MATHEMATICAL MODELING OF PROCESSES AND SYSTEMS

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[1]**M. Ye. Fryz,** Ph. D.,
[2]**M. A. Stadnyk**, post-graduate student

JUSTIFICATION OF THE MATHEMATICAL MODEL OF STEADY-STATE VISUAL EVOKED POTENTIALS AS A LINEAR RANDOM PROCESS

Ternopil National Technical University Ivan Pul'uj, e-mail: [1]m_fryz@ukr.net,
[2]maria.stadnyk@gmail.com

Abstrakt. *The mathematical model of steady-state visual evoked potential as a linear stochastic process is justified, which includes biophysical peculiarities of the potential generation. The mechanism of visual evoked potential creation by separate neurons of the main human brain is described. The expression of characteristic function of proposed mathematical model is presented.*

Keywords: visual analyzer, visual evoked potential, impulse, mathematical model, linear random process, action potential, excitatory postsynaptic potential, inhibitory postsynaptic potential, characteristic function.

Introduction. Taking into consideration the present ecological conditions and significant harmful influence of computer work, the problem of qualitative estimation of visual analyzer state is particularly substantial. At present time, in order to solve the above specified problem ophthalmologists use the latest methods of diagnosis, namely electroretinography, electro-oculography, the registration of visual evoked potentials (VEPs). The last one allows us estimate not only the work of the central or peripheral circuit, but also the visual system in total. VEPs are a particular case of the electroencephalogram signal (EEGs), namely the reaction of the visual cortex of the brain to external stimuli (photic stimulus, spatially structured stimuli). Depending on the frequency of stimulation VEