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ALGORITHMS OF GYRO-FREE ACCELEROMETER-BASED SATELLITE-INERTIAL NAVIGATION SYSTEM

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The basic problem of creation of accelerometer-based Strapdown Inertial Navigation System is the conversion of the redundant linear accelerometers readouts in the estimation of the angular accelerations, angular rates and the attitude determination. The relation between amount of redundant sensors and performance (accuracy) of signal processing software is investigated. It is shown that increasing amount of sensors from minimal value 6 to 12 gives additional possibilities of correction of estimation results, thus increasing the estimation accuracy. Simulation of such systems shows the efficiency of estimation especially in a case of the moving vehicle, spinning with high angular rate, when traditional usage of gyros might be very problematic.

Key words: linear accelerometers; rate gyros; Euler angles; integrated navigation systems; rotational mechanization, Global Positioning System.

Introduction. Nowadays many researchers all over the world are concentrating their efforts in order to create gyro-free Inertial navigation system (INS) based on the usage of accelerometers only [1 – 8]. There are at least two reasons of this activity. The first reason is a growth of the Micro Electro-Mechanical Sensors (MEMS) – technology, producing low-cost sensors with enhanced accuracy characteristics [1], thus facilitating creation of Accelerometer-based Strapdown Inertial Navigation System (ABSINS). The second reason is the emerging of some difficulties of attitude determination based on gyros application in the case of navigation and control of the rapidly rotating moving vehicle (MV) and the large accelerations values [1; 3]. Hereafter we will consider the MV as a rigid body. It is clear that feasibility of ABSINS is based on the sensors redundancy, their allocation over the MV and directions of their sensitivity axes [2; 5, and 6]. From this point of view it is necessary to make special mention of results obtained in [5; 6], where it was proposed very successful method of 6 accelerometers allocation with corresponding directions of their axes over the faces of cube having center coinciding with the body frame origin. However, usage only 6 accelerometers doesn't allow correction of the results of integration of the angular acceleration without external aids. This paper shows that increasing of sensors redundancy from 6 to 12 gives possibility of updating of integration results for increasing the accuracy of the angular rates estimates. In other words, there is possibility of trade off between the sensors redundancy and the software performance. Numerical example demonstrates that it is especially effective in a case of MV, spinning with high angular rate, when usage of gyros might be very problematic. Practical implementation of ABSINS requires its fusion with multi-antenna Global Positioning System (GPS), which is able to estimate MV attitude. Such fusion was considered in [2] in details and the basic modern principles of GPS/INS architecture are specified in [10; 14; 15; 17; 18]; that is why this topic is omitted in this paper, which is devoted to attitude determination only. However it is necessary to mention that the basic relations in any ABSINS are equations allowing determination of the MV angular accelerations using linear accelerometers readouts. Numerical integration of these equations gives possibility to estimate MV angular velocities and then its attitude. For starting the integration procedure it is necessary to have initial values of angular velocities, which could be estimated by multi-antenna GPS in the integrated navigation system. It requires special procedure of the angular velocities estimation using the measurements of the linear velocities, which is considered below.

As far as ABSINS cannot be classified as high-precision system, for the sake of simplicity its algorithms are considered in suggestion that Coriolis acceleration can be neglected. However its including in the consideration is not the fundamental issue.

Determination of Angular Rates and Accelerations.

A. Determination of the angular rates.

The problem of determination of the MV angular rate vector and velocity vector of one of its point on the basis of observation of its three points velocities was considered by several authors (see [16; 20], where further references were cited). The problem is formulated in the following way (see fig. 1). Three vectors r_1, r_2, r_3 define points of MV, where the linear velocities are measured. Using results of these measurements, it is necessary to determine the angular rate vector ($\omega = [\omega_1\ \omega_2\ \omega_3]^T$) in the body frame and the linear velocity vector ($v_0 = [v_1\ v_2\ v_3]^T$) of the body frame origin O_1. There is known expression (see, for instance, (2.7.8) in [12], and (2) in [20]), which defines velocity of the *i*-th point of MV determined by vector r :

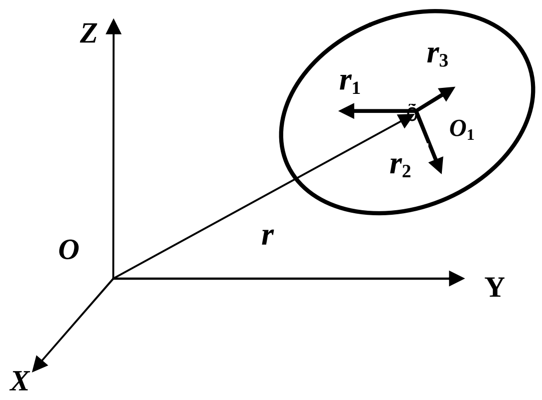

Fig. 1. Translational and rotational motions of rigid body

$$v_i = v_0 + \omega \times r_i;\ i = 1,\ 2,\ 3. \tag{1}$$

Using equations (6) in [20] and (4) in [16], it is possible to transform expression (1) in the following linear expression, which combines the searched components of the ω, v vectors and the results of the velocity observations in the given three points of rigid body:

$$V = \Omega P_V + v_0 h^T , \tag{2}$$

where

$\Omega = \omega\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$, $\mathrm{P}_V = [r_1\ r_2\ r_3]$, $h = [1\quad 1\quad 1]^T$, $\mathbf{V}$ is the matrix, which columns are velocity vectors in three points, determined by vectors $r_1,\ r_2,\ r_3$. Let $\beta_1,\ \beta_2,\ \beta_3$ and $\gamma_1,\ \gamma_2,\ \gamma_3$ are the columns of the P_V^T и V^T matrices:

$$\mathrm{P}_V^T = [\beta_1,\ \beta_2,\ \beta_3],\ \ \mathrm{V}^T = [\gamma_1,\ \gamma_2,\ \gamma_3].$$

