UDC 629.7.058.47: 629.783(045)

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USE OF LEAST SQUARE CRITERION IN TRIAD ALGORITHM

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In this paper a using of least squares criterion is proposed to reduce the measurement errors influence to the TRIAD algorithm accuracy. The method testing was carried out to verify the proposed formulas. The case of constant attitude was considered and the results of its estimation are presented.

Keywords: attitude determination method; TRIAD; least squares criterion;vector measurements.

Introduction. All existing satellite attitude determination methods are divided into two big groups: deterministic and stochastic [1]. First group consists of methods which are based on the measurements of two or more sensors in a single point of time. Furthermore, the attitude information derived at previous points of time is not used. Stochastic methods (also known as recursive estimation methods) form the second group. They use information from successive time points, as well as spacecraft dynamics and/or kinematics equations.

Deterministic attitude determination methods also can be divided into two groups. First of them is formed by the TRIAD algorithm and its variants. The algorithms estimating direction cosine matrix based on least-squares criterion are referred to the second group. The last ones are also named optimal as far as they treat measurements in optimal way as opposed to the TRIAD.

The TRIAD Algorithm. The TRIAD algorithm is the simplest deterministic way to find the attitude matrix which maps from reference frame to a body frame [2]. The algorithm uses only two vectors, where each vector has a reference frame vector in a reference frame and a measurement vector in a body frame. The vector measurements in the spacecraft body frame are denoted as $\vec{b}_1$ and $\vec{b}_2$, and the vectors in the reference frame are denoted as $\vec{r}_1$ and $\vec{r}_2$.

In accordance with the TRIAD algorithm triads of orthonormal unit vectors are constructed in the reference frame and in the body frame:

$$\vec{v}_1 = \vec{r}_1,\ \vec{v}_2 = \frac{\vec{r}_1 \times \vec{r}_2}{\left|\vec{r}_1 \times \vec{r}_2\right|},\ \vec{v}_3 = \vec{v}_1 \times \vec{v}_2, \tag{1}$$

$$\vec{w}_1 = \vec{b}_1,\ \vec{w}_2 = \frac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|},\ \vec{w}_3 = \vec{w}_1 \times \vec{w}_2. \tag{2}$$

Then based on constructed triads two matrices M_0 and M are composed:

$$M_0 = \left[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3\right], \tag{3}$$

$$M = \left[\vec{w}_1 \,|\, \vec{w}_2 \,|\, \vec{w}_3\right]. \tag{4}$$

As components of matrices M_0 and M are orthogonal unit vectors so these matrices are orthogonal matrices. The attitude matrix based on these matrices can be written as shown

$$A = MM_0^{-1} = MM_0^T \tag{5}$$

It should be noted that as the first vector of triad $\vec{b}_1$ (and $\vec{r}_1$ respectively) should be selected a vector that is measured in body frame more precisely. Based on the attitude matrix A angles of rotation can be calculated if a sequence of rotations is known.

There are some variants of the TRIAD algorithm which differ one from another essentially by a manner of constructing triads of vectors [3]. But the idea of the triads' construction is the same for all variants.

This algorithm is very simple to be realized and very fast. The main drawback of the TRIAD algorithm is that only two vector measurements can be used to obtain attitude matrix. Some accuracy is lost when more measurements are available.

In the ideal case the next equation is true

$$\vec{b}_i = \mathrm{A}\vec{r}_i. \tag{6}$$

But it is not generally true in reality which caused by the presence of measurement errors.

Wahba's Problem and its Solutions. The problem of finding the best estimate of the **A** matrix was posed by Grace Wahba [4] who was the first to choose a least squares criterion to define the best estimate, i. e. to find the orthogonal matrix **A** with determinant +1 that minimizes the loss function

$$L(A) = \frac{1}{2}\sum_{i=1}^{n} a_i \left|\vec{b}_i - A\vec{r}_i\right|^2, \tag{7}$$

where a_i is a set of positive weights assigned to each measurement. It was proven that the loss function can be rewritten as

$$L(A) = a_0 - tr\left(AB^T\right) \tag{8}$$

with

$$a_0 = \sum_{i=1}^{n} a_i \text{ and } B = \sum_{i=1}^{n} a_i \vec{b}_i \vec{r}_i^{\,T} . \tag{9}$$

The loss function will be minimal when the trace of the matrix product AB^T is maximal, under constraint

$$A^T A = I . \tag{10}$$

It can be seen that such problem formulation allows incorporating more than two measurements for the estimation of an attitude matrix. Moreover, measurements derived by means of different sensors take into account in different way through the coefficients a_i.

The first solutions of Wanba's problem were presented in [5]. Wessner and Brock independently proposed the solution

$$A_{opt} = \left(B^T\right)^{-1}\left(B^T B\right)^{1/2} = B\left(B^T B\right)^{-1/2} \tag{11}$$

but the matrix inverses in (11) exist only if B is non-singular, which means that a minimum of three vectors must be observed. It is well known that two vectors are sufficient to determine the attitude; and the method of Farrell and Stuelpnagel, as well as the other existing methods only require B to have rank two. They were earliest solutions. But almost all deterministic attitude determination methods are based on quaternion representation of the loss function derived by *P*. Davenport [6]. Majority of deterministic attitude determination methods solve the problem, i. e. estimate attitude matrix in optimal way [7].

Using the Least Squares Criterion for the TRIAD Algorithm. Lets consider possibility of using the least squares criterion for the TRIAD algorithm. Matrix A has to minimize the loss function:

$$L(A) = \frac{1}{2}\sum_{i=1}^{n}\left\|M_i - \mathrm{A}M_{0i}\right\|^2 , \tag{12}$$

where matrices M_i and M_{0i} are defined in according to the TRIAD algorithm (see (3) – (4)). An index i can be considered as a notation for different points of time or as notation for different vector pairs using to calculate these matrices. Equation (12) can be rewritten as:

$$\begin{aligned} L(A) &= \frac{1}{2}\sum\left\|\mathrm{M}_i - \mathrm{AM}_{oi}\right\|^2 = \frac{1}{2}\sum tr\left[\left(\mathrm{M}_i - \mathrm{AM}_{oi}\right)^T\left(\mathrm{M}_i - \mathrm{AM}_{oi}\right)\right] = \\ &= \frac{1}{2}\sum_{i=1}^{n} tr\left(\mathrm{M}_i^T\mathrm{M}_i + \mathrm{M}_{0i}^T\mathrm{M}_{0i} - 2\mathrm{M}_i^T\mathrm{AM}_{0i}\right). \end{aligned} \tag{13}$$

It can be shown that $tr\left(\mathrm{M}_{0i}^T\mathrm{A}^T\mathrm{M}_i\right) = tr\left(\mathrm{M}_i^T\mathrm{AM}_{0i}\right)$. Since the first two terms are independent of A, $L(A)$ is minimized by maximizing $J_1(A) = \sum_{i=1}^{n} tr(\mathrm{M}_i^T\mathrm{AM}_{0i})$. The last expression can be recast into the following form:

$$J_1(A) = \sum tr(\mathrm{M}_i^T\mathrm{AM}_{0i}) = \sum tr\left(\mathrm{AM}_0\mathrm{M}_i^T\right) = tr\left(AS\right), \tag{14}$$

where

$$S = \sum \mathrm{M}_{0i}\mathrm{M}_i^T . \tag{15}$$

The constraint (10) can be included in the cost function via the Lagrange multiplier matrix Λ, Λ is assumed to be a symmetric matrix. So consider the following scalar function:

$$J_2(A) = tr\left[\left(AS\right) - \frac{1}{2}\Lambda\left(\mathrm{A}^T\mathrm{A} - I\right)\right], \tag{16}$$

