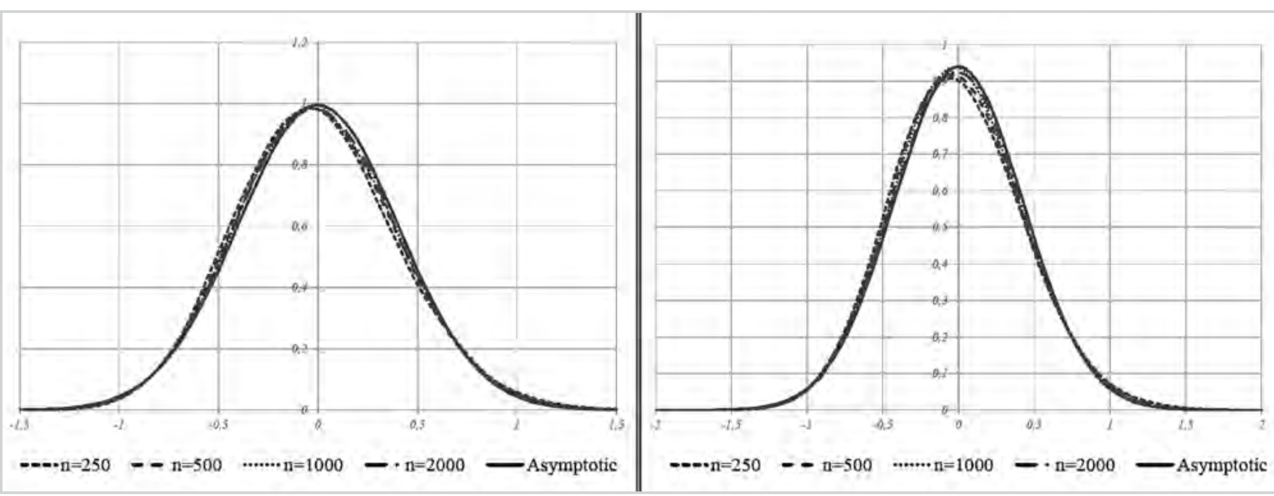

Fig. 3: **Empirical and asymptotic densities of $\sqrt{n}(\hat{\Delta}^*_{ra}-\Delta_{ra})$ for k=5 (left) and k =10 (right) and n = {250, 500, 1000, 2000}.**
Source: Developed by the authors based on data from finance.yahoo.com in program R

restructurization. The portfolio restructurization process is rather costly, both in terms of financial costs and in time consuming. That is why the actual problem is to explore the tools that confirm the necessity of portfolio restructurization and the rationale of investing resources in this process. On the other hand, these tools give the possibility to decide that changes in portfolio characteristics are not significant and there is no need for portfolio restructurization that will save resources.

In the paper, the difference between the main characteristics of the minimum VaR and the minimum $CVaR$ portfolios is calculated and appropriate sample estimates of these indicators are constructed. Taking into consideration that sample estimators are random variables, we investigate the probabilistic properties of the estimators of the differences between the main characteristics of the minimum VaR and the minimum $CVaR$ portfolios. We find the asymptotic distributions of these estimators and justify the correctness of the use of sample estimators of the asymptotic variances for these distributions. Based on the constructed distributions, we develop a toolkit for testing the significance of value of differences between the considered characteristics of the minimum VaR and the minimum $CVaR$ portfolios. By means of a simulation study, we show that both sample estimators are biased. In the case of a sample estimator of the difference between the expected returns of portfolios, we find that the use of this estimator is correct only with large sample sizes. Thus, we show that with the number of assets in the portfolio $k = 5$ and the size of the sample of historical values of $n = 2000$ the empirical density is significantly asymmetric and biased. Instead, the sample estimator of the difference between the investors' risk aversion coefficients that corresponds to each portfolio does not have such drawbacks. The convergence rate of empirical distributions to the asymptotic one is satisfactory, and the bias is not large compared to the exact value. Moreover, we construct an adjusted estimator for this difference for which the convergence rate of empirical variances to the asymptotic one is slightly slower while the empirical biases are close to zero.

We conclude that to testing the optimality of the minimum VaR portfolio under $CVaR$ as a risk proxy and make a decision on the necessity of portfolio restructurization the estimator of the difference between investor's risk aversion coefficients that correspond to the minimum VaR and the minimum $CVaR$ portfolios is appropriate. It should be noted that if one uses the sample estimator, then its bias should be taken into account. We can avoid this by using the adjusted estimator presented in the paper.

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