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ANALYTICAL SOLUTION AND NEUTRAL CURVES OF THE STATIONARY LINEAR RAYLEIGH PROBLEM WITH RIGID OR MIXED BOUNDARY CONDITIONS IN CYLINDRICAL GEOMETRY

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On the basis of the Navier-Stokes equations in the Boussinesq approximation in the linear approximation the classical problem of Rayleigh, dedicated to the study of stable stationary solutions to the horizontal plane layer of a viscous, incompressible fluid heated from below in the case of execution on the upper and lower boundaries of the layer of solid boundary conditions is considered. The analytical solutions describing the perturbed velocity and temperature of the fluid in a cylindrical convection cell with solid boundary conditions are obtained. The obtained analytical solution to build similar solutions of the problem with mixed boundary conditions is used. The analytical expressions for the neutral curves in the case of solid and mixed boundary conditions based on the obtained solutions are built. A comparison of the analytically constructed neutral curves with numerical simulations obtained by other authors is carried out. The derived analytical expressions for the neutral curves of solid and mixed boundary conditions with sufficient accuracy correspond to the numerical calculations is shown.
KEYWORDS: stationary linear of Rayleigh problem, cylindrical geometry, rigid or mixed boundary conditions, analytic solution, neutral curves.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ И НЕЙТРАЛЬНЫЕ КРИВЫЕ СТАЦИОНАРНОЙ ЛИНЕЙНОЙ ЗАДАЧИ РЭЛЕЯ С ТВЕРДЫМИ ИЛИ СМЕШАННЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ В ЦИЛИНДРИЧЕСКОЙ ГЕОМЕТРИИ

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На основе уравнений Навье - Стокса в приближении Буссинеска в линейном приближении рассмотрена классическая задача Рэлея, посвященная исследованию устойчивых стационарных решений для горизонтального плоского слоя вязкой, несжимаемой, подогреваемой снизу жидкости в случае выполнения на верхней и нижней границах слоя твердых граничных условий. Для рассмотренной задачи получены аналитические решения, описывающие возмущенные скорость и температуру жидкости в цилиндрической конвективной ячейки с твердыми граничными условиями. Найденное аналитическое решение использовано для построения решений аналогичной задачи со смешанными граничными условиями. На основе найденных решений построены аналитические выражения для нейтральных кривых в случае твердых и смешанных граничных условий. Проведено сравнение аналитически построенных нейтральных кривых с результатами численного моделирования, которые получены другими авторами. Показано, что аналитические выражения для нейтральных кривых с твердыми и смешанными граничными условиями с достаточной степенью точности соответствуют численным расчетам.
КЛЮЧЕВЫЕ СЛОВА: стационарная линейная задача Рэлея, цилиндрическая геометрия, твердые или смешанные граничные условия, аналитическое решение, нейтральные кривые.

АНАЛІТИЧНЕ РІШЕННЯ І НЕЙТРАЛЬНІ КРИВІ СТАЦІОНАРНОЇ ЛІНІЙНОЇ ЗАДАЧІ РЕЛЕЯ З ТВЕРДИМИ АБО ЗМІШАНИМИ ГРАНИЧНИМИ УМОВАМИ В ЦИЛІНДРИЧНІЙ ГЕОМЕТРІЇ

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На основі рівнянь Нав'є - Стокса в наближенні Буссінеска в лінійному наближенні розглянута класична задача Релея, присвячена дослідженню стійких стаціонарних рішень для горизонтального плоского шару в'язкої, нестисливої рідини, що підігрівається знизу, в разі виконання на верхній і нижній межах шару твердих граничних умов. Для розглянутої задачі отримані аналітичні рішення, що описують збурені швидкість і температуру рідини в циліндричному конвективному

осередку з твердими граничними умовами. Знайдене аналітичне рішення використано для побудови рішень аналогічної задачі зі змішаними граничними умовами. На основі знайдених рішень побудовані аналітичні вирази для нейтральних кривих в разі твердих і змішаних граничних умов. Проведено порівняння аналітично побудованих нейтральних кривих з результатами чисельного моделювання, які отримані іншими авторами. Показано, що аналітичні вирази для нейтральних кривих з твердими і змішаними граничними умовами з достатнім ступенем точності відповідають чисельним розрахункам.
КЛЮЧОВІ СЛОВА: стаціонарна лінійна задача Релея, циліндрична геометрія, тверді або змішані граничні умови, аналітичне рішення, нейтральні криві.

