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Periodic solutions in the dynamics of an optimal resource extraction model

Abstract

The model presented in this paper studies the presence of closed orbits that signal economic fluctuations and periodic solutions around the steady state. The problem is to understand the mechanism that leads to an indeterminate equilibrium in presence of natural resource use, and therefore suggest the emergence of a poverty-environment trap.

Keywords: indeterminacy, Hopf bifurcation, renewable resources.
JEL Classification: C62, O41, Q20.

Introduction

The sustainable use of natural resources is one of the most debated and intriguing social issues. Trying to understand whether an economy can grow along an optimal path without sacrificing its natural capital is not an easy task. A common view of nowadays economies basically assumes that productions highly depend on natural resource overuse, and no growth is therefore possible without this input.

Some problems, however, emerge when we try to model this stylized fact. Even though standard neoclassical theory predicts the existence of a unique (i.e. determinate) saddle path stable equilibrium. It is well known that the rise of market imperfections may create the problem of an indeterminate equilibrium.

In fact, dynamic models of renewable resources with two state variables can easily exhibit multiple steady states and give rise to some ecological management problems (Wirl, 2004). This may consequently explain the cross-country differences in optimal resource use due to market imperfections, associated with the indeterminacy of the equilibrium. Moreover, the rise of multiple equilibria in presence of an overexploitation of natural resources can be the major cause for a vicious poverty-environment trap situation, where policies (i.e. technological innovations) aimed at alleviating the overexploitation and exhaustion of the environment might not be able to avoid a still unsustainable use of natural resources (see, for example, Finco, 2009).

The determinants of endogenous growth have been widely investigated in the literature, with particular focus on two-sector continuous-time growth models, with infinitely-lived agents and some form of market imperfections, where a large strand of analysis demonstrates how a *continuum* of equilibrium trajectories, existing in the neighborhood of the steady state, can emerge whenever some parametric conditions are verified. This phenomenon is commonly known as '*local indeterminacy'* (see, inter alia, Slobodyan, 2007; Chamley, 1993; Benhabib and Farmer, 1994 and 1996, Benhabib and Perli, 1994; Benhabib, Perli and Xie, 1994; and Benhabib, Meng and Nishimura, 2000).

However, only very few attempts have been made to analyze the conditions under which these indeterminacy problems arise outside such small neighborhood of the steady state, to whom we refer to as global *indeterminacy*. The presence of non-linear functions increases the difficulties in handling these models. Examples of this analysis can be found, for instance, in Nishimura and Shigoka (2006), where the emergence of an Andronov-Hopf bifurcation is used to prove the presence, in a bounded parametric region, of periodic orbits surrounding an equilibrium, that is said to be globally indeterminate (see, also, Mattana and Venturi, 1999; and Mattana, Nishimura and Shigoka, 2009).

The model presented in this paper studies the presence of closed orbits, that signal economic fluctuations and periodic solutions around the steady state. The purpose is to understand the mechanism that leads to such problem in presence of natural resource use. That is, when the orbit is attracting, trajectories on the center manifold are locally captured by the orbit which becomes itself an (indeterminate) limit set. On the contrary, when the orbit is repelling, trajectories around the BGP tend to converge to the long-run equilibrium. The presence of such periodic solutions can therefore suggest the emergence of a poverty-environment trap.

The rest of the paper is organized as follows. In Section 1, we present the model, derive the steady state conditions, and study the local dynamics. In Section 2, we characterize the parametric space, where periodic solutions emerge, and the equilibrium becomes indeterminate. The final Section concludes, and a subsequent Appendix provides all the necessary proofs.

1. The model

Let us consider the following optimal control problem:

$$\underset{c(t)}{Max} \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \qquad (P)$$

subject to the following constraints:

$$\dot{k} = Ak^\alpha (nR)^{1-\alpha} R_a^\gamma - c$$
$$\dot{R} = \delta R(1-nR),$$

and given initial positive values:

$$k(0) = k_0 > 0, \text{and } R(0) = R_0 > 0,$$

where c is consumption, k is physical capital, and A measures the stock of existing technology. Let also assume a fraction $n \in [0,1]$ of the natural resources at disposal R, be allocated both with physical capital k, to the production of new output Y, whereas the remaining part is left for future recreation. Moreover, $\alpha \in [0,1]$ is the share of physical capital in the same sector, ρ is a time preference rate, and σ is the inverse of the intertemporal elasticity of substitution.

In particular, the level of investment in physical capital is given by the usual functional form $\dot{K} = Y - C$, where output Y is produced according to the function:

$$Y = Ak^\alpha (nR)^{1-\alpha} R_a^\gamma \qquad (1)$$

with physical capital k, entering as an input along with natural resources R. Moreover, R_a represents an external effect due to the presence of a common pool natural resource, that no one will take account for when deciding how to allocate it in time, and γ is an externality parameter. Basically, we are assuming that, in addition to the individual effects coming from the use of natural resources on his own productivity – what we may call the *internal environmental effect* – room is left for some external effects either, denoted by R_a^γ. Specifically, we call this effect external because, even though all benefit (if positive) from it, no individual will take it into account when making his optimal decision plan[1].

Henceforth we assume that, the dynamics of the stock of natural resources R, follow a logistic-like law of motion, given by:

$$\dot{R} = \delta R(1-nR), \qquad (2)$$

where δ is a positive parameter for natural productivity.

The current value Hamiltonian of problem P is given by:

$$H_C = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda\left[Ak^\alpha (nR)^{1-\alpha} R_a^\gamma - c\right] + $$
$$+ \mu\left[\delta R(1-nR)\right],$$

where λ and μ represent the shadow prices of physical capital and natural resources, respectively. Solution to this optimal control problem implies the following necessary first order conditions:

$$c^{-\sigma} = \lambda$$
$$\lambda(1-\alpha)Ak^\alpha n^{-\alpha} R^{1-\alpha+\gamma} = \mu\delta R^2, \qquad (3)$$

accompanied by the equation of motion for each costate variable, that can be obtained with a bit of mathematical manipulation:

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha A k^{\alpha-1} n^{1-\alpha} R^{1-\alpha+\gamma},$$
$$\frac{\dot{\mu}}{\mu} = \rho - \delta(1-nR), \qquad (4)$$

and the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}[\lambda_t k_t + \mu_t R_t] = 0 \qquad (5)$$

has jointly constitute the canonical system.

The next Section, thus, moves directly to the analysis of the transitional dynamics around the equilibrium solution.

1.1. The reduced model. The standard procedure is conducted in this Section to study the transitional dynamics of problem *P*.

Proposition 1. The maximum principle associated with the decentralized optimization problem *P* implies the following four-dimensional system of first order differential equations:

$$\xi_k = \frac{\dot{k}}{k} = Ak^{\alpha-1} n^{1-\alpha} R^{1-\alpha+\gamma} - \frac{c}{k},$$
$$\xi_R = \frac{\dot{R}}{R} = \delta(1-nR), \qquad (S1)$$
$$\xi_c = \frac{\dot{c}}{c} = -\frac{\rho}{\sigma} + \frac{\alpha}{\sigma} Ak^{\alpha-1} n^{1-\alpha} R^{1-\alpha+\gamma},$$

[1] This specific production function exhibits constant returns to scale at a disaggregate level because each firm takes R_a as given. On the contrary, a social planner can internalize this kind of externality, thus obtaining increasing returns.

$$\xi_n = \frac{\dot{n}}{n} = (\gamma - \alpha)\frac{\delta}{\alpha}(1 - nR) - \frac{c}{k}.$$

Proof (see Appendix).

Lemma 1. The system S1 implies also the following reduced version:

$$\dot{x} = -\frac{\rho}{\sigma}x + \left(\frac{\alpha - \sigma}{\sigma}\right)mx + x^2,$$

$$\dot{q} = \frac{\gamma\delta}{\alpha}(1-q)q - xq, \qquad \text{(S2)}$$

$$\dot{m} = (\alpha - 1)m^2 + \frac{\gamma\delta}{\alpha}(1-q)m$$

by means of the convenient variable substitutions: $x = \frac{c}{k}$, $q = nR$, and $m = \frac{y}{k}$.

Proof (see Appendix).

Lemma 2. The steady state is a triplet (x^*, q^*, m^*) which solves the reduced system S2:

$$m^* = \frac{\rho}{\alpha(1-\sigma)},$$

$$x^* = \frac{\rho(1-\alpha)}{\alpha(1-\sigma)},$$

$$q^* = 1 - \frac{\rho(1-\alpha)}{\gamma\delta(1-\sigma)},$$

given $0 < \sigma < 1$.

Since the Jacobian matrix associated with S2 is:

$$J^* \equiv J_{(x^*,q^*,m^*)} = \begin{bmatrix} \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} & 0 & \frac{\rho(1-\alpha)(\alpha-\sigma)}{\alpha\sigma(1-\sigma)} \\ 1-\frac{\rho(1-\alpha)}{\gamma\delta(1-\sigma)} & \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha} & 0 \\ 0 & -\frac{\rho\gamma\delta}{\alpha^2(1-\sigma)} & \frac{\rho(\alpha-1)}{\alpha(1-\sigma)} \end{bmatrix}$$

let

$$\det(\lambda I - J^*) = \lambda^3 - trJ^*\lambda^2 + BJ^*\lambda - DetJ^*$$

be the characteristic polynomial of J^*, where I is the identity matrix and trJ^*, BJ^*, and $DetJ^*$ are trace, determinant and sum of principal minors associated with J^*, respectively. Algebraic computation easily shows:

$$trJ^* = \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha},$$

$$BJ^* = -\frac{\rho^2(1-\alpha)^2}{\alpha^2(1-\sigma)^2}, \qquad (6)$$

$$DetJ^* = \frac{\gamma\delta(1-\sigma)}{\sigma}m^*x^*q^*.$$

Studying the local dynamics of this economy while converging to the steady state is very direct and straightforward. The neat Routh-Hurwitz criterion applies to this case, and confirms the possibility of emergence of periodic solutions. The interior steady state can therefore be indeterminate.

