THEORY AND METHODS OF SIGNAL PROCESSING

UDC 621.372.061.9:517/9.001.57 (045)

[1]O. V. Vasyliev,
[2]V. V. Vasylyev,
[3]L. A. Simak

S-TRANSFORM IN DIGITAL SIGNAL PROCESSING OF FRACTIONAL DYNAMICS OF CONTINUOUS SYSTEMS

[1]Dept. of hybrid modelling and controlling systems in energetics, Pukhov Institute for Modelling in Energy Engineering (PIMEE), NAS of Ukraine, e-mail: oleksii.vasyliev@gmail.com
[2]Dept. of hybrid modelling and controlling systems in energetics, PIMEE, NAS of Ukraine, e-mail: vsvv06@gmail.com
[3]Dept. of hybrid modelling and controlling systems in energetic, PIMEE, NAS of Ukraine, e-mail: liliya.simak@mail.ru

Application of approximating – operational method of S-transform based on the local and global versions of Legendre polynomials for estimation of average values of signals, average values of derivatives of the first and second orders, and also an estimation of Riemann-Liouville and Caputo fractional derivatives of various orders is considered. Illustrative examples of method's application in programming environment of system "Mathematica®" are given.

Keywords: fractional calculus, approximation and processing of signals, operational calculus, *S*-transform, dynamic system, fractional Caputo derivative, integral of Riemann-Liouville.

Introduction. When researching nonstationary processes in dynamic systems of different purposes the wide circulation was received by the operational methods allowing to algebraize integro-differential models of such systems. To these methods Laplace transform, Fourier transform, *Z*-transform, various integral and differential transforms belong [3; 6]. However, use of the specified methods in case of nonlinear dynamic systems encounters considerable difficulties. Development of the theory of fractal dynamics and fractional calculus [2; 4; 5] has led to creation of the numerically-analytical and approximate methods of research of nonstationary processes, to which belong Pukhov's differential transforms [3] and *S*-transform [1; 2]. The last is based on use polynomial approximations of signals with various systems of the basis functions forming an approximating polynomial. It is possible to interpret the expressions making a basis of approximating methods, as an operational calculus, and various basis systems of functions generate various variants of operational calculuses. Use of orthogonal polynomials as systems of basis functions allows us to solve variety of problems of digital signal processing with algebraization of mathematical models of dynamic systems. The given work is devoted to reviewing of some problems of digital processing of continuous signals.

Approximating methods of processing of continuous signals allow us to fulfil effective compression of the information of a signal, signal digitization, ensure a low-frequency filtration and allocation of useful signal against noises, identification of parametres of dynamic systems, an estimation of a signal and its derivative of various integer and fractional orders. In the given work reviewing is limited to programs and solution examples of problems of specified parametres of signals estimation and is organised as follows. The second section contains retrospective reviewing of a polynomial approximation, as operational method. The concept is introduced and expressions for operational matrices of differentiation of various integer orders are received. It has allowed to receive expressions for operational analogues of operators of Riemann-Liouville and Caputo fractional differentiation [2; 4; 5]. In the third section programs and examples of methods realization of digital signals processing, basically using systems of basis functions on the base of local and global versions of Legendre polynomials are given. In the final section there are the analysis of the results received during computing experiments, and recommendations about their further use.

Polynomial approximation of signals as an operational calculus. Under polynomial approximation of $x(t)$ signal it's implied expression:

$$x_a(t) = \sum_{i=1}^{m} X_i s_i(t) = \vec{\mathbf{X}}^* \cdot \vec{\mathbf{S}}(t)\,, \qquad (1)$$

where $\vec{\mathbf{X}} = \{X_1, X_2, \cdots, X_m\}^*$ – vector of approximating polynomial factors; $\vec{\mathbf{S}}(t) = \{s_1(t), s_1(t), \cdots, s_m(t)\}^*$ – vector of basis functions system; (*) – a symbol of vectors and matrices transposition.

It is supposed that the signal and system of basis functions are defined on the same interval of argument $0 \le t \le T$, and the functions organising basis system, are linearly independent. The best approximation is reached, if the vector of factors of an approximating polynomial is chosen from a condition of a minimum of integral of approximation error function square $\varepsilon(t) = x(t) - x_a(t)$ at approximation interval

$$\mu(\vec{\mathbf{X}}) = \int_0^T \varepsilon^2(t)dt \to \min\,. \qquad (2)$$

The condition (2) leads to system of the linear algebraic equations

$$\mathbf{W} \cdot \vec{\mathbf{X}} = \vec{\mathbf{Q}}\,. \qquad (3)$$

Matrix elements $\mathbf{W}$ depend only on system of basis functions

$$\mathbf{W} = \int_0^T \vec{\mathbf{S}}(t) \cdot \vec{\mathbf{S}}(t)^* dt\,. \qquad (4)$$

And the vector $\vec{\mathbf{Q}}$ is defined by expression

$$\vec{\mathbf{Q}} = \int_0^T \vec{\mathbf{S}}(t) x(t) dt\,. \qquad (5)$$

In formulas (4), (5) and further in the given work integration of vector and matrix functions is fulfilled element by element. The solution of equations system (3) looks like:

$$\vec{\mathbf{X}} = \mathbf{W}^{-1} \cdot \vec{\mathbf{Q}}\,. \qquad (6)$$

It is possible to interpret given above formulas of polynomial signals approximation (1), (4) – (6) as an operational calculus of the approximating type which basic relations look like:

$$\vec{\mathbf{X}} = \left(\int_0^T \vec{\mathbf{S}}(t) \cdot \mathbf{S}(t)^* dt \right)^{-1} \cdot \left(\int_0^T \vec{\mathbf{S}}(t) x(t) dt \right), \qquad (7)$$

$$x_a(t) = \sum_{i=1}^{m} X_i s_i(t) = \vec{\mathbf{X}}^* \cdot \vec{\mathbf{S}}(t) = \vec{\mathbf{S}}(t)^* \cdot \vec{\mathbf{X}}\,. \qquad (8)$$

Direct transform (7) compares to a signal $x(t)$ its operational analogue or the image in the form of a vector of factors $\vec{\mathbf{X}}$ of the approximating polynomial, and inverse transform (8) realises reconstruction of a signal in the form of its approximation:

$$x(t) \Rightarrow \vec{\mathbf{X}} \Rightarrow x_a(t)\,. \qquad (9)$$

The chain of transforms given here has received a name of *S*-transform [2]. For *S*-transform as an operational method, there are rules of mathematical operations performed on images of signals in domain of the images that are equivalent to set operations on signals in domain of originals. We will consider some of these rules in more detail.

– Linear combination of signals. *To a linear combination of signals corresponds the same combination of their images:*

$$ax(t) \pm by(t) \Rightarrow a\vec{\mathbf{X}} \pm b\vec{\mathbf{Y}}. \quad (10)$$

– Integration of a signal with a variable upper limit. *To this operation corresponds multiplication of the image of intergrand function to an operational matrix of integration:*

$$y(t) = \int_0^t x(\tau)d\tau \Rightarrow \vec{\mathbf{Y}} = \mathbf{P}_S^1 \cdot \vec{\mathbf{X}}. \quad (11)$$

The operational matrix of integration is defined by expression

$$\mathbf{P}_S^1 = \mathbf{W}^{-1} \cdot \left(\int_0^T \vec{\mathbf{S}}(t) \cdot \left(\int_0^t \vec{\mathbf{S}}(\tau)^* d\tau \right) dt \right). \quad (12)$$

– Riemann-Liouville integration with fractional order . *To fractional integration of a signal corresponds also multiplication of its image to an operational matrix of integration:*

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \cdot x(\tau)d\tau \Rightarrow \vec{\mathbf{Y}} = \mathbf{P}_S^{\beta} \cdot \vec{\mathbf{X}}. \quad (13)$$

The operational matrix of integration of a fractional order is defined by expression:

$$\mathbf{P}_S^{\beta} = \mathbf{W}^{-1} \cdot \left(\int_0^T \vec{\mathbf{S}}(t) \cdot \left(\frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \vec{\mathbf{S}}(\tau)^* d\tau \right) dt \right). \quad (14)$$

– Differentiation of a signal with the integer order *n*. *To the taking of n – th derivative of a signal there corresponds multiplication of its image to operational matrix of differentiation:*

$$y(t) = \frac{d^n x(t)}{dt^n} \Rightarrow \vec{\mathbf{Y}} = \mathbf{Ldn} \cdot \vec{\mathbf{X}}. \quad (15)$$

The operational matrix of differentiation is defined by expression:

$$\mathbf{Ldn} = \mathbf{W}^{-1} \cdot \left(\int_0^T \vec{\mathbf{S}}(t) \cdot \frac{d^n \vec{\mathbf{S}}(t)^*}{dt^n} dt \right). \quad (16)$$

For existence of a differentiation operational matrix the basis system functions should suppose differentiation with corresponding order.

– Riemann-Liouville fractional derivative of order determination. *To Riemann–Liouville signal differentiation with order* ($n-1 \le \beta < n$) *corresponds the expression:*

$$y(t) = \frac{d^n}{dt^n} \left(\frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} x(\tau)d\tau \right) \Rightarrow \vec{\mathbf{Y}} = \mathbf{Ldn} \cdot \mathbf{P}_S^{n-\beta} \cdot \vec{\mathbf{X}}. \quad (17)$$

The operational matrix of differentiation is defined by expression:

$$^{RL}\mathbf{Ld}^{\beta} = \mathbf{Ldn} \cdot \mathbf{P}_S^{n-\beta}. \quad (18)$$

– Caputo fractional derivative of order determination. *To Caputo signal differentiation with order* ($n-1 \le \beta < n$) *corresponds expression:*

$$y(t) = \left(\frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{d^n x(\tau)}{d\tau^n} d\tau \right) \Rightarrow \vec{\mathbf{Y}} = \mathbf{P}_S^{n-\beta} \cdot \mathbf{Ldn} \cdot \vec{\mathbf{X}}. \quad (19)$$

The operational matrix of differentiation is defined by expression:

$$^{C}\mathbf{Ld}^{\beta} = \mathbf{P}_{S}^{n-\beta} \cdot \mathbf{Ldn}. \tag{20}$$

Programs and examples of digital processing of continuous signals within the limits of *S*-transform with basis systems of Legendre polynomials.

The program 1. Forming the systems of basis functions on the base of Legendre polynomials.

– *The setting of a digitization step, amount of digitization intervals, order of polynomials and range of an argument modification:*

```
h = 1 / 25; m = 25; r = 10; T = 1;
```

– *Shaping of the displaced Legendre polynomials, orthogonal on a digitization step or a range of a signal argument modification:*

```
s[j_, i_, t_, h_] :=
  If[(i - 1) * h ≤ t < i * h, LegendreP[j - 1, 1 - 2 i + 2 t / h], 0];
```

– *Shaping of a subsystem of basis functions on the base of local Legendre polynomials of a zero order:*

```
V1 = Table[s[1, i, t, h], {i, m}];
```

– *Shaping of a subsystem of basis functions on the base of local Legendre polynomials of the first order:*

```
V2 = Table[s[2, i, t, h], {i, m}];
```

– *Shaping of a subsystem of basis functions on the base of local Legendre polynomials of the second order:*

```
V3 = Table[s[3, i, t, h], {i, m}];
```

– *Shaping of system of basis functions on the base of the displaced Legendre polynomials, orthogonal on a modification interval* $0 \le t \le T$ *of argument t:*

```
V = Table[s[j, 1, t, T], {j, r}];
```

– *Visualisation of some functions of local basis subsystems* (*fig.* 1):

```
po = Plot[{V1[[1]], V2[[3]], V3[[5]], V1[[7]], V2[[7]], V1[[9]],
   V2[[9]], V3[[9]]}, {t, 0, 0.41}]
```

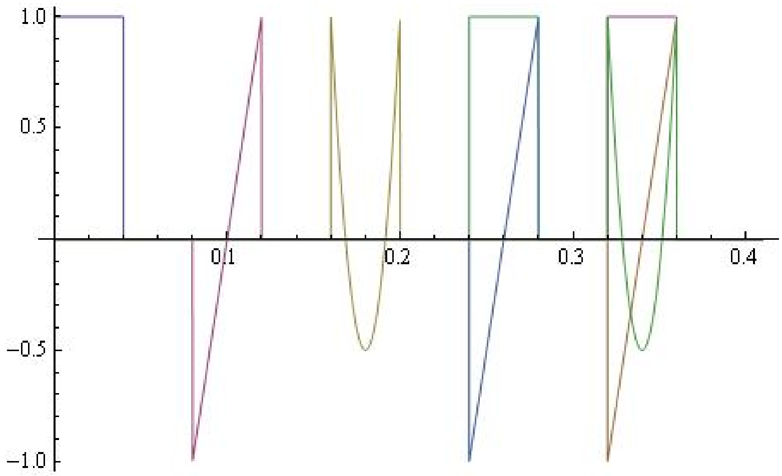

Fig. 1. Some functions of local basis subsystems

– Visualisation of system of the global version of Legendre polynomials of the 10-th order (*fig.* 2):