Minimizing this function (taking the partial derivative with respect to A and equating the result to the null matrix) yields the following:

$$\frac{\partial J_2(A)}{\partial A} = S^T - A\Lambda = 0_{3\times 3} . \tag{17}$$

The value of A that minimizes the cost function becomes

$$\mathrm{A} = S^T\Lambda^{-1}. \tag{18}$$

Next, the constraint (10) above is used to eliminate Λ^{-1} in the result just obtained. Substituting that result into the constraint equation gives the solving:

$$\mathrm{A} = S^T\left(\sqrt{SS^T}\right)^{-1}, \tag{19}$$

where matrix square root $\sqrt{SS^T}$ is the positive definite (principal) square root of a positive definite symmetric matrix.

As was mentioned above the summation in (15) can be done both for several measurements and for subsequent points of time. In the first case it allows to combine more that two measurements. But in such case the products $\mathrm{M}_{0i}\mathrm{M}_i^T$ should be weighted to take into account different accuracies of measurements. In the second case the summation can reduce an influence of random measurement errors to the attitude determination accuracy.

Consider the second case in details. When a satellite has been stabilized in the reference frame its angular rates are small. It allows us to make an assumption that the satellite attitude does not change during the short period of time. That is reference vectors $\vec{r}_1$ and $\vec{r}_2$ computed at different orbital points during the short time will be approximately identical. By analogy measurement vectors derived by sensors at different orbital points will be the same with the exception of noise component. The summation will reduce the noise level of measurements. It by-turn will increase an accuracy of attitude determination.

It should be noted that a moving total can be used in (15) instead of a simple summation. Using of moving total allows obtaining attitude information at every point of time, while the summation allows obtaining result only after necessary amount of measurements will be accumulated.

The Method Testing. In order to test the introduced algorithm an estimation of an attitude matrix for case of constant rotation angles was done. The rotation sequence which transforms the reference frame to the body frame is: a rotation about axis 3 by yaw angle · first; then a rotation about axis 2 by pitch angle g is performed; finally a rotation about axis 1 by roll angle ¨ is performed. The rotation angles are assumed to be constant and are given by $\cdot = 20°$, $g = 15°$, $¨ = 10°$ respectively.

The case of using only two vector measurements is considered as far as it is a common case for microsatellites [8]. The reference vectors are assumed to be invariable. It is assumed that measurement vectors are given by

$$\tilde{\vec{b}}_i = \vec{b}_i + \vec{\nu}_i, \qquad (20)$$

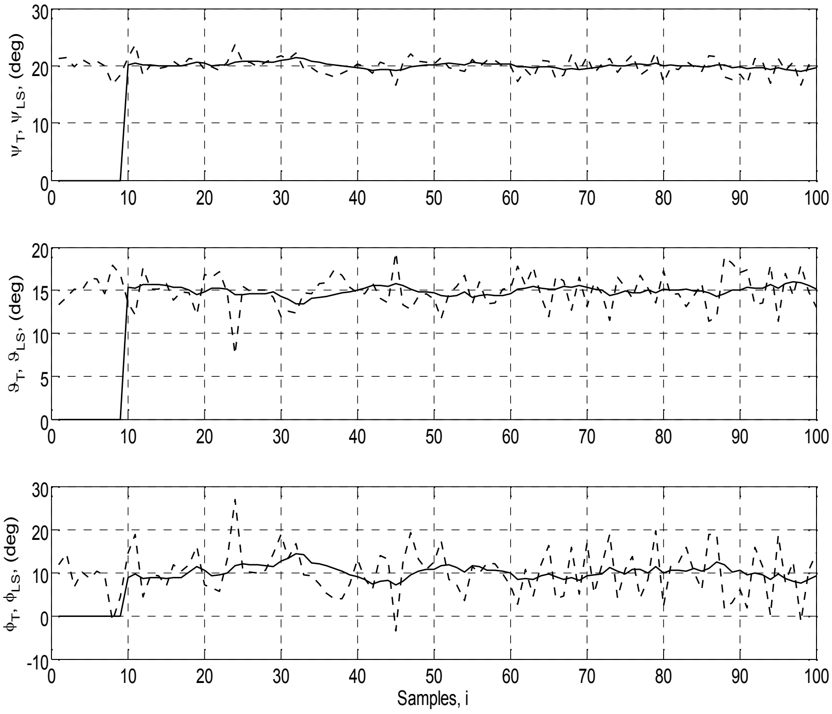
where $\vec{v}_i$ are vectors of measurement errors which components are assumed to be Gaussian with zero mean. The first vector is measured more precisely than the second one that is the components standard deviations $\vec{\nu}_i$ are given by $\sigma_{\nu 1} = 0,01$, $\sigma_{\nu 2} = 0,1$ respectively. The vector triads in the reference frame and in the body frame are constructed based on the first vector. The attitude matrix is estimated by means of the TRIAD algorithm (5) and the introduced algorithm (19) which uses the least squares criterion.