In detail, the bifurcation analysis allows us to verify if a parameter value, $\sigma = \sigma_c$, there exists at which a structure of closed orbits Hopf bifurcating from the steady state solution would appear[1]. This occurs when a complex conjugate pair of eigenvalues do emerge. To prove this, we need to check the sign of the following expression:

$$G(\sigma) = -BJ \cdot trJ + DetJ = \frac{\rho^2(1-\alpha)(\sigma-\alpha)}{\alpha^3(1-\sigma)^3} \times$$
$$\times[\rho(1-\alpha) - \gamma\delta(1-\sigma)], \qquad (7)$$

which vanishes

at $\sigma_c^* = \alpha$ or $\sigma_c^{**} = \frac{\gamma\delta - \rho(1-\alpha)}{\gamma\delta}$.

The following Figure 1 shows the presence of multiple solutions for $G = G(\sigma)$. We analyze, for example, the case where $\rho = 0.002$, $\alpha = 0.66$, $\gamma = 2$, and $\delta = 0.04$. Therefore, $G(\sigma) = 0$ when $\sigma_c^* = 0.66$ and $\sigma_c^{**} = 0.975$.

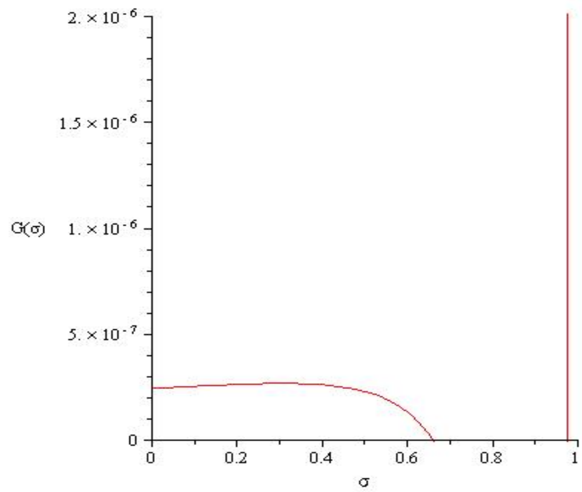

Fig. 1. The Hopf bifurcation curve

Thus, multiple values of σ are able to annihilate $G(\sigma)$. If this happens, we will show that any variation of σ around σ_c can force the variables associated to the complex conjugate eigenvalues to oscillate

[1] The choice of σ as a bifurcation parameter is conveniently made to avoid the particular case of a zero eigenvalue that would appear if using the externality parameter γ. This is out of the scope of the present paper, and is left for further research.

around a common constant value. This means also that, an invariant cycle (a closed orbit) may emerge around the steady state, or collapses crucial role in the characterization of an optimal solution to our maximization problem, and thus matters in the process of a long run sustainable growth.

To get in a deep investigation of this concern the next Section is devoted to this.

2. Periodic orbits and indeterminacy

Studying the properties of an equilibrium outside the small neighborhood of the steady state is not an easy task, especially, when dealing with non-linear functions that complicate the algebraic calculations behind it.

This Section is aimed at showing the emergence of periodic solutions by means of the Hopf bifurcation theorem, which requires several necessary steps to be followed.

Firstly, we need to put the system S2 in an appropriate canonical form to work with. To do this, we translate the equilibrium fixed point to the origin, by assuming:

$$\tilde{x} = x - \bar{x}^*,$$

$$\tilde{m} = m - \bar{m}^*,$$

$$\tilde{q} = q - \bar{q}^*$$

which transforms the original systems S2 into S3:

$$\dot{\tilde{x}} = -\frac{\rho}{\sigma}(\tilde{x}+\tilde{x}^*) + \left(\frac{\alpha-\sigma}{\sigma}\right)(\tilde{m}+\bar{m}^*)(\tilde{x}+\bar{x}^*) + (\tilde{x}+\bar{x}^*)^2,$$

$$\dot{\tilde{q}} = \frac{(\bar{\gamma}+\mu)\delta}{\alpha}\left[1-(\tilde{q}+\bar{q}^*)\right](\tilde{q}+\bar{q}^*) - (\tilde{x}+\bar{x}^*)(\tilde{q}+\bar{q}^*),$$

$$\dot{\tilde{m}} = (\alpha-1)(\tilde{m}+\bar{m}^*)^2 + \frac{\delta(\bar{\gamma}+\mu)}{\alpha}\left[1-(\tilde{q}+\bar{q}^*)\right](\tilde{m}+\bar{m}^*).$$

The second order Taylor expansion of this vector field allows us to put S2 in a Jordan normal form, given by:

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{q}} \\ \dot{\tilde{m}} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{q} \\ \tilde{m} \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(\tilde{x},\tilde{q},\tilde{m}) \\ \tilde{f}_2(\tilde{x},\tilde{q},\tilde{m}) \\ \tilde{f}_3(\tilde{x},\tilde{q},\tilde{m}) \end{pmatrix}, \quad (8)$$

where J corresponds to the Jacobian matrix associated with the linearization of the original system S2, $J = J^*(0)$, whereas the $\tilde{f}_i$ terms represent the non-linear terms (of order 2). In detail:

$$\tilde{f}_1(\tilde{x},\tilde{q},\tilde{m}) = \frac{1}{2}\tilde{x}^2 + \left(\frac{\alpha-\sigma}{\sigma}\right)\tilde{x}\tilde{m},$$

$$\tilde{f}_2(\tilde{x},\tilde{q},\tilde{m}) = -\tilde{x}\tilde{q} - \frac{\gamma\delta}{\alpha}\tilde{q}^2, \quad (9)$$

$$\tilde{f}_3(\tilde{x},\tilde{q},\tilde{m}) = (\alpha-1)\tilde{m}^2 - \frac{\gamma\delta}{\alpha}\tilde{q}\tilde{m}.$$

Proposition 2. Assume J be characterized by one real and a pair of purely imaginary eigenvalues, that is: $\lambda_1 = trJ$, $\lambda_{2,3} = \pm\omega i$. Let T be a matrix of the eigenvectors (u,v,z) associated with the structure of the aforementioned eigenvalues of J at the bifurcation point. Then, it is possible to put the system in the following Jordan normal form:

$$w = T^{-1}J^*(0)Tw + F_i,$$

where $F_i = T^{-1}\tilde{f}_i(Tw)$, given the associated change in coordinates:

$$\begin{pmatrix} \tilde{x} \\ \tilde{q} \\ \tilde{m} \end{pmatrix} = T \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

which transforms system S3 into S4:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \mathrm{Tr}(J^*(0)) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} F_1(w_1,w_2,w_3) \\ F_2(w_1,w_2,w_3) \\ F_3(w_1,w_2,w_3) \end{pmatrix},$$

where F_i are the transformed second order non-linear terms.

Proof (see Appendix).

Thus, we are able at this step to restrict the vector field in (*S4*) to the plane (w_1, w_2) whose eigenspace, at the bifurcation value $\sigma = \sigma_c$, corresponds to the complex pair of eigenvalues, $\lambda_{1,2} = \pm\omega i$, which is topologically invariant with respect to the original system S1[1]. A center manifold reduction of the linearized vector field allows us to investigate this case.

Proposition 3. A second order approximation of the center manifold which reduces the vector field in S3 is given by the following equation.

$$w_3 = h(w_1, w_2) = \frac{1}{2}[\tau_1 w_1^2 + \tau_2 w_1 w_2 + \tau_3 w_2^2],$$

where τ_i are coefficients that satisfy the stability condition $\dot{w}_3 = 0$.

Proof (see Appendix).

[1] If we substitute $BJ \cdot trJ = DetJ$ in the characteristic equation at the bifurcation point, one eigenvalue is real and positive, and equal to the trace, $\lambda_r = trJ$, while the other two eigenvalues are complex conjugate, $\lambda_c = \pm\omega i$, assuming $\omega = \sqrt{BJ}$.

The vector field at the center manifold therefore reduces to:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \overline{F}^1(w_1, w_2, h(w_1, w_2)) \\ \overline{F}^2(w_1, w_2, h(w_1, w_2)) \end{pmatrix}, \quad \text{(S5)}$$

where $\overline{F}^i$ represents the second order nonlinear terms of the vector field at the center manifold.

The restricted vector field (S5) allows us to properly investigate the presence of periodic solutions in the two-dimensional phase space (w_1, w_2). In fact, an application of the Hopf bifurcation theorem follows immediately. Let

$$q = \frac{1}{16}\left[\overline{F}^1_{w_1w_1w_1} + \overline{F}^1_{w_1w_2w_2} + \overline{F}^2_{w_1w_1w_2} + \overline{F}^2_{w_2w_2w_2}\right] + \frac{1}{16\omega}\left[\overline{F}^1_{w_1w_2}\left(\overline{F}^1_{w_2w_2} + \overline{F}^1_{w_1w_1}\right) - \overline{F}^2_{w_1w_2}\left(\overline{F}^2_{w_2w_2} + \overline{F}^2_{w_1w_1}\right] - \overline{F}^1_{w_1w_1} \quad (10)$$

be the explicit calculation of the Andronov-Hopf bifurcation coefficient[1].

Remark 1. If $q < 0$ the emerging cycle around the steady state is attracting, i.e. a supercritical Hopf bifurcation occurs.

The value of q, at the two bifurcation points, can be either positive or negative. Therefore, both bifurcations can be super-critical or sub-critical. The fixed points are unstable and the orbits are attracting on the center manifold. This is shown by means of the following numerical example.

Example 1. Assume $\rho = 0.002$, $\alpha = 0.66$, $\gamma = 2$, $\delta = 0.04$. If $\sigma_c^* = 0.66$, then $q = -2.40 \cdot 10^{12} < 0$, that is to say the bifurcation is super-critical, the steady state is unstable and the periodic orbits are attracting on the center manifold. On the contrary, in correspondence of $\sigma_c^{**} = 0.975$, we have $q = 8.37 \cdot 10^{14} > 0$, that is to say the bifurcation is sub-critical, and the periodic orbits start repelling.

Thus, we are able to conclude that different periodic solutions may emerge in presence of optimal resource extraction, which leads consequently to the rise of some indeterminacy problems, which might be able to explain the rise and fall of different nowadays economies that, even though endowed with the same initial conditions, may at some point start to perform differently in growth rate terms and thus follow different long run equilibrium paths.

Conclusions

The emergence of multiple equilibria has been used in the literature to explain the diversity of growth rates across countries. It is so worth noting that depending on the values of the inverse of the intertemporal elasticity of substitution, either multiple or unique equilibria (i.e., determinate versus indeterminate solutions) may consequently arise.

The implications of indeterminacy concerns can be synthesized in follows: two identically endowed economies with identical initial conditions may consume, and invest in the production of natural and physical capital, at completely different rates. Only in the long run those economies will converge to the same growth rate, but not to the same level of output and natural and physical capital. It is therefore possible to consider other cultural, historical or non-economic factors as the means for equilibria to differ on the transition path to be followed towards the long run steady state. Indeed, we refer here to local indeterminacy, and the coexistence of multiple balanced growth paths, as the device to theoretically reinterpret the possibility for different regions, identically endowed in terms of existing natural resources, to exhibit uneven economic developments in a sustainable way.

The positive implication of this paper can be the following: given the different allocation of natural resources across countries, and assumed that multiple equilibria may exist. It is no wonder that a clear convergence among the world economies is not observed. In the management of their natural resources, we may notice instead that, meanwhile, some countries have lagged permanently behind due to short-sighted policies, some others have experienced higher growth rates due to a more sustainable behavior. It might be so that an historically stagnant region continue to be so, while other regions, perhaps historically more active, may continue to flourish, even though they are the same in all other aspects. History matters then, and the management of the natural resources may act as a selection device among these different equilibria.

To shed some light in this field, we presented a model to answer the question on whether similar countries may exhibit very different growth experiences, arguing that a crucial aspect for the occurrence of both indeterminacy and cyclical adjustment towards the steady state might be the presence of particular bifurcation values of the inverse of the intertemporal elasticity of substitution. Conclusions

[1] The whole calculation of all coefficient values in equation (10) though computed is omitted from the Appendix for convenience of space. They are available upon request.

to our analysis confirm that such parameter matters in the transition towards a long-run sustainable equilibrium, thus leaving space to other more complicated dynamic phenomena characterized by periodic solutions and closed orbits to trap the economy in a low level equilibrium.

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Appendix

1. The current value Hamiltonian associated with problem P is given by:

$$H_C = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda\left[Ak^{\alpha}(nR)^{1-\alpha}R_a^{\gamma} - c\right] + \mu\left[\delta R(1-nR)\right],$$

where λ and μ represent the shadow prices of physical capital and natural resources, respectively. The first order condition for a maximum requires that the discounted Hamiltonian be maximized with respect to its control variables, that is:

$$\frac{\partial H_C}{\partial c} = c^{-\sigma} - \lambda = 0,$$

$$\frac{\partial H_C}{\partial n} = \lambda(1-\alpha)Ak^{\alpha}n^{-\alpha}R^{1-\alpha+\gamma} - \mu\delta R^2 = 0$$

with the associated log-derivatives:

$$-\sigma\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}, \tag{1}$$

$$\frac{\dot{\lambda}}{\lambda} + \alpha\frac{\dot{k}}{k} - \alpha\frac{\dot{n}}{n} - (1+\alpha-\gamma)\frac{\dot{R}}{R} = \frac{\dot{\mu}}{\mu} \tag{2}$$

accompanied by the law of motion of each costate variable

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha Ak^{\alpha-1}n^{1-\alpha}R^{1-\alpha+\gamma}, \tag{3}$$

$$\frac{\dot{\mu}}{\mu} = \rho - \delta(1-nR). \tag{4}$$

To add more, given the constraints on both physical and natural resources:

$$\dot{k} = Ak^{\alpha}(nR)^{1-\alpha}R_a^{\gamma} - c,$$

$$\dot{R} = \delta R(1-nR) \tag{5}$$

and by means of the set of equations (1-4), we can derive, with a little bit of mathematical manipulation, the following four-dimensional system of first order differential equations, S1:

$$\xi_k = \frac{\dot{k}}{k} = Ak^{\alpha-1}n^{1-\alpha}R^{1-\alpha+\gamma} - \frac{c}{k},$$

$$\xi_R = \frac{\dot{R}}{R} = \delta(1-nR),$$

$$\xi_c = \frac{\dot{c}}{c} = -\frac{\rho}{\sigma} + \frac{\alpha}{\sigma}Ak^{\alpha-1}n^{1-\alpha}R^{1-\alpha+\gamma},$$

$$\xi_n = \frac{\dot{n}}{n} = (\gamma-\alpha)\frac{\delta}{\alpha}(1-nR) - \frac{c}{k}$$

or rather the more tractable reduced three-dimensional system, S2:

$$\dot{x} = -\frac{\rho}{\sigma}x + \left(\frac{\alpha-\sigma}{\sigma}\right)mx + x^2,$$

$$\dot{q} = \frac{\gamma\delta}{\alpha}(1-q)q - xq,$$

$$\dot{m} = (\alpha-1)m^2 + \frac{\gamma\delta}{\alpha}(1-q)m$$

by means of the convenient variable substitution: $x = \frac{c}{k}$, $q = nR$, and $m = \frac{y}{k}$. This implies in growth rate terms that: $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$, $\frac{\dot{q}}{q} = \frac{\dot{n}}{n} + \frac{\dot{R}}{R}$, and $\frac{\dot{m}}{m} = \frac{\dot{y}}{y} - \frac{\dot{k}}{k}$.

2. The Jacobian matrix of the reduced system S2 is then

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

here:

$$J_{11} = x, \quad J_{21} = -q, \quad J_{31} = 0,$$

$$J_{12} = 0, \quad J_{22} = -\tfrac{\gamma\delta}{\alpha}q, \quad J_{32} = -\tfrac{\gamma\delta}{\alpha}m,$$

$$J_{13} = \left(\tfrac{\alpha-\sigma}{\sigma}\right)x, \quad J_{23} = 0, \quad J_{33} = (\alpha-1)m$$

hence, the Jacobian evaluated at the steady state finally becomes

$$J_{(x^*,q^*,m^*)} = \begin{bmatrix} \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} & 0 & \frac{\rho(1-\alpha)(\alpha-\sigma)}{\alpha\sigma(1-\sigma)} \\ 1-\frac{\rho(1-\alpha)}{\gamma\delta(1-\sigma)} & \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha} & 0 \\ 0 & -\frac{\rho\gamma\delta}{\alpha^2(1-\sigma)} & \frac{\rho(\alpha-1)}{\alpha(1-\sigma)} \end{bmatrix}$$

3. Translation to the origin. Substitute $\tilde{x} \equiv x - \overline{x}^*$, $\tilde{m} \equiv m - \overline{m}^*$, $\tilde{q} = q - \overline{q}^*$ in the original system S2:

$$\dot{\tilde{x}} = -\frac{\rho}{\sigma}(\tilde{x} + \overline{x}^*) + \left(\frac{\alpha - \sigma}{\sigma}\right)(\tilde{m} + \overline{m}^*)(\tilde{x} + \overline{x}^*) + (\tilde{x} + \overline{x}^*)^2,$$

$$\dot{\tilde{q}} = \frac{(\overline{\gamma} + \mu)\delta}{\alpha}\left[1 - (\tilde{q} + \overline{q}^*)\right](\tilde{q} + \overline{q}^*) - (\tilde{x} + \overline{x}^*)(\tilde{q} + \overline{q}^*),$$

$$\dot{\tilde{m}} = (\alpha - 1)(\tilde{m} + \overline{m}^*)^2 + \frac{\delta(\overline{\gamma} + \mu)}{\alpha}\left[1 - (\tilde{q} + \overline{q}^*)\right](\tilde{m} + \overline{m}^*),$$

we find:

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{q}} \\ \dot{\tilde{m}} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{q} \\ \tilde{m} \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(\tilde{x}, \tilde{q}, \tilde{m}) \\ \tilde{f}_2(\tilde{x}, \tilde{q}, \tilde{m}) \\ \tilde{f}_3(\tilde{x}, \tilde{q}, \tilde{m}) \end{pmatrix}$$

with the following non-linear terms in Taylor expansion:

1) $\tilde{f}_1(\tilde{x}, \tilde{q}, \tilde{m}) = \frac{1}{2}\tilde{x}^2 + \left(\frac{\alpha-\sigma}{\sigma}\right)\tilde{x}\tilde{m}$,

2) $\tilde{f}_2(\tilde{x}, \tilde{q}, \tilde{m}) = -\tilde{x}\tilde{q} - \frac{\gamma\delta}{\alpha}\tilde{q}^2$,

3) $\tilde{f}_3(\tilde{x}, \tilde{q}, \tilde{m}) = (\alpha - 1)\tilde{m}^2 - \frac{\gamma\delta}{\alpha}\tilde{q}\tilde{m}$.