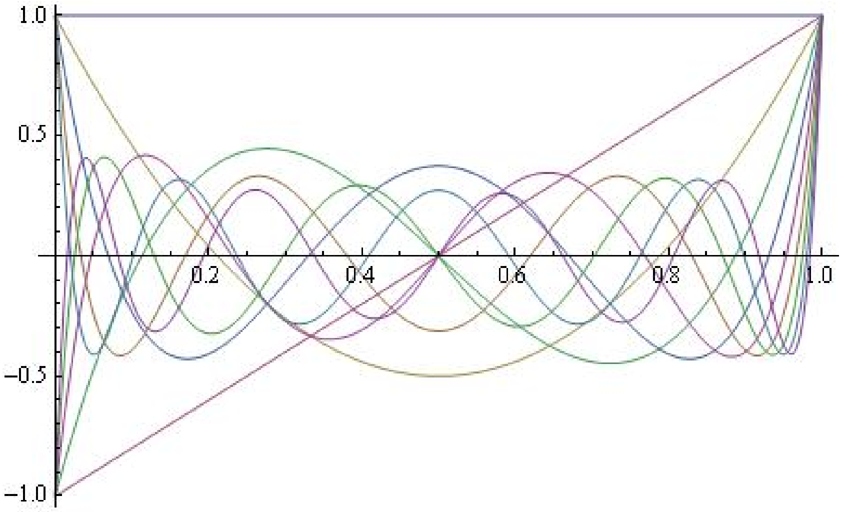

Fig. 2. System of Legendre polynomials of 10-th order, that are orthogonal on interval $T = 1$

The program 2. Approximation of a test signal, digitization and estimation of average values of a signal and its derivatives of the first and second orders. (It is supposed that results of the program 1 are kept working and active).

– Determination of a test signal and its derivatives of the first and second orders:

```
x = Sin[2 π * t^2];
x1 = D[x, t]
4 π t Cos[2 π t^2]
x2 = D[x1, t]
4 π Cos[2 π t^2] - 16 π^2 t^2 Sin[2 π t^2]
```

– Determination of elements of reciprocal matrices of local subsystems and matrix ***W*** (*see the formula* (*3*)):

```
m1 = (∫_0^h s[1, 1, t, h]^2 ⅆt)^-1;
m2 = (∫_0^h s[2, 1, t, h]^2 ⅆt)^-1;
m3 = (∫_0^h s[3, 1, t, h]^2 ⅆt)^-1;
W = DiagonalMatrix[Table[∫_0^T (s[j, 1, t, T])^2 ⅆt, {j, r}]];
```

– Determination of fragments of factors vectors of the polynomials approximating a signal, for local basis subsystems:

```
X1 = m1 * N[Table[∫_0^T x * V1[[i]] ⅆt, {i, m}]]
{0.00335101, 0.0234542, 0.0636198, 0.123645, 0.202923, 0.300086,
 0.412556, 0.536025, 0.663952, 0.787156, 0.893676, 0.969071, 0.99737,
 0.962855, 0.852771, 0.660828, 0.391118, 0.0616387, -0.293737,
 -0.626264, -0.877618, -0.990168, -0.921564, -0.661076, -0.242975}
X2 = m2 * N[Table[∫_0^T x * V2[[i]] ⅆt, {i, m}]]
```

```
{0.00502649, 0.0150749, 0.0250791, 0.0349085, 0.0442818,
 0.0527138, 0.0594728, 0.0635637, 0.0637523, 0.0586524,
 0.0468982, 0.0274181, -0.00018914, -0.0352038, -0.0753315,
 -0.116368, -0.15216, -0.175056, -0.17703, -0.151526,
 -0.0958666, -0.0137245, 0.0831577, 0.175088, 0.237362}
```

```
X3 = m3 * N[Table[∫_0^T x * V3[[i]] dt, {i, m}]]
```

```
{0.00167547, 0.00167317, 0.00165842, 0.00161103, 0.00150121,
 0.00129092, 0.000936745, 0.000395314, -0.000368123, -0.00136729,
 -0.00257904, -0.00392578, -0.00525965, -0.00635464, -0.00691467,
 -0.00660674, -0.00512643, -0.00229657, 0.00181183, 0.00676444,
 0.0117007, 0.0153819, 0.0164041, 0.0135946, 0.00654038}
```

– *Determination of a factors vector of the polynomial approximating a signal for global basis system:*

```
X = Inverse[W].N[Table[∫_0^T x * s[j, 1, t, T] dt, {j, r}]]
```

```
{0.171708, -0.515124, -0.945975, 0.288801, 0.902188,
 0.350798, -0.095126, -0.130934, -0.0400863, 0.00441312}
```

– *Visualisation a signal and its average values on a grid with a step h* (*fig. 3*):

```
p12 = ListPlot[Table[{(i - 0.5) / m, X1[[i]]}, {i, m}],
  Filling → Axis, FillingStyle → Red]
p13 = Plot[x, {t, 0, 1}]
p14 = Show[p12, p13]
```

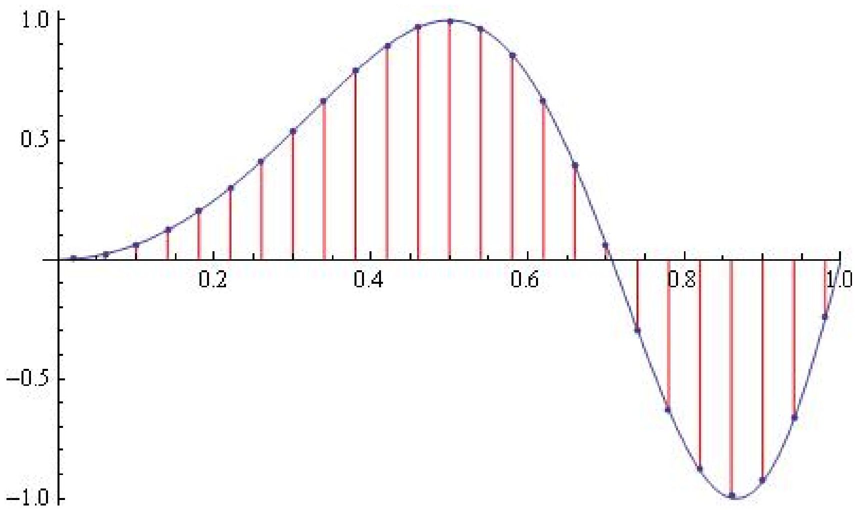

Fig. 3. A test signal and results of its digitization (estimation)

– *Visualisation of the first derivative of a signal and its average values (estimation) on a grid with a step h* (*fig. 4*):

```
p15 = Plot[x1, {t, 0, 1}]
p17 = ListPlot[Table[{(i - 0.5) / m, 2/h * X2[[i]]}, {i, m}],
  Filling → Axis, FillingStyle → Red]
p19 = Show[p15, p17]
```

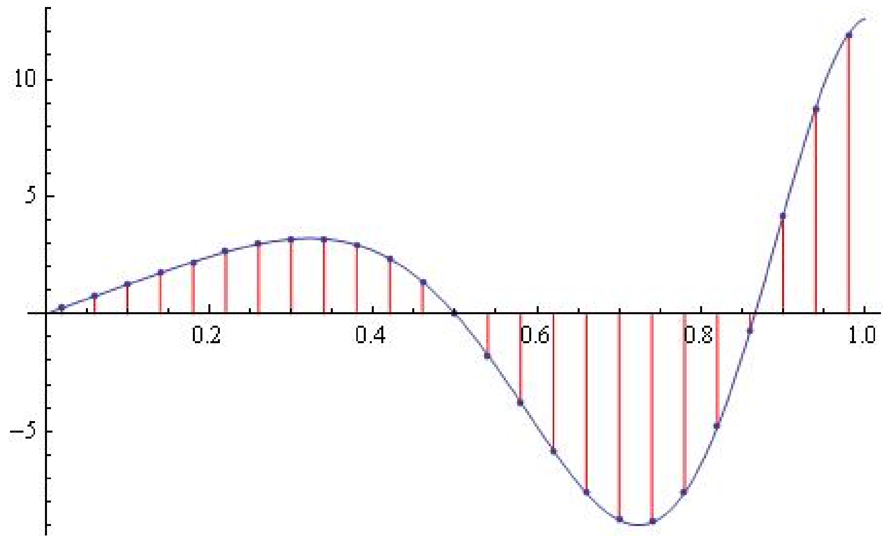

Fig. 4. The first derivative of a a test signal and its digitization (estimation)

– Visualisation of a second derivative of a signal and its average values (estimation) on a grid with a step h (*fig. 5*):

```
p16 = Plot[x2, {t, 0, T}]
p18 = ListPlot[Table[{(i - 0.5) / m, 12/h^2 * X3[[i]]}, {i, m}],
  Filling → Axis, FillingStyle → Red]
p20 = Show[p16, p18]
```

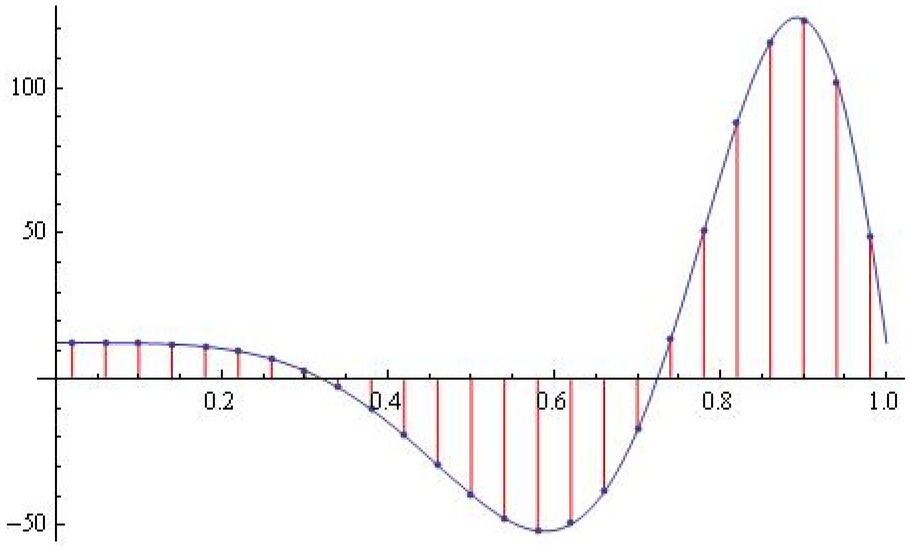

Fig. 5. The second derivative of a test signal and its digitization (estimation)

The program 3. Definition of operational matrices of various integer and fractional orders in base of Legendre polynomials.

– The representation of an order of basis system and range of argument modification:

```
r = 10; T = 1;
```

– Determination of basis system functions on the basis of the global version of Legendre polynomials:

```
so[t_, T_, i_] := LegendreP[i - 1, -1 + 2 t / T];
So = Table[so[t, T, i], {i, r}];
```

– *Determination of operational matrix* **W***:*

```
W = DiagonalMatrix[Table[∫_0^T (so[t, T, j])^2 dt, {j, r}]];
```

– *The setting of numerical value of a fractional order integral (differential) operators:*

```
β = 1/2;
```

– *Determination of an operational matrix of integration of a fractional order:*

```
Pβ :=
  N[Inverse[W].Table[∫_0^T so[t, T, i] * (1/Gamma[β] * ∫_0^t (t - τ)^(β-1) * so[τ, T, j] dτ) dt,
    {i, r}, {j, r}]];
```

– *Determination of vectors of derivatives of the first and second orders from basis system of functions:*

```
S1 = D[So, t]; S2 = D[So, {t, 2}];
```

– *Determination of a vector of derivatives of order n from basis system of functions* **So** *of order r* (*n* <*r*):

```
sn := D[So, {t, n}];
```

– *Determination of differentiation operational matrix of n-th order in basis of Legendre polynomials system of order r:*

```
Ldn := Inverse[W].Table[∫_0^T so[t, 1, i - 1] * sn[[j]] dt, {i, r}, {j, r}];
```

– *Determination of operational matrices of differentiation of the first and second orders* (*fig. 6, 7*):

```
Ld1 = Inverse[W].Table[∫_0^T so[t, 1, i] * S1[[j]] dt, {i, r}, {j, r}];
Ld1 // MatrixForm

Ld2 = Inverse[W].Table[∫_0^T so[t, 1, i] * S2[[j]] dt, {i, r}, {j, r}];
Ld2 // MatrixForm
```

0	2	0	2	0	2	0	2	0	2
0	0	6	0	6	0	6	0	6	0
0	0	0	10	0	10	0	10	0	10
0	0	0	0	14	0	14	0	14	0
0	0	0	0	0	18	0	18	0	18
0	0	0	0	0	0	22	0	22	0
0	0	0	0	0	0	0	26	0	26
0	0	0	0	0	0	0	0	30	0
0	0	0	0	0	0	0	0	0	34
0	0	0	0	0	0	0	0	0	0

Fig. 6. An operational matrix of differentiation of 1-st order

0	0	12	0	40	0	84	0	144	0
0	0	0	60	0	168	0	324	0	528
0	0	0	0	140	0	360	0	660	0
0	0	0	0	0	252	0	616	0	1092
0	0	0	0	0	0	396	0	936	0
0	0	0	0	0	0	0	572	0	1320
0	0	0	0	0	0	0	0	780	0
0	0	0	0	0	0	0	0	0	1020
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Fig. 7. An operational matrix of differentiation of 2-nd order

– *Determination of Riemann-Liuville and Caputo operational matrices of differentiation of a fractional order* ($\beta\ n-1 \le \beta < n$) (*in this case it is chosen* $n = 2$):

```
DRL = Ld2.Pβ;
DC = Pβ.Ld2;
```

Example 1. To define the image of the first and a second derivatives of a test signal $y = \sin(\pi t)$ ***in base of Legendre polynomials of 10-th order, having using expressions of operational matrices of differentiation, and to compare them to the images received by approximation of corresponding derivatives of a test signal.***

The representation of a test signal and its two derivatives:

```
y = Sin[π * t]
y1 = D[y, t]
y2 = D[y, {t, 2}]
```

– Determination of images of a test signal and its derivatives:

```
Y = Inverse[W].N[Table[∫_0^T so[t, 1, i] * y dt, {i, r}]]
{0.63662, 0., -0.687085, 0., 0.051779, 0., -0.00133046, 0., 0.0000171322, 0.}
Y1 = Inverse[W].N[Table[∫_0^T so[t, 1, i] * y1 dt, {i, r}]]
{0., -3.81972, 0., 0.706517, 0., -0.028896, 0., 0.000510015, 0., -5.00604×10^-6}
Y2 = Inverse[W].N[Table[∫_0^T so[t, 1, i] * y2 dt, {i, r}]]
{-6.28319, 0., 6.78126, 0., -0.511038, 0., 0.0131311, 0., -0.000169088, 0.}
```

– Determination of images of the first and second derivatives of a test signal in operational domain with the use of operational matrices of differentiation:

```
Y1a = Ld1.Y
{0., -3.81972, 0., 0.706519, 0., -0.0288931, 0., 0.000513967, 0., 0.}
Y2a = Ld2.Y
{-6.28316, 0., 6.7814, 0., -0.510825, 0., 0.0133631, 0., 0., 0.}
```

The analysis of the received results shows satisfactory accuracy of images estimation of the first and second derivatives: $\mathbf{Y1} \approx \mathbf{Y1a}$, $\mathbf{Y2} \approx \mathbf{Y2a}$.

Example 2. To define images of Riemann-Liuville and Caputo fractional derivatives of a test signal $z(t) = e^{-t}t^5$ ***of order 1.5 in basis of Legendre polynomials of 10-th order, having used expressions of operational matrices of differentiation, and to compare them to the images received by approximation of corresponding derivatives of a test signal. A range of argument modification T=1.***

The representation of a test signal and its derivative of the second order:

```
z[t_] := t^5 * e^-t; z2[t_] := D[z[t], {t, 2}];
```

– Determination of images of a test signal and its derivative of the second order:

```
Z = Inverse[W].N[Table[∫_0^T so[t, 1, i] * z[t] dt, {i, r}]]
{0.0713022, 0.145695, 0.108181, 0.0391033,
 0.00456341, -0.000892997, -0.0000991821, 0., 0., 0.}
Z2 = Inverse[W].N[Table[∫_0^T so[t, 1, i] * z2[t] dt, {i, r}]]
{1.47152, 2.20728, 0.599336, -0.203916,
 -0.0443434, 0.019659, -0.00314903, 0., 0., 0.}
```

– Determination of the image of Caputo derivative of order 1.5 of the test signal by approximation:

```
zc = 1/Gamma[1/2] * ∫_0^t (t - τ)^(-1/2) * z2[τ] dτ
```

$$\text{ConditionalExpression}\Big[\frac{1}{16\sqrt{\pi}}\Big(-\sqrt{t}\ (45+4\,t\,(15+2\,t\,(9+2\,(-7+t)\,t)))+(45+2\,t\,(45+4\,t\,(15+2\,t\,(15+t\,(-15+2\,t)))))\ \text{DawsonF}\big[\sqrt{t}\,\big]\Big),\ t>0\Big]$$

$$\text{z1c}=\frac{1}{16\sqrt{\pi}}\Big(-\sqrt{t}\ (45+4\,t\,(15+2\,t\,(9+2\,(-7+t)\,t)))+(45+2\,t\,(45+4\,t\,(15+2\,t\,(15+t\,(-15+2\,t)))))\ \text{DawsonF}\big[\sqrt{t}\,\big]\Big)$$

$$\text{ZC}=\text{Inverse}[\text{W}].\text{N}\Big[\text{Table}\Big[\int_0^T \text{so}[t,\ 1,\ i]*\text{z1c}\,dt,\ \{i,\ r\}\Big]\Big]$$

```
{0.76356, 1.292, 0.570328, -0.000786606,
 -0.0403096, 0.00383376, 0.00094223, 0., 0., -9.5}
```

– Determination of the image of a Caputo derivative of order 1.5 of the test signals by use of an operational matrix of differentiation:

```
ZC1 = DC.Z
{0.765956, 1.28889, 0.571318, -0.00369742, -0.041201,
 0.00229511, -0.00192962, 0.00119086, -0.000803159, 0.000581764}
```

– Determination of the image of a Riemann–Liuville derivative of order 1.5 of the test signals on the basis of approximation of a derivative signal:

$$\text{zrl}=\text{D}\Big[\frac{1}{\text{Gamma}[0.5]}*\int_0^t (t-\tau)^{-0.5}*\text{z}[\tau]\,d\tau,\ \{t,\ 2\}\Big]$$

$$10.3166\,t^{3.5}\ \text{Hypergeometric1F1}[6,\ 6.5,\ -t]-4.23246\,t^{4.5}\ \text{Hypergeometric1F1}[7,\ 7.5,\ -t]+0.359117\,t^{5.5}\ \text{Hypergeometric1F1}[8,\ 8.5,\ -t]$$

$$\text{ZRL1}:=\text{Inverse}[\text{W}].\text{N}\Big[\text{Table}\Big[\int_0^T \text{so}[t,\ 1,\ i]*\text{zrl}\,dt,\ \{i,\ r\}\Big]\Big]$$

```
{0.76356, 1.292, 0.570328, -0.000786606, -0.0403096,
 0.00383359, 0.000918912, -0.000389609, 0.000106094, -0.0000317606}
```

– Rimann–Liuville derivative determination of order 1.5 of the test signals by use of an operational matrix of differentiation:

```
ZRL = Ld2.(Pβ.Z)
{0.76356, 1.292, 0.570328, -0.000786606, -0.0403096,
 0.00383359, 0.000918912, -0.000389609, 0.000106094, -0.0000317606}
```

The analysis of the received results shows satisfactory estimation accuracy of images of Riemann–Liuville and Caputo fractional derivatives: $\mathbf{ZC} \approx \mathbf{ZC1}, \mathbf{ZRL} \approx \mathbf{ZRL1}$.

Conclusion. Use of basis systems on the basis of the local and global version of Legendre polynomials has allowed to generate expressions for operational matrices of differentiation of the integer orders. Expressions for operational matrices of Riemann–Liuville and Caputo fractional differentiation are deduced. On a series of test examples the application of the *S*-transforms for estimation of average signal values and its derivatives of the first and second orders, and also the estimation of Riemann–Liuville and Caputo derivatives of fractional orders is shown. The programs written in language of the system Mathematica®, despite of the particular character of the presented illustrative examples, suppose a modification of a basis systems sort, key parametres and type of signals and, in our opinion, can be useful when using the methods of *S*-transform in digital processing of continuous signals and modeling of fractional dynamics systems problems.

References

1. *Васильєв В. В.* Аналіз та математичне моделювання динамічних систем на базі некласичних операційних числень / В. В.Васильєв, Л. О. Сімак, О. А. Зеленков та ін. – К.: НАН України, 2006. – 184 с.
   [*Vasyliev V. V.* Analysis and mathematical modeling of dynamic systems based on nonclassical operational calculi / V. V. Vasyliev, L. O. Simak, O. A. Zelenkov and others. – K.: NAN Ukraine, 2006. – 184 p.] (in Ukrainian).
2. *Васильев В. В.* Дробное исчисление и аппроксимационные методы в моделировании динамических систем / В. В. Васильев, Л. А. Симак – К.: НАН Украины, 2008. – 256 с.
   [*Vasyliev V. V.* Fractional calculus and approximation methods for modeling of dynamic systems / V. V. Vasyliev, L. O. Simak – K.: NAN Ukraine, 2008. – 256 p.] (in Russian).
3. *Пухов Г. Е.*, Преобразования Тейлора и их применение в электротехнике и электронике / Г. Е. Пухов. // Наукова Думка, К., 1978. – 260 с.
   [*Pukhov G. E.* Taylor's conversion and their use in electrical and electronic engineering / G. E. Pukhov – K.: Naukova dumka, K., 1978. – 260 p.] (in Russian).
4. *Учайкин В. В.* Метод дробных производных / В. В. Учайкин – Ульяновск, изд-во «Артишок», 2008. – 512 с.
   [*Uchaykin V. V.* The method of fractional derivatives / V. V. Uchaykin – Ulyanovsk: "Artishok", 2008. – 512 p.] (in Russian).
5. *Oldham K.B.* The Fractional Calculus / Oldham K. B.,Spanier J. – Academic Press, 1974. – 234 p.
6. *Poularikas A.D* . (Editor-in-Chief), The Transforms and Applications Handbook, CRC Press & IEEE Press, 2000.
7. *Wolfram Stephen* The Mathematica book. – Wolfram Media / Cambridge University Press, 1996. – 1403 p.

О. В. Васильєв, В. В. Васильєв, Л. О. Сімак

***S*-перетворення в цифрової обробці сигналів дробової динаміки безперервних систем**

Розглянуто застосування аппроксімаційно-операційного методу *S*-перетворення на основі локальної та глобальної версій поліномів Лежандра для оцінювання сигналів: середніх значень, середніх значень похідних першого і другого порядків, а також оцінювання дробових похідних різних порядків за Ріманом–Ліувіллем та Капуто. Наведено ілюстративні приклади застосування методу в програмному середовищі системи «Mathematica ®».

А. В. Васильев, В. В. Васильєв, Л. А. Симак

***S*-преобразование в цифровой обработке сигналов дробной динамики непрерывных систем**

Рассмотрено применение аппроксимационно-операционного метода *S*-преобразования на основе локальной и глобальной версий полиномов Лежандра для оценивания сигналов: средних значений, средних значений производных первого и второго порядков, а также оценивания дробных производных разных порядков по Риману–Лиувиллю и Капуто. Приведены иллюстративные примеры применения метода в программной среде системы «Mathematica ®».