In this case, expression (2) can be written as the system of linear equations with respect to ω, v_0:

$$A_v x = B, \tag{3}$$

where $x = \begin{bmatrix} \omega \\ v_0 \end{bmatrix}$, $A_v = \begin{bmatrix} o & \beta_3 & \beta_2 & h & o & o \\ -\beta_3 & o & \beta_1 & o & h & o \\ \beta_2 & -\beta_1 & o & o & o & h \end{bmatrix}$, $B = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}^T$, and symbol "$o$" stands hereafter for the (3×1) zero matrix. Taking into account that velocity measurements are corrupted with noises we rewrite (10) in the following form:

$$A_v x = B_0 + n_v, \tag{4}$$

where n_v is the measurement errors vector and vector $\mathbf{B}_0$ is formed from true values of velocities of considered points.

B. Determination of the angular accelerations by 6 sensors.

Likewise to the previous problem it is possible to consider the problem of determination of the MV angular acceleration vector and the linear acceleration of one of its points based on the results of observation of accelerations in three points of MV. Let three vectors ρ_1, ρ_2, ρ_3 define three points of the rigid body. In each of these points three accelerometers are located, which allow to record acceleration vector components in the given points. Using results of these measurements and the angular rate vector value ($\omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$) it is necessary to determine the angular acceleration vector $\varepsilon = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}^T = d\omega / dt$ and the linear acceleration vector $w_0 = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T$ of the body frame origin. Concerning considered problem, the expression (8) might be considered as an analog of the expression (2.17.9) in [12], which calculates the acceleration (w_i) of the rigid body's i-th point determined by vector ρ_i:

$$w_i = w_0 + \varepsilon \times \rho_i + \omega \times (\omega \times \rho_i);\ i = 1,\ 2,\ 3. \tag{5}$$

Denoting $U = \left[W_1 W_2 W_3 \right]$, where W_i are acceleration vectors of the points, which are determined by ρ_i $(i = 1,\ 2,\ 3)$, it is possible on the basis of (12) to write the analog of the expression (2):

$$U = \Omega^2 P_w + E P_w + w_0 h^T. \tag{6}$$

Here $P_w = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 \end{bmatrix}$, $E = \varepsilon \times = \begin{bmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$, and matrices Ω, h are similar to the matrices in the expression (2).

Likewise to (2) it is possible to represent expression (6) as the system of linear equations with respect to ε, w_0. Let $\alpha_1, \alpha_2, \alpha_3$; $\delta_1, \delta_2, \delta_3$; $\sigma_1, \sigma_2, \sigma_3$ are the columns of the matrices $\mathrm{U}^T, \mathrm{P}_w^T, (\Omega^2 \mathrm{P}_w)^T$, i. e.

$$U^T = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix},\quad P_w^T = \left[\delta_1 \delta_2 \delta_3 \right],\ (\Omega^2 P_w)^T = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}.$$

Then it is possible to write expression (13) in the form, which is similar to (10), namely

$$A_w x = B_\omega + B_w, \tag{7}$$

where $x = \begin{bmatrix} \varepsilon \\ w_0 \end{bmatrix}$, $A_w = \begin{bmatrix} o & \delta_3 & \delta_2 & h & o & o \\ -\delta_3 & o & \delta_1 & o & h & o \\ \delta_2 & -\delta_1 & o & o & o & h \end{bmatrix}$,

$\mathbf{B}_{\omega}^T = [\sigma_1 \quad \sigma_2 \quad \sigma_3]^T$, $\mathbf{B}_W^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3]^T$, and o is the 3×1-size zero matrix, as it was accepted in (3). Likewise to the expression (3), we suggest, that accelerometers outputs are corrupted with noises. Then (14) could be rewritten in the following form:

$$A_w x = B_{\omega} + B_{wo} + n_w, \qquad (8)$$

where n_w is vector of acceleration errors and components of vector $\mathbf{B}_{wo}$ are formed by true values of accelerations. As far as the size of matrix $\mathbf{A}_w$ in (8) is equal 9×6, it is possible to eliminate from consideration in system (8) three rows (to exclude the readouts of three accelerometers). In other words, it is possible to create the inertial navigation system using only 6 accelerometers. For illustration of this statement let us consider the scheme of allocation of accelerometers depicted in fig. 2. Here X_1, Y_1, Z_1 are points of the axes OX, OY, OZ, where couples of accelerometers are installed.

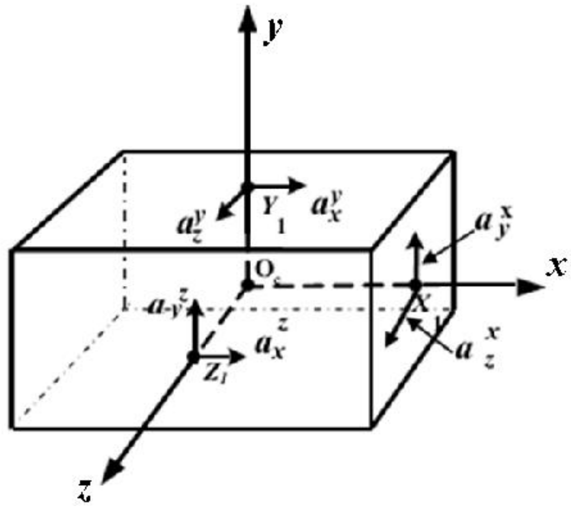

Fig. 2. The scheme of allocation of accelerometers in the rigid body

The orientation of their sensitivity axes is represented at this Figure. For example, a_x^y denotes, that this accelerometer measures the acceleration of the point Y_1 in the direction of the axis OX. Using this scheme of allocation of accelerometers, it is possible in the system (8) consisting of 9 equations to remain 6 equations, deleting the 1st, 5th and the 9th rows. Thus, if the angular velocity vector B_{ω} is known, then it is sufficient to install 6 accelerometers in order to determine vectors ε and w_0 (below Example 1).

Note that in [5, 6] another scheme of the accelerometers allocation is given, which allows determining directly the angular acceleration vector ε as a linear combination of the accelerometers readouts. However it is obvious, that the accuracy of the determining of the current value of the angular rate vector as a result of integration of the angular acceleration will significantly depend on the accuracy of definition of the initial value of the angular rate vector in the initial moment of time. For decreasing this dependence it is expedient to increase the amount of accelerometers and to use obtained redundant information for increasing the accuracy of estimation of the current value ω. That is why we will consider the cases of usage of 9 and 12 accelerometers.

C. Determination of the angular accelerations by 9 sensors.

Let system depicted in fig. 2 will be augmented with 3 accelerometers located in the point O. Their sensitivity axes are directed along the axes OX, OY, OZ respectively, so these accelerometers are measuring the components of acceleration vector of origin. The readouts of these accelerometers are denoted as a_x^0, a_y^0, a_z^0. Consider that the distance from the origin to each point X_1, Y_1, Z_1 is equal L. Denote also: $n_y^x = a_y^x - a_y^0, n_x^y = a_x^y - a_x^0$, $n_z^x = a_z^x - a_z^0$, $n_z^y = a_z^y - a_y^0$, $n_x^z = a_x^z - a_x^0$, $n_y^z = a_y^z - a_y^0$. If this scheme of the accelerometers allocation is used, it is possible to derive from (5) or (6) the following equations:

$$2L\varepsilon_1 = n_z^y - n_y^z,\ 2L\varepsilon_2 = n_x^z - n_z^x,$$
$$2L\varepsilon_3 = n_y^x - n_x^y, \tag{9}$$
$$2L\omega_2\omega_3 = n_z^y + n_y^z,\ 2L\omega_1\omega_3 = n_x^z + n_z^x,$$
$$2L\omega_1\omega_2 = n_y^x + n_x^y. \tag{10}$$

Note that equations (9), (10) coincide with equations (3.390) in [8]. So in a case of 9 accelerometers equations (10) determine additional 3 variables: $\omega_1\omega_2$, $\omega_1\omega_3$, $\omega_2\omega_3$. It is expedient to use this information for correction of the results of angular acceleration ε integration. Note also, that if 2 from 3 components of vector ω are equal to zero (the rotation takes place with respect to only one fixed axis), the relations (10) can't be used for correction of the integration results. From this point of view it is expedient to augment the 9-accelerometer measuring system, which is described above, with additional three accelerometers.

D. Determination of the angular accelerations by 12 sensors.

Additional 3 accelerometers are located in the following way: in the point X_1 the acceleration along axis OX is now measured, as well as in the points Y_1 and Z_1 the accelerations along axes OY and OZ respectively are measured also (see fig. 3). Note that scheme of allocation of accelerometers coincides with the scheme, shown in fig. 3.7 in [8]. Let the readouts of these accelerometers are equal to a_x^x, a_y^y, a_z^z. Denote

$$n_x^x = a_x^x - a_x^0,\ n_y^y = a_y^y - a_y^0,\ n_z^z = a_z^z - a_z^0.$$

For this 12-accelerometer measuring system relations (9), (10) must be augmented as follows:

$$2L\omega_1^2 = n_x^x - n_y^y - n_z^z,$$
$$2L\omega_2^2 = -n_x^x + n_y^y - n_z^z, \tag{11}$$
$$2L\omega_3^2 = -n_x^x - n_y^y + n_z^z.$$

So in the considered case of 12 accelerometers the relations (10), (11) could be used for correction of results of integration.

Implemetation of the angular acceleration estimates in ABSINS algorithms. Taking into account, that $w_0 = dv_0 / dt$ and $\varepsilon = d\omega / dt$, it is possible to consider system (8) as a system of nonlinear differential equations with respect to ω and $v_0(t)$. In other words, considering accelerometers signals (B_w) as known external input signals, it is possible to find $\omega(t)$ and $v_0(t)$ integrating system (8) under the given initial conditions. Thereby the considered approach allows obtaining information about MV angular rate without usage of the angular rate sensors. However it is necessary to take into account the following circumstances, when such ABSINS will be designed. Vector $\mathbf{x}$ appearing in (8) is given in the moving frame. So far as we are interesting in MV position determination in the inertial frame, it is expedient to map the second component of the vector x (vector w_0) in the inertial frame and to perform further integration in this frame, thus allowing to determine in the moving frame velocity and the coordinates of the MV point, which is accepted as the origin of the moving frame. Meanwhile the first part of the vector $\mathbf{x}$ (vector $\boldsymbol{\omega}$) can be used for the determination of the current attitude of MV, which is defined by classical Euler angles ψ, ϑ, φ (precession, nutation and rotation) [12; 13], which determine the attitude of the rigid body, i. e. its transition from initial position in the fixed frame $Oxyz$ to the final position in the moving (body) frame $Ox'y'z'$. Other definition of the Euler angles ψ, ϑ, φ (yaw, pitch, roll) is also possible [10 – 12], but it is not used in this paper. The MV attitude with respect to the inertial frame can be determined by quaternion vector $\lambda = \begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}^T$ or Rodriguez–Hamilton parameters as well as by direction cosine matrix (DCM). The relations between them are very well known [9 – 12]. During the estimation of the angular rate vector by integrating (8) with known rigid body initial attitude given by the value of initial quaternion $\lambda(0)$, we can determine current Rodriguez-Hamilton parameters integrating following known equation (rotational mechanization) [9 – 12]:

$$\dot{\lambda} = 0,5 \cdot \Omega \cdot \lambda, \tag{12}$$

where $\Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}$, $\|\lambda\|^2 = \lambda^T \lambda = 1$.

Then the values of matrix A entries, which are used for mapping of vector w_0 into inertial frame, can be determined using known relation between Rodriguez–Hamilton parameters (quaternion $\lambda(t)$) and DCM $A(\lambda(t))$. After that it is possible to determine current values of MV velocity and position in the inertial frame.

If Euler angles are small and trihedrons $Oxyz$ and $Ox'y'z'$ are very close to each other, then is possible to use approximate expression for DCM A based on relation (26) in [21]:

$$A \cong \begin{bmatrix} 1 & \mu_3 & -\mu_2 \\ -\mu_3 & 1 & \mu_1 \\ \mu_2 & -\mu_1 & 1 \end{bmatrix}, \tag{13}$$

where μ_1, μ_2, μ_3 are small angles of turn of the trihedron $Oxyz$ with respect to the axes x, y, z respectively. So the implementation of such kind of inertial system includes:

– finding $\omega(t)$ by integrating 3 differential equations (the first three equations of system (8));

– finding quaternion λ, which in turn determines DCM A allowing to map acceleration vector $\mathbf{w}_0$ in inertial frame, by integration of system of 4 equations (12);

– determination of the MV velocity and position by integration of known 6 equations of translational mechanization.

In other words, it is necessary to integrate the 13-th order system of the nonlinear differential equations. Initial conditions for this system would be the values in the initial moment of time of the following variables: MV position ($r_0 = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T$), MV initial attitude: quaternion $\bar{\lambda}_0$ or DCM $A(\bar{\lambda}_0)$, velocity corresponding DCM vector ($\bar{v}_0 = \begin{bmatrix} v_{x0} & v_{y0} & v_{z0} \end{bmatrix}^T$), and initial MV angular rate vector ($\omega_0 = \begin{bmatrix} \omega_{x0} & \omega_{y0} & \omega_{z0} \end{bmatrix}^T$). From the point of view of practical implementation of such SINS it is expedient to consider the “time sampling” of this system, when the acquisition of the readouts of sensors is made with equidistant time periods Δt or with the sampling frequency $f = 1/\Delta t$. Correspondingly, the following navigation parameters: DCM $A(\lambda(t))$, velocity $\boldsymbol{v}$, and position $\boldsymbol{r}$, which are sought for, can be computed after sampling time $\Delta \boldsymbol{t}$. As far as different "time sampling" procedures may be used, we will consider each of them separately. We begin with determination of quaternion estimates in the sampling moments $t_i, t_i - t_{i-1} = \Delta t$, $i = 1, 2, 3, \ldots$ [9; 19]. So let the quasi-coordinates (components of vector $\nabla\theta_i = \int_{t_i}^{t_i+\Delta t} \omega\, dt$) are known on the time period $\Delta \boldsymbol{t}$. Expressing the elementary quaternion (or quaternion increment) $\delta\lambda(t_i)$ as function of this quasi-coordinates (i. e. computing quaternion corresponding to the small turn of the rigid body during time period Δt), we can determine the solution of the equation (12) on the time period $\Delta \boldsymbol{t}$ with initial condition $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$ as consequent multiplications of the quaternion $\lambda(t_{i-1})$ by elementary quaternion $\delta\lambda(t_i)$ [9; 12; 13; 19]:

$$\delta\lambda(t_i) = \begin{bmatrix} 1 - \|\nabla\theta_i\|^2 / 12 \\ 0,5 \cdot \nabla\theta_i - (\nabla\theta_i \times \nabla\theta_{i-1}) / 24 \end{bmatrix}. \tag{14}$$

Likewise to [19], for computing $\nabla\theta_i$ it is possible to use the quadratic spline approximation of the angular rate vector $\omega(t)$. So if the values of $\omega(t_{i-2})$, $\omega(t_{i-1})$, $\omega(t_i)$ are known, then:

$$\nabla\theta_i = \Delta t \cdot (5\omega(t_i) + 8\omega(t_{i-1}) - \omega(t_{i-2})) / 12, \tag{15}$$

$$\omega(t_1) = \omega(t_{i-1}) + \varepsilon(t_1)\Delta t\,. \tag{16}$$

In expression (16) $\varepsilon(t_i)$ stands for angular acceleration vector, which is defined by expression (7) using the accelerometer readouts in the moments t_i. Components of vector $\omega(t_{i-1})$ are used for computing the entries of matrix Ω in expression (6). Having the elementary quaternion estimate $\delta\lambda(t_i)$ from (14), we can find at first quaternion $\lambda(t_i)$ using (12), (13) and then matrix $\mathbf{A}(\lambda(t_i))$. Mapping vector $\mathbf{w}_0(t_i)$ determined by (7) to the inertial frame using matrix $\mathbf{A}(\lambda(t_i))$, we can find the estimates of velocity $(v(t_i))$ and position $(r(t_i))$ of MV [19]. So far as the last procedures are known, we can restrict ourselves by estimation of the accuracy of matrix $\mathbf{A}(\lambda)$, which is the result of execution of rotational mechanization procedure.

Increasing Accuracy of Angular Velocity Estimation. Consider briefly the problem of relations (10) usage for increasing of accuracy of determination ω for case of 9 accelerometers and similar problem in a case of 12 accelerometers. In the last case besides aforementioned relations (10) expressions (11) are used as well. Consider 9-accelerometer case, which readouts determine the angular acceleration vector ε from expressions (9), as well as vector $\Omega_n = \begin{bmatrix}\omega_2\omega_3 & \omega_1\omega_3 & \omega_2\omega_1\end{bmatrix}^T$ from expressions (10). Let current value ω is determined by expression (16), which can be rewritten as follows $\omega(t_1) = \omega(t_{i-1}) + \Delta\omega_i$. Then estimation of the vector $\omega(t_{i-1})$ increment, obtained as a result of measurements ε, could be determined by relation $\Delta\overline{\omega}_i = 0,5(\varepsilon(t_i) + \varepsilon(t_{i-1}))\Delta t$. From the other hand, considering $\Delta\overline{\omega}_i$ as a small value, it is possible to write the following relations:

$$\Omega_n = H\Delta\omega_i + \Omega_{n0},\ H = \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & \omega_1 \\ \omega_2 & \omega_1 & 0 \end{bmatrix},$$

$$\Omega_{n0} = \begin{bmatrix}\omega_2\omega_3 & \omega_1\omega_3 & \omega_1\omega_2\end{bmatrix}^T. \tag{17}$$

Components of vector $\mathbf{\Omega}_n$ and the components of vector ω, which determine H and Ω_{n0}, correspond to the values of vector $\omega(t_{i-1})$ components are determined by (17). In other words, as a result of previously made assumption about small value of $\Delta\omega_i$, we have the standard problem of the parameter estimation by weighted least square method [22]. In accordance with (24) the following vector $\mathbf{z}$ is observed:

$$z = \Omega_n - \Omega_{n0} = H\Delta\omega + \nu, \tag{18}$$

where $\mathbf{v}$ is the vector of the measurement errors. The estimation of the $\Delta\hat{\omega}_i$ value is determined by relation (12.2.7) from [22]:

$$\Delta\hat{\omega}_i = \Delta\bar{\omega}_i + PH^T R^{-1}(z - H\Delta\bar{\omega}_i), \tag{19}$$

$$P^{-1} = M^{-1} + H^T R^{-1} H.$$

Here M is the covariance matrix of the estimation errors of $\Delta\hat{\omega}_i$, and R is covariance matrix of measurement errors ν in (18). Finally, the value of vector ω in the moment t_i is determined by relation:

$$\omega(t_i) = \omega(t_{i-1}) + \Delta\hat{\omega}_i, \tag{20}$$

where $\Delta\hat{\omega}_i$ can be found from (19).

Note, that matrix $\mathbf{P}^{-1}$ might be ill-conditioned. Then for determination of matrix $\mathbf{P}$ in (23) it could be expedient to use the approach, described in [17; 19]. As far as the matrices $\mathbf{M}, \mathbf{R}$ are symmetric and positively defined, they can be represented in the following form of Cholesky factorization:

$\mathrm{M} = m^2$, $\mathrm{R} = r^2$, or $m = \mathrm{M}^{1/2}$, $r = \mathrm{R}^{1/2}$. Correspondingly it is possible to represent the expressions for matrix $\mathbf{P}^{-1}$ as follows:

$$\mathrm{P}^{-1} = \begin{bmatrix} m^{-1} & H^T r^{-1} \end{bmatrix} \begin{bmatrix} m^{-1} & H^T r^{-1} \end{bmatrix}^T. \tag{21}$$

Using QR-factorization procedure, we can transform matrix $\begin{bmatrix} m^{-1} & H^T r^{-1} \end{bmatrix}^T$ to the following form:

$$\begin{bmatrix} m^{-1} & H^T r^{-1} \end{bmatrix}^T = Q\begin{bmatrix} \rho & 0 \end{bmatrix}^T, \tag{22}$$

where Q is orthogonal matrix, and ρ is the invertible matrix. Taking into account, that $Q^T Q = I$ and substituting (29) in (28), we obtain:

$$P^{-1} = \rho^T \rho, \text{ or } P = \rho^{-1}\rho^{-T}.$$

Thus the expression (26) can be represented in the following form:

$$\Delta\hat{\omega}_i = \Delta\bar{\omega}_i + \rho^{-1}\rho^{-T} H^T R^{-1}(z - H\Delta\bar{\omega}_i). \tag{23}$$

If we suppose, that $\mathrm{M} = \mu^2 \mathrm{I}$, $\mathrm{R} = \gamma^2 \mathrm{I}$, then the relation (23) can be written as:

$$\Delta\hat{\omega}_i = \Delta\bar{\omega}_i + \rho^{-1}\rho^{-T} H^T (z - H\Delta\bar{\omega}_i), \tag{24}$$

where ρ is determined by QR-factorization of the following matrix:

$$\begin{bmatrix} \Lambda I & H^T \end{bmatrix}^T, \quad \Lambda = \gamma / \mu. \tag{25}$$

Note, that described algorithm of correction can be used in a case of 12 accelerometers. In this case the matrix H and the vector Ω_{n0} in (24) have the following forms:

$$\mathrm{H}=\begin{bmatrix} 0 & \omega_3 & \omega_2 & 2\omega_1 & 0 & 0 \\ \omega_3 & 0 & \omega_1 & 0 & 2\omega_2 & 0 \\ \omega_2 & \omega_1 & 0 & 0 & 0 & 2\omega_3 \end{bmatrix}^T,$$

$$\Omega_{n0}=\begin{bmatrix} \omega_2\omega_3 & \omega_1\omega_3 & \omega_1\omega_2 & \omega_1^2 & \omega_2^2 & \omega_3^2 \end{bmatrix}^T.$$

Here, likewise to the 9 accelerometers case, entries of H and Ω_{n0} are determined by components of vector $\omega(t_{i-1})$. Components of vector Ω_n are determined by expressions (10), (11).

Examples.