4. Computation of the eigenvectors. Given:

$$J^* = \begin{bmatrix} j_{11}^* & 0 & j_{13}^* \\ j_{12}^* & j_{22}^* & 0 \\ 0 & j_{32}^* & j_{33}^* \end{bmatrix}$$

For eigenvalues, $\lambda_{1,2} = \pm\omega i$, and $\lambda_3 = tr(J^*)$, we need:

$$J^* v = \lambda v$$

that is:

$$u = \begin{bmatrix} -\frac{j_{22}^*}{\omega} \\ \frac{j_{21}^*}{\omega} \\ \frac{j_{11}^* j_{22}^* - \omega^2}{\omega j_{13}^*} \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} \frac{\gamma\delta}{\alpha} - \frac{\rho(1-\alpha)}{\alpha(1-\sigma)} \\ 1 - \frac{\rho(1-\alpha)}{\gamma\delta(1-\sigma)} \\ \frac{\sigma\left[\frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha}\right]}{(\alpha-\sigma)} - \frac{\alpha\sigma(1-\sigma)\omega^2}{\rho(1-\alpha)(\alpha-\sigma)} \end{bmatrix},$$

$$v = \begin{bmatrix} 1 \\ 0 \\ \frac{j_{11}^* j_{22}^* - \omega}{j_{13}^* j_{33}^*} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma\alpha(1-\sigma)\left[\frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha}\right]}{(\alpha-\sigma)\rho(\alpha-1)} + \frac{\alpha^2\sigma(1-\sigma)^2\omega}{\rho^2(1-\alpha)^2(\alpha-\sigma)} \end{bmatrix},$$

$$z = \begin{bmatrix} \frac{j_{11}^* + j_{33}^*}{j_{21}^* j_{32}^*} \\ \frac{1}{j_{32}^*} \\ \frac{1}{j_{11}^* + j_{22}^*} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\alpha^2(1-\sigma)}{\rho\gamma\delta} \\ \frac{1}{2\frac{\rho(1-\alpha)}{\alpha(1-\sigma)} - \frac{\gamma\delta}{\alpha}} \end{bmatrix}.$$

The eigenvectors are thus the basis for the following transformation matrix, *T*, which transforms the equilibrium coordinates:

$$\begin{pmatrix} \tilde{x} \\ \tilde{q} \\ \tilde{m} \end{pmatrix} = T \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

$$T = \begin{bmatrix} -\frac{j_{22}^*}{\omega} & 1 & \frac{j_{11}^*+j_{33}^*}{j_{21}^* j_{32}^*} \\ \frac{j_{21}^*}{\omega} & 0 & \frac{1}{j_{32}^*} \\ \frac{j_{11}^* j_{22}^* - \omega^2}{\omega j_{13}^*} & \frac{j_{11}^* j_{22}^* - \omega}{j_{13}^* j_{33}^*} & \frac{1}{j_{11}^*+j_{22}^*} \end{bmatrix} = \begin{bmatrix} u_1 & 1 & z_1 \\ u_2 & 0 & z_2 \\ u_3 & v_3 & z_3 \end{bmatrix}$$

that implies:

$$\begin{pmatrix} \tilde{x} \\ \tilde{q} \\ \tilde{m} \end{pmatrix} = \begin{bmatrix} -\frac{j_{22}^*}{\omega} & 1 & \frac{j_{11}^*+j_{33}^*}{j_{21}^* j_{32}^*} \\ \frac{j_{21}^*}{\omega} & 0 & \frac{1}{j_{32}^*} \\ \frac{j_{11}^* j_{22}^* - \omega^2}{\omega j_{13}^*} & \frac{j_{11}^* j_{22}^* - \omega}{j_{13}^* j_{33}^*} & \frac{1}{j_{11}^*+j_{22}^*} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{bmatrix} u_1 & 1 & z_1 \\ u_2 & 0 & z_2 \\ u_3 & v_3 & z_3 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

and also:

$$\tilde{x} = u_1 w_1 + w_2 + z_1 w_3,$$
$$\tilde{q} = u_2 w_1 + z_2 w_3, \qquad .$$
$$\tilde{m} = w_1 + z_3 w_3.$$

5. To put system S2 in Jordan normal form, we need to calculate:

$$\dot{w} = T^{-1} J^*(0) T w + T^{-1} \tilde{f}_i(Tw)$$

and derive:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \mathrm{Tr}(J^*(0)) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} \overline{F}_1(w_1, w_2, w_{3,}) \\ \overline{F}_2(w_1, w_2, w_{3,}) \\ \overline{F}_3(w_1, w_2, w_{3,}) \end{pmatrix},$$

where:

$$\overline{F}_1(w_1, w_2, w_{3,}) = \tfrac{1}{D}\left[-v_3 z_2 \tilde{f}_1(Tw) + (-z_3 + v_3 z_1)\tilde{f}_2(Tw) + z_2 \tilde{f}_3(Tw)\right],$$

$$\overline{F}_2(w_1, w_2, w_{3,}) = \tfrac{1}{D}\left[(-u_2 z_3 + u_3 z_2)\tilde{f}_1(Tw) + (u_1 z_3 - u_3 z_1)\tilde{f}_2(Tw) + (-u_1 z_2 + u_2 z_1)\tilde{f}_3(Tw)\right],$$

$$\overline{F}_3(w_1, w_2, w_{3,}) = \tfrac{1}{D}\left[u_2 v_3 \tilde{f}_1(Tw) + (u_3 - u_1 v_3)\tilde{f}_2(Tw) - u_2 \tilde{f}_3(Tw)\right],$$

with $D = \frac{1}{-u_2 z_3 + u_3 z_2 - u_1 v_3 z_2 + u_2 v_3 z_1}$, and:

$$\tilde{f}_1(Tw) = \tfrac{1}{2}(u_1 w_1 + w_2 + z_1 w_3)^2 + \left(\tfrac{\alpha - \sigma}{\sigma}\right)(u_1 w_1 + w_2 + z_1 w_3)(w_1 + z_3 w_3),$$

$$\tilde{f}_2(Tw) = -(u_1 w_1 + w_2 + z_1 w_3)(u_2 w_1 + z_2 w_3) - \tfrac{\gamma\delta}{\alpha}(u_2 w_1 + z_2 w_3)^2,$$

$$\tilde{f}_3(Tw) = (\alpha - 1)(w_1 + z_3 w_3)^2 - \tfrac{\gamma\delta}{\alpha}(u_2 w_1 + z_2 w_3)(w_1 + z_3 w_3).$$

6. To allow a center manifold reduction, we assume the following relationship

$$w_3 = h(w_1, w_2)$$

to be stable over time, which implies also:

$$\dot{w}_3 - \frac{\partial h}{\partial w_1}\dot{w}_1 - \frac{\partial h}{\partial w_2}\dot{w}_2 = 0 .$$

We explicitly assume:

$$w_3 = h(w_1, w_2) = \frac{1}{2}[\tau_1 w_1^2 + \tau_2 w_1 w_2 + \tau_3 w_2^2],$$

where:

$$\tau_1 = \frac{2\omega}{Tr^2(J^*(0))+4\omega^2}\left[-C_4 + \frac{2\omega C_1}{Tr(J^*(0))} - \frac{2\omega C_2}{Tr(J^*(0))}\right] - \frac{2C_1}{Tr(J^*(0))},$$

$$\tau_2 = \frac{\left[4\omega(C_1 - C_2) - 2C_4 Tr(J^*(0))\right]}{\left[Tr^2(J^*(0))+4\omega^2\right]},$$

$$\tau_3 = -\frac{2C_2}{Tr(J^*(0))} - \frac{2\omega}{Tr^2(J^*(0))+4\omega^2}\left[-C_4 + \frac{2\omega C_1}{Tr(J^*(0))} - \frac{2\omega C_2}{Tr(J^*(0))}\right].$$

To study the stability of periodic orbits around the steady state, we consider the Andronov-Hopf bifurcation coefficient:

$$q = \frac{1}{16}\left[\overline{F}^1_{w_1w_1w_1} + \overline{F}^1_{w_1w_2w_2} + \overline{F}^2_{w_1w_1w_2} + \overline{F}^2_{w_2w_2w_2}\right] + \left[\overline{F}^1_{w_1w_2}(\overline{F}^1_{w_2w_2} + \overline{F}^1_{w_1w_1}) + -\overline{F}^2_{w_1w_2}(\overline{F}^2_{w_2w_2} + \overline{F}^2_{w_1w_1}) - \overline{F}^1_{w_1w_1}\overline{F}^2_{w_1w_1} + \overline{F}^1_{w_2w_2}\overline{F}^2_{w_2w_2}\right]$$

or explicitly:

$$q = \frac{1}{8\omega}\left[A_3(A_1 + A_2) - B_3(B_1 + B_2) - 2A_1B_1 + 2A_2B_2\right],$$

where:

$$A_1 = -\frac{v_3 z_2}{D}\left[\frac{1}{2} + \left(\frac{\alpha-\sigma}{\sigma}\right)\right]u_1 - \frac{\gamma\delta}{\alpha}\frac{(-z_3 + v_3 z_1)}{D}u_2^2 + \frac{z_2}{D}\left[(\alpha - 1) - \frac{\gamma\delta}{\alpha}u_2\right],$$

$$A_2 = -\frac{1}{2}\frac{v_3 z_2}{D},$$

$$A_3 = -\frac{v_3 z_2}{D}\left[u_1 + \left(\frac{\alpha-\sigma}{\sigma}\right)\right] - \frac{(-z_3 + v_3 z_1)}{D}\left[u_1 u_2 + u_2\right],$$

$$B_1 = \frac{(-u_2 z_3 + u_3 z_2)}{D}\left[\frac{1}{2} + \left(\frac{\alpha-\sigma}{\sigma}\right)\right]u_1 - \frac{\gamma\delta}{\alpha}\frac{(u_1 z_3 - u_3 z_1)}{D}u_2^2 + \frac{(-u_1 z_2 + u_2 z_1)}{D}\left[(\alpha - 1) - \frac{\gamma\delta}{\alpha}u_2\right],$$

$$B_2 = \frac{1}{2}\frac{(-u_2 z_3 + u_3 z_2)}{D},$$

$$B_3 = \frac{(-u_2 z_3 + u_3 z_2)}{D}\left[u_1 + \left(\frac{\alpha-\sigma}{\sigma}\right)\right] - \frac{(u_1 z_3 - u_3 z_1)}{D}\left[u_1 u_2 + u_2\right].$$