Thermal convection occurs in nature [1,2], and in many technological processes [3]. For example, the convective mass transfer of air is used in agriculture [4], in the process of crystals growth in microelectronics [5]. Description of the cellular structures formation is important for applications of laser processing of materials [6].

Navier-Stokes equations in the Boussinesq approximation (NSBA) are used to describe the convection processes in the layers of the viscous incompressible fluid heated from below. There are three types of convection problems which differ by the boundary conditions: "free" with no tangential stresses at the layer boundaries; rigid or mixed (non-free) when the vertical velocity and its gradient are equal to zero [1,7-9].

The solutions of the stationary linearized NSBA equations with non-free boundary conditions for the normal perturbed velocity and temperature, in contrast to the free boundary conditions, have no analytical form and are obtained by numerical methods [6,8]. Therefore the search for analytical solutions of stationary linearized NSBA equations with non-free boundary conditions has certain scientific and practical interest.

The aim of this work is to obtain analytical solutions and neutral curves for linear stationary system of equations consisting of the Navier-Stokes equations in the Boussinesq approximation and the heat conduction equation in the presence of rigid or mixed boundary conditions.

THE THEORY OF CYLINDRICAL CELL WITH RIGID BOUNDARY CONDITIONS

In the problem of heat convection in the viscous medium with rigid boundaries [1] there was studied viscous liquid layer with the thickness h and it was infinite in both axis x and y directions. Axis z was directed upwards perpendicularly to the layer boundaries $z=0$ and $z=h$. Distribution of temperature inside the layer $T_0(z)$ was given in such a way that the lower boundary temperature was higher the one of the upper boundary: $T_0(0)=T_2$, $T_0(h)=T_1$, $(T_2>T_1)$. Let's consider, that in equilibrium state the dependence of the layer temperature on z coordinate is described by linear function:

$$\nabla T_0(z) = -\frac{\Theta}{h}\vec{e}_z\,, \tag{1}$$

where $\Theta = T_2 - T_1$ - the difference of temperatures between lower and upper planes, $\vec{e}_z$ - unit vector directed along the axis z.

Basing on the experiment results one can make two conclusions [10]:

- the form of convection cells is a cylindrical one;
- the internal velosity distribution in the convection cell doesn't depend on azimuthal angle ϕ.

On the ground of these conclusions we'll search for the solutions of linearized NSBA [7] in cylindrical geometry. In this case in the layer with flat boundaries the initial equations for perturbed vertical velocity v_z and temperature T have the form:

$$\frac{\partial}{\partial t}\Delta \mathrm{v}_z = \Delta\Delta \mathrm{v}_z + R\Delta_\perp T\,, \tag{2}$$

$$P\frac{\partial T}{\partial t} = \Delta T + \mathrm{v}_z\,, \tag{3}$$

where $\Delta = \Delta_\perp + \dfrac{\partial^2}{\partial z^2}$ - Laplacian operator, $\Delta_\perp = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right)$ - transverse Laplacian in which on the basis of the axial symmetry of the cells is not a term that characterizes the dependence of the perturbations of the azimuthal angle is missing, i.e. everywhere we suppose $\partial .../\partial\phi = 0$, $R = g\beta h^3\Theta/(\nu\chi)$ - Rayleigh number, g – gravitational acceleration directed against axis z, $P=\nu/\chi$ - Prandtl number, ν and χ - coefficients of kinematic viscosity and thermal conductivity of liquid correspondingly, β - volumetric coefficient of thermal expansion of the liquid, v_z, T - perturbations of vertical velocity and temperature correspondingly.

For reducing the system of equations (2) - (3) to the dimensionless type there were used the following characteristic measurement units: unit of length – layer thickness h; unit of time - $\tau = h^2\nu^{-1}$; unit of temperature - Θ. It has to be mentioned that for the chosen unit of length z coordinate changes within the interval $0 \le z \le 1$.

System of equations (2) - (3) can be applied to define "normal" perturbations in the viscous liquid layer heated from below under the condition that this system has to be complemented by the boundary conditions. In the current study we'll consider rigid boundary conditions – the case when on the boundaries at $z=0$ and $z=1$ perturbed velocity projections, temperature and vertical speed derivatives have the satisfies next conditions [1,7]:

$$\mathrm{v}_r = \mathrm{v}_z = 0; T = 0; \frac{d\mathrm{v}_z}{dz} = 0 . \tag{4}$$

RECEIVING OF ANALYTICAL SOLUTIONS

The solutions of the linearized NSBA equations for the perturbations of vertical velocity and temperature in the cylindrical geometry have the form of the cylindrical cell [9]:

$$\mathrm{v}_z(r,z,t) = \mathrm{v}(z) J_0(k_r r)\exp(-\lambda t),$$
$$T(r,z,t) = \vartheta(z) J_0(k_r r)\exp(-\lambda t), \tag{5}$$

where λ - the stability parameter [1,7]; $\mathrm{v}(z)$ and $\vartheta(z)$ - amplitude of perturbations of vertical velocity and temperature, $J_0(x)$ - the Bessel function of the first kind of zero order of the argument x, k_r - the radial wave number.

In the steady state ($\lambda = 0$) substitution of (5) in the linearized NSBA equations leads to the characteristic equation [7]:

$$\left(q^2 - b\right)^3 = -a^3, \tag{6}$$

where $a = (k_r^2 R)^{\frac{1}{3}}, b = k_r^2$.

The roots of the characteristic equation (6) are given by

$$q_{1,2} = \pm\sqrt{b-a}; q_{3,4} = \sqrt{b+0.5a(1+i\sqrt{3})} = \pm\left(X_+ + iX_-\right),\ q_{5,6} = \sqrt{b+0.5a(1-i\sqrt{3})} = \pm\left(X_+ - iX_-\right), \tag{7}$$

where $X_\pm = \left[0.5\left(\sqrt{\chi^2 + 0.75a^2} \pm \chi\right)\right]^{\frac{1}{2}}$, $\chi = 0.5a + b$, $i = \sqrt{-1}$ - the imaginary unit.

The amplitude of the vertical speed in (5) is expressed in the terms of the characteristic equation roots, and describes neutral perturbations. We'll seek for it in the form of:

$$\mathrm{v}(z) = \sum_{m=1}^{6} C_m \exp\left(q_m z\right), \tag{8}$$

where C_m - constants defined by the boundary conditions:

$$\mathrm{v}(0) = \mathrm{v}(1) = 0, \partial\mathrm{v}(0)/\partial z = \partial\mathrm{v}(1)/\partial z = 0,\ \vartheta(0) = \vartheta(1) = 0 \tag{9}$$

Equation (8) determines the critical Rayleigh numbers and an amplitude of neutral perturbation which can be found only as an approximate numerical solution of transcendental equations [7].

The method obtaining of solution (8) with (9) in a simple analytical form is submitted below [10].

To do this, in (8) let's suppose given the following relationships between the constants C_m : $C_1 = C_2 = A_1/2$, $C_3 = C_4 = C_5 = C_6 = -A_1/\left(4ch\left(z_0 X_+\right)\right)$, where A_1 - arbitrary constant, $z_0 = 0.5$ - layer's half-width coordinate. In this case (8) takes the form:

$$\mathrm{v}(z) = A_1\left[ch\left(q_1(z-z_0)\right) - \cos\left(X_-(z-z_0)\right)ch\left(X_+(z-z_0)\right)ch^{-1}\left(z_0 X_+\right)\right].$$

By setting $q_1 = i\sqrt{a-b} = in\pi$ and demanding the equality $X_- = \sqrt{a-b} = n\pi$, the amplitude of the vertical velocity should be determined by the formula:

$$\mathrm{v}(z) = A_1\left[1 - ch\left(\left(z-z_0\right)X_+\right)ch^{-1}\left(z_0X_+\right)\right]\sin\left(z\sqrt{a-b}\right). \tag{10}$$

It is easily to show that the expression (6) satisfies the boundary conditions (9).

The equality of $X_- = \sqrt{a-b}$ is valid for $a = 8\left(n\pi\right)^2$ and $b = 7\left(n\pi\right)^2$. Since the solution (6) describes the cylindrical cell's steady state ($\lambda = 0$) , the parameters a and b , or their scale-shift counterparts (see below.) should determine the point k_r lying on the neutral curve $R_n^{rigid}\left(k_r\right)$.

Solution (10), as well as received on its basis the expression for the amplitude of perturbed temperature $\vartheta\left(z\right)$ (see (5)) can be used for comparison with heat and mass transfer indexes in the convective cell with free boundaries [11].

THE CONSTRUCTION OF THE NEUTRAL CURVES FOR THE PROBLEM WITH RIGID BOUNDARY CONDITIONS

Let's use the solution (10) for the construction of the neutral curves $R_n^{rigid}\left(k_r\right)$ of the cylindrical cell's stationary $\left(\lambda = 0\right)$ states. The neutral curves for a given n split the plane (R_n^{rigid}, k_r) and separate stable solutions (5) (below the curve) and unstable ones (above the curve) [7].

In order to construct the neutral curves let's use the solutions (10) invariance with respect to the scale-shift transformation of the problem's parameters: a , b . The term "invariance with respect to the scale-shift transformation" responds to the immutability of the solutions (10) and boundary conditions form (9) to addition to a and b of a constant (shear invariance) and multiplication on a constant (scale invariance).

The shift invariance. Shift of the parameters a and b on an arbitrary value x_0 :

$$a - x_0 = 8\left(n\pi\right)^2, \quad b - x_0 = 7\left(n\pi\right)^2, \tag{11}$$

where $-7\left(n\pi\right)^2 \le x_0 < \infty$ - arbitrary number, does not change the expression $\sqrt{a-b}$ in (6). It does not change the values $X_\pm$ as according to (11) changed parameters $a - x_0$ and $b - x_0$ retain their values. Thus, the use of shift transformation provides an analytical expression for the Rayleigh number:

$$R_n^{rigid} = \frac{a^3}{b} = \frac{\left(n^2\pi^2 + \left(\beta_n \cdot k_r\right)^{\mu_n}\right)^3}{\left(\beta_n \cdot k_r\right)^{\mu_n}}, \tag{12}$$

where $7\left(n\pi\right)^2 - x_0 = \left(\beta_n \cdot k_r\right)^{\mu_n}$, β_n and μ_n - the positive integers which depend on the modes' number.

The scale invariance. The solution (10) is characterized by the scale invariance in relation to a and b . To confirm this, let's perform following replacement: $(z, z_0) \to \alpha_n^{-1} \cdot (z, z_0)$ in (10). Such replacement can be interpreted as a change in the layer's thickness $h \to \alpha_n \cdot h$, where $\alpha_n \ne 0$ - arbitrary positive number depending on the mode's number, $n \ge 1, 2, 3...$. Then, to ensure the scale invariance of the expression (10) should be satisfied following conditions of parameter's change: $a \to \alpha_n^2 \cdot a$ and $b \to \alpha_n^2 \cdot b$. As a result of the described above scale-shift transformation, the expression (12) can be written as:

$$R_n^{rigid} = \frac{\left(a\alpha_n^2\right)^3}{b\alpha_n^5} = \alpha_n \frac{\left(n^2\pi^2 + \left(\beta_n \cdot k_r\right)^{\mu_n}\right)^3}{\left(\beta_n \cdot k_r\right)^{\mu_n}}. \tag{13}$$

Comparing the neutral curve (13) $n = 1$ (curve 1, Fig. 1) to the numerical data obtained by other authors [7] (points on the curve 1) shows good quantitative agreement. It allows to reliably determine the value of the constants in the expression (13): $\alpha_1 = 2.597,\ \alpha_2 = 1.674$, $\beta_1 = 0.7$, $\mu_1 = 2.085$. Herewith the maximum relative deviation of the Rayleigh number (13) of the numerical results [7] is about percent (lays in the range from -1.35% to 0.67%).

The circle (○) in Fig. 1 marks the point $k_r = \pi\sqrt{7}/1.038 = 8$,

$R_1^{rigid} = \alpha_1^{-3}\left(8\pi^2 - 5.087\right)^3 \Big/ \left(7\pi^2 - 5.087\right) = 7062.177$ corresponding to the parameters $a = 8\pi^2$ and $b = 7\pi^2$ after scale-shift transformations, where: $\alpha_1 = 0.963, x_0 = 5.087$. It shows a fairly accurate location of points on the neutral curve.

Thus, the expression (13) for $n = 1$ defines the functional dependence of the neutral curve on the wave number.

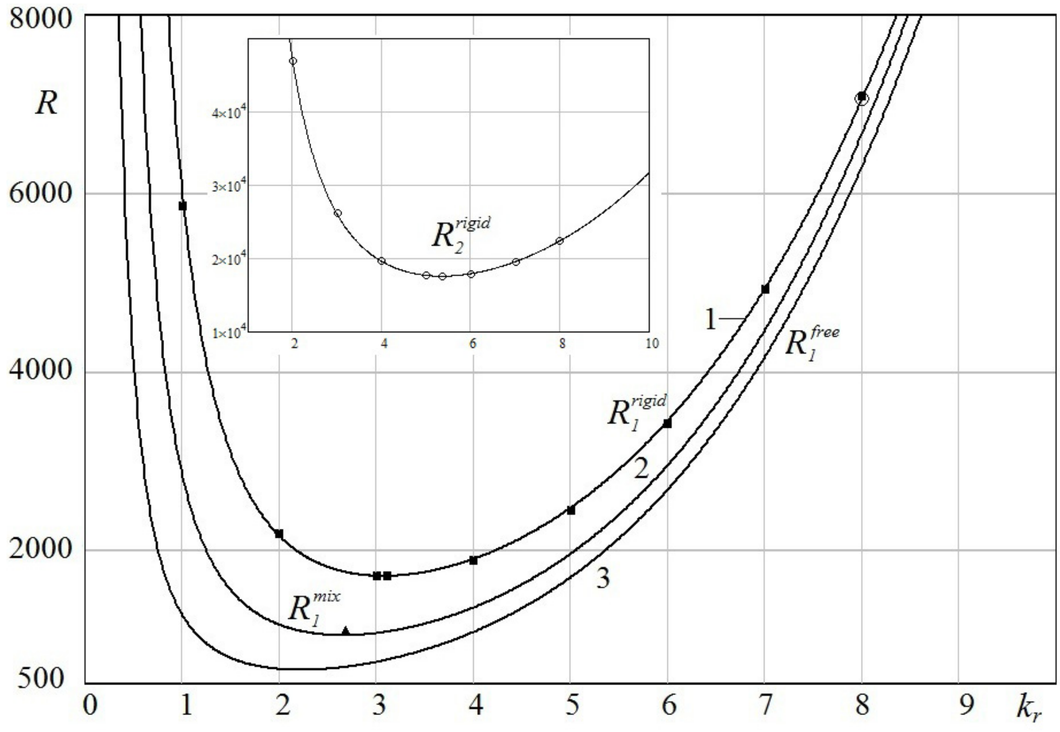

Fig. 1. Neutral curves for mode number $n = 1$.
1 – two boundaries are rigid, 2 – rigid and free boundaries, 3 – two boundaries are free.

For the mode numbers $n > 1$, using the minimum critical Rayleigh numbers for the horizontal layer with rigid boundaries [7], one can show that the neutral curves are determined by the expression (13), wherein the constants α_n and β_n should be given in the form:

$$\alpha_n = \alpha_2\left(\frac{2}{n}\right)^{0.37}\left(n \geq 2\right), \quad \beta_n = \frac{\beta_1 n^{\frac{2}{\mu_n}}\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{2}{\mu_n}-\frac{2}{\mu_1}}}{1+\beta_1\left(n-1\right)\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{\mu_1-2}{\mu_1}}}. \tag{14}$$

Let's define the exponent μ_n in the expression for β_n.

It is known that for large mode numbers *n* the difference in the critical number of cell with solid boundaries and free cell decreases, i.e. the exponent μ_n with the mode numbers n increase should aspire to the exponent for the free cell $\mu_n = 2$ [7]. There under the exponent μ_n can be approximated by the expression:

$$\mu_n = \frac{49.06n}{24.53n - 1}. \tag{15}$$

For $n = 1,2$ the expression (15) provides $\mu_1 = 2.085$ and $\mu_2 = 2.04$ that with sufficient degree of accuracy corresponds to the results of numerical calculations in [7]. If $n \to \infty$, as noted above, the exponent μ_n aspires from above to 2. Fig. 1 shows good agreement between the numerical calculations of the neutral curve points for $n = 2$ and its theoretical estimation (13):

$$R_2^{rigid} = \alpha_2 \frac{\left(2^2\pi^2 + \left(\beta_2 \cdot k_r\right)^{\mu_2}\right)^3}{\left(\beta_2 \cdot k_r\right)^{\mu_2}}, \beta_2 = 0.814\,. \qquad (16)$$

In this section, using the analytical solutions of the stationary linear Rayleigh problem with rigid boundary conditions a family of neutral curves is built for the mode numbers $n \ge 1$ which with high degree of accuracy correspond to the results of numerical calculations performed by other authors.

THE CONSTRUCTION OF THE NEUTRAL CURVES FOR THE PROBLEM WITH MIXED BOUNDARY CONDITIONS

Basing on the analytical dependence (13) let's find the form of the neutral curve for the case of mixed boundary conditions, when the upper boundary is free, and the bottom one is rigid. It can be shown that the form of the curve (curve 2 in Figure 1) is given by: $R_1^{mix} = \alpha_2 \frac{\left(2^2\pi^2 + \left(\beta_2 \cdot k_r\right)^{\mu_2}\right)^3}{2^4\left(\beta_2 \cdot k_r\right)^{\mu_2}}$. Curve 2 has minimal value $\left(R_1^{mix}\right)_{\min} = 1045.60$ that accurately (up to 5%) corresponds to the critical point, calculated by numerical method $R_m = 1100.657$ [7].

To compare the analytical dependences for neutral curves with rigid boundary conditions with neutral curves with free boundaries the curve 3 in Fig.1 of Rayleigh problem is presented. It is shown that the analytical dependencies (13), (16) (curves 1 and 2 respectively) for large arguments k_r do not intersect, and asymptotically aspire to the neutral curve of the cell with free boundaries.

In conclusion, we'll note that obtained results are useful in solving the stationary Rayleigh problem with rigid boundaries in the Cartesian coordinate system. To do this, one should replaced in (5) the k_r by $k = \sqrt{k_x^2 + k_y^2}$ and the $J_0\left(k_r r\right)$ by $\exp\left(ik_x x + ik_y y\right)$.

CONCLUSION

Thus, in present paper analytical solutions of the stationary linear Rayleigh problem with rigid boundary conditions are submitted. On the basis of those solutions the analytical dependencies are built for the families of neutral curves of considered problem with rigid and mixed boundary conditions.

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