Example 1, [2]. Consider the following attitude determination problem in order to illustrate described above algorithms. Let frame $Oxyz$ is fixed to the Earth surface. In this frame the vehicle associated with frame $Ox'y'z'$ is moving along the circle in the xy -plane with velocity $\bar{v}=30\ \mathrm{m/sec}$ and the period $T=60\ \mathrm{sec}$. In the motion process its attitude (in the system $Ox'y'z'$) is determined by the following Euler angles as the functions of time (t): $\psi=\dot{\psi}t,\ \dot{\psi}=2\pi/T,$ and $\varphi=0,\ \vartheta=0$. In accordance with known expressions for components of the angular rate vector in the moving frame [12; 13; 19]:

$$\begin{aligned} \omega_1 &= \dot{\psi}\ \sin\vartheta\ \sin\varphi+\dot{\vartheta}\cos\varphi, \\ \omega_2 &= \dot{\psi}\ \sin\vartheta\ \cos\varphi-\dot{\vartheta}\sin\varphi, \\ \omega_3 &= \dot{\psi}\ \cos\vartheta+\dot{\varphi}, \end{aligned} \tag{26}$$

these components will be equal to the following values: $\omega_1=\omega_2=0,\ \omega_3=\dot{\psi}$. We suppose zero initial conditions of attitude: $\psi(0)=\varphi=\vartheta=0.$ In accordance with expressions for quaternion components $\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ as function of classical Euler angles [9 – 12]:

$$\begin{aligned} \lambda_0 &= \cos(\vartheta/2)\cdot\cos((\varphi+\psi)/2), \\ \lambda_1 &= \sin(\vartheta/2)\cdot\cos((\varphi-\psi)/2), \\ \lambda_2 &= \sin(\vartheta/2)\cdot\sin((\varphi-\psi)/2), \\ \lambda_3 &= \cos(\vartheta/2)\cdot\sin((\varphi+\psi)/2), \end{aligned} \tag{27}$$

the initial MV attitude is determined by quaternion $\lambda(0)=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$. Direction cosine matrix can be expressed via Rodriguez–Hamilton parameters $\lambda_0,\lambda_1,\lambda_2,\lambda_3$ [9 – 12]:

$$A(\lambda)=\begin{bmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3+\lambda_0\lambda_2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{bmatrix}. \tag{28}$$

In accordance with expression (28) DCM would be the unit matrix. Due to (27) the current time-depended true value of quaternion determining true attitude of MV equals:

$$\bar{\lambda}(t)=\begin{bmatrix}\cos(\pi t/T) & 0 & 0 & \sin(\pi t/T)\end{bmatrix}^T. \tag{29}$$

Substituting (29) in expression (28) we obtain true value of corresponding DCM $\bar{A}(\bar{\lambda})$. At the moment $t=0$ the MV is located on the y axis at the distance $R_0=T\bar{v}/2\pi$ from the origin. Its velocity vector $\mathbf{v}_0$ in (2) and acceleration vector $\mathbf{w}_0$ in (13) are determined by the following expressions:

$$\mathbf{v}_0 = \left[-\bar{v}\ \ 0\ \ 0\right]^T,\ \mathbf{w}_0 = \left[0\ \ -\bar{v}^2/R_0\ -g\right]^T. \tag{30}$$

Here $g = -9,81\ \text{m}/\sec^2$ is the gravity acceleration. Six accelerometers are installed along the trihedron axes, as it is represented in fig. 2, and the distance $L = OX_1 = OY_1 = OZ_1$ is accepted as 0,1 m. Accelerometers errors (n_w in (8)) are simulated with uniformly distributed uncorrelated random numbers having zero mean values and variance $\sigma_w = 10^{-3}\ \text{m}/\sec^2$.

So accepted initial conditions and suggestions about accelerometers errors allow performing simulation of the considered SINS operation without the rate gyros. However, in order to illustrate the algorithm described in the item B, we will consider situation, when initial values of the vehicle's velocities (angular and linear ones) are determined with the aids of GPS by the data processing of the MV velocity measurement results in its three points, when $t = 0$. For this case we suppose that the matrix P_V in (2) has the following form:

$$P_V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

The same assumptions could be done for matrix P_W in (6). This case corresponds to the location of sensors in one plane. Note, that so far as matrices P_V^{-1} and P_W^{-1} don't exist, then it is impossible to use the algorithm [20]. The measurements errors (components of vector n_v in (11)) are supposed to be uncorrelated uniformly distributed random numbers with zero mean values and variance $\sigma_v = 0,1\,\text{m}/\sec$. Using these initial data and the algorithm of item A, the following estimates were obtained (note that values of the angular rates are given in rad/sec):

$$\omega(0) = \left[-10^{-3}\quad 6,4\cdot 10^{-3}\quad 0,1024\right]^T,\ v_0(0) = \left[-29,9955\quad -3,4\cdot 10^{-3}\quad 0,0101\right]^T. \tag{32}$$

These estimates were accepted as corresponding initial conditions during the SINS simulation. True values of the rest of parameters, namely parameters determining the MV initial position and attitude, were accepted as initial values, when $t = 0$. The simulation of the system operation during 15 sec was done with usage the Matlab procedures, namely: the procedure "ode 45.m" was applied for systems of differential equations integration, and for the random number generation the procedure "rand.m" was used. The simulation results are represented in fig. 3, where errors of the attitude determination $\mu_x,\ \mu_y,\ \mu_z$ (measured in degrees) are represented. They are upper diagonal entries of the matrix $\bar{A}^T(\bar{\lambda})A(\lambda)$ approximated with expression (13):

$$\bar{A}^T(\bar{\lambda})A(\lambda) \cong \begin{bmatrix} 1 & \mu_z & -\mu_y \\ -\mu_z & 1 & \mu_x \\ \mu_y & -\mu_x & 1 \end{bmatrix}.$$

Here λ stands for quaternion obtained via integration, meanwhile $\bar{\lambda}$ stands for its true value, defined by expression (29). $\bar{A}(\bar{\lambda})$, and $A(\lambda(t))$ stand respectively for true DCM value, defined by quaternion (29), and its evaluation, obtained as a result of integration. In fig. 3 and in the next figures the solid line corresponds to μ_x, the dash-line corresponds to μ_y, and the dot-and-dash line corresponds to μ_z. Note that results represented in the fig. 3 correspond to the results represented in fig. 9 – 11 in [2].

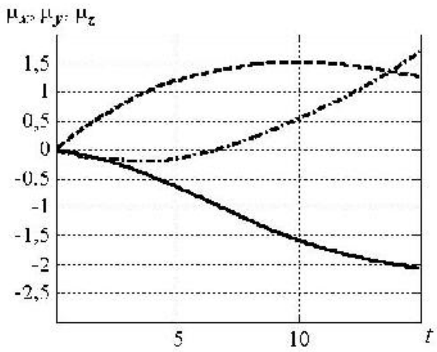

Fig. 3. Errors of attitude determination of system with 6 accelerometers

Example 2. 9 accelerometers are installed on the MV in a following way: six accelerometers are installed in accordance with fig. 2 and 3 additional accelerometers are installed in the origin of the body frame as it was described in the item C. Position of the origin of the trihedron associated with the moving frame is determined by vector $\mathrm{R} = [0\ 1\ 0]^{\mathrm{T}}$. The attitude of the moving frame is given by the following initial values of the Euler angles (in radians) and their time-dependent angular rates (in rad/sec): $\psi(0) = 0,\ \theta(0) = \dfrac{\pi}{4},\ \varphi(0) = 0,\ \ \dot\psi = 1,\ \dot\theta = 0,\ \dot\varphi = 10$. Components of the angular rate vector in the moving frame are determined by (26). Note that in this example the module of the angular velocity vector equals ≈ 615 deg/sec; meanwhile in the Example 1 it was equal to ≈5,87 deg/sec. That is why in the considered example the sampling time Δt was accepted as $\Delta t = 10^{-3}$ sec. The initial attitude (quaternion) is determined by relations (1). Likewise to the example 1, it was accepted, that $\sigma_w = 10^{-3}$ m/sec^2, $L = 0{,}1$m. Initial error of the angular rate (likewise to (32)) was simulated as follows. As the initial value of $(\tilde\omega(0))$ it was accepted the following magnitude:

$$\tilde\omega(0) = 0{,}5 \cdot \omega(0)\,, \tag{33}$$

where $\omega(0)$ is the true value, determined by the expression (5). It is accepted, that $\Lambda = 0{,}1$ in the expression (32). Estimations of errors of definition of the kinematical parameters of motion obtained via motion simulation during 15 sec are shown in fig. 4 and 5. In fig. 4 the magnitudes of the value $\mathrm{dom}\,(t_k)$ having dimension deg/sec:

$$\mathrm{dom}(t_k) = \left(\sum_{i=1}^{k} \left\| \omega(t_i) - \tilde\omega(t_i) \right\| \right) \Big/ k$$

are given. The value $<\mathrm{dom}>$ characterizes the accuracy of the estimation of the angular rate vector current value, which was obtained in accordance with (20).

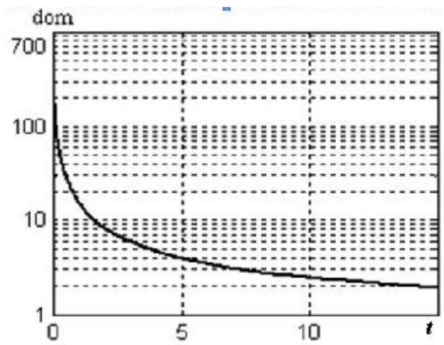

Fig. 4. Accuracy of estimation of the angular rate (9 sensors)

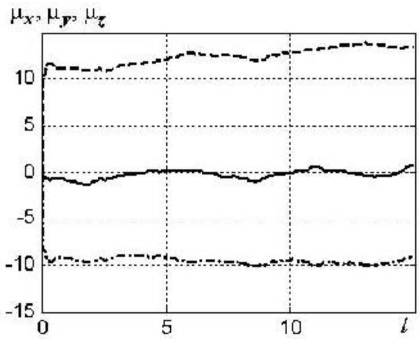

Fig. 5. Errors of the attitude determination (9 sensors)

In fig. 5 the values of estimations of the attitude determination are given, at that types of lines coincide with those, which are accepted in fig. 3. Evaluating obtained results, it is possible to state, that usage of the algorithm of correction (20) allows increasing significantly the accuracy of the determination of the current value ω, and, as a consequence, to increase the accuracy of the attitude determination. Really, in accordance with fig. 4 the value of angular rate ω error (dom) after 15 sec has order 2 deg/sec, meanwhile in accordance with (33) the error of determination of the initial value of ω has order 300 deg/sec. As a result of this fact it is possible to state, that the accuracy of the attitude determination is increasing (see fig. 5). Thus in the initial period different increasing of the attitude error is observed, which is explained by coarse initial value estimation of ω. However, after further decreasing of the current value of error of the ω determination it is possible to state, that errors of the attitude determination are not changed essentially. It is relevant to state that for such large value of the initial angular rate error (300 deg/sec) previous system in Example 1 is inoperable. It is necessary to underline also that in this case, when the error of the initial angular rate determination has order 300 deg/sec, the error of the MV attitude determination has order 10 degrees after 15 sec.

Example 3. Consider the system having 12 accelerometers, which is described in the item *D*. Here the initial data of Example 2 (parameters of motion, the errors of the initial values, etc.) are preserved besides the values of σ_w and Λ only. In this case it is accepted that $\sigma_w = 10^{-1}$ m/sec, $\Lambda = 700$, i. e. the accuracy of accelerometers is deteriorated in 100 times. The simulation results are given in fig. 6 and 7 (denotations coincide with those accepted in fig. 4 and 5). These results demonstrate essentially higher accuracy of the measurement system consisting of 12 accelerometers.

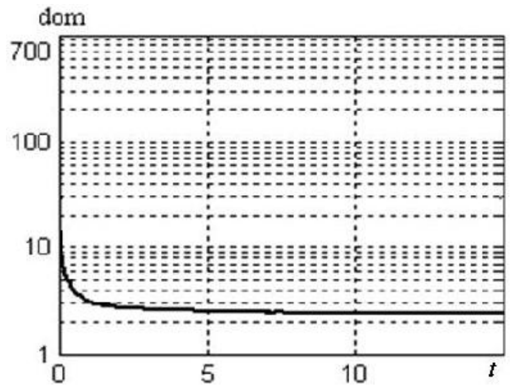

Fig. 6. Accuracy of angular rate estimation (12 sensors)

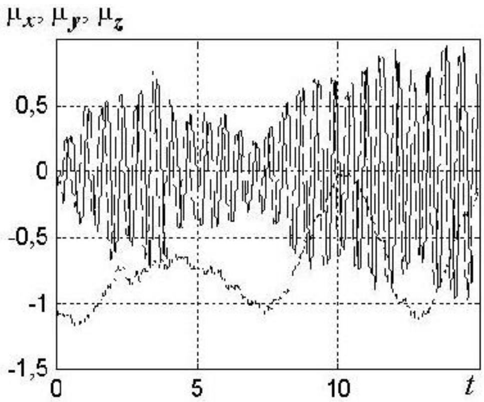

Fig. 7. Errors of the attitude determination (12 sensors)

Thus in the Figure 6 the value of error of the current value ω determination has practically the same order as in Example 2 despite of increasing the errors of acceleration measurements in 100 times. It is possible to state that in this system the decreasing of the error of initial value of the angular rate ω is essentially faster than in the previous case. As a conclusion from this circumstance, the errors of the attitude determination decrease significantly (compare fig. 7 with fig. 5).

Example 4. Now we will return back to Example 1 preserving all initial data of this example (parameters of motion, the accuracy of accelerometers, etc.), however considering that system contains not 6 but 12 accelerometers likewise to Example 3. We accept that value μ in (25) equals $\mu = 7$. Results of simulation are represented in fig. 8.

Comparing them with results represented in fig. 3 it is possible to state that the same accuracy of attitude determination can be achieved, but as opposed to Example 1 in this case the procedure of integrating of system of nonlinear differential equations (“ode 45.m”) was not used. Instead of this, much simpler finite difference scheme (16) was applied.

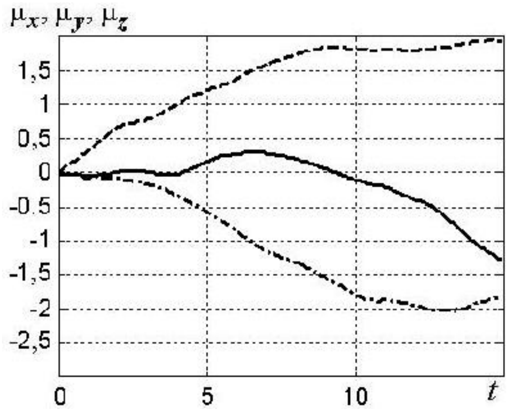

Fig. 8. Errors of the attitude determination (12 sensors under conditions of Example 1)

Conclusion. The main algorithms of operation of the accelerometer-based autonomous inertial navigation systems, which don't use the gyros, are given for cases of application of 6, 9, and 12 accelerometers. In the first case usage of 6 accelerometers is quite enough for measurement of the angular acceleration of the rigid body, which then is used for the attitude determination. However in this case it is impossible to correct results of integration of the estimated angular accelerations. Meanwhile the usage of 9 accelerometers can increase the accuracy of the attitude determination due to additional procedure of correction of the aforementioned integration results. Further increasing the amount of sensors up to 12 allows increasing the accuracy of the attitude determination significantly. So the trade-off between the redundancy of sensors and the accuracy of attitude determination can be achieved.

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Алгоритми супутникової інерціальної навігаційної системи на основі акселерометрів без використання гіроскопів

Основна проблема, що виникає при створенні акселерометричних безплатформенних інерціальних навігаційних систем, є перетворення показань надлишкових лінійних акселерометрів в оцінку кутових прискорень, кутових швидкостей та вирішення, таким чином, завдання кутової орієнтації. Досліджено співвідношення між кількістю надлишкових датчиків і якістю (точністю) алгоритмів оброблення сигналів. Показано, що збільшення кількості датчиків від мінімальної величини шість до дванадцяти дає додаткові можливості для корекції результатів оцінювання та збільшення їх точності. Моделювання цих систем показує ефективність оцінювання особливо для випадку рухомих об'єктів, що обертаються з великою кутовою швидкістю, коли традиційне використання гіроскопів може бути досить проблематичним.

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Алгоритмы спутниковой инерциальной навигационной системы на основе акселерометров без использования гироскопов

Основная проблема, возникающая при создании акселерометрических бесплатформенных инерциальных навигационных систем, является преобразование показаний избыточных линейных акселерометров в оценку угловых ускорений, угловых скоростей и решение, таким образом, задачи угловой ориентации. Исследовано соотношение между количеством избыточных датчиков и качеством (точностью) алгоритмов обработки сигналов. Показано, что увеличение количества датчиков от минимальной величины 6 до 12 даёт дополнительные возможности для коррекции результатов оценивания и увеличения его точности. Моделирование этих систем показывает эффективность оценивания в особенности для случая подвижных объектов, вращающихся с большой угловой скоростью, когда традиционное использование гироскопов может быть весьма проблематичным.