Table contains the results of the rotation angles estimation performed by two methods for different amount N of measurements which are incorporated in the matrix S (see (15)). The index T denotes the angle estimate in according to (5) and the index LS denotes the angle estimate in according to (19).

The estimation results for the TRIAD algorithm and the introduced algorithm

N	**Value**	$\cdot_T$, **(deg)**	$\cdot_{LS}$, **(deg)**	g_T, **(deg)**	g_{LS}, **(deg)**	$¨_T$, **(deg)**	$¨_{LS}$, **(deg)**
3	mean	20,097	20,171	14,675	14,715	10,856	10,892
	max	23,028	22,323	19,369	17,689	25,098	18,147
	min	15,684	17,914	8,095	11,445	-4,928	1,392
	std. dev.	2,989	0,946	2,413	1,251	3,706	3,409
5	mean	19,727	19,783	15,118	15,231	9,475	9,328
	max	23,244	21,952	20,341	17,444	23,243	18,270
	min	15,244	17,879	8,980	11,978	-6,464	2,459
	std. dev.	3,978	0,819	3,160	1,015	3,382	2,899
10	mean	19,952	20,027	14,890	14,943	10,019	10,089
	max	23,695	21,341	19,367	15,971	26,903	14,452
	min	16,627	19,060	7,569	13,424	-3,582	7,221
	std. dev.	5,778	0,480	4,326	0,517	3,219	1,462

The figure represents the estimation process for the case $N = 10$. From this figure, it is clear that the using of (19) to estimate the attitude matrix allows reducing the influence of measurement errors to the attitude determination accuracy. The using of moving total allows obtaining a satellite attitude for every point of time.



Comparison the results of the TRIAD(dotted line) and the TRIAD with *LS* criterion using (solid line) for the case $N = 10$

Conclusion. The using of the least squares criterion to improve the attitude matrix estimate derived by the TRIAD algorithm is presented. The introduced method can be used for the case of small angular rates of satellite. In this case a satellite attitude can be considered as constant during a short period of time. A possibility to obtain analytical solutions for a satellite angular rates constraint should be considered. Also introduced formulae can be simplified in consideration of the estimated value, i. e. an attitude, is constant.

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Використання критерію найменших квадратів в алгоритмі TRIAD

Запропоновано використання критерію найменших квадратів з метою зменшення впливу похибок вимірювачів на точність оцінки орієнтації, отриманої за допомогою алгоритму TRIAD. З метою перевірки отриманих співвідношень проведене чисельне моделювання роботи алгоритму. Розглянутий випадок повільного кутового руху та представлено результати оцінювання його параметрів.

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Использование критерия наименьших квадратов в алгоритме TRIAD

Предложено использование критерия наименьших квадратов с целью уменьшения влияния погрешностей измерителей на точность оценки ориентации, полученной с помощью алгоритма TRIAD. С целью проверки полученных соотношений проведено численное моделирование работы алгоритма. Рассмотрен случай медленного вращения и представлены результаты оценивания его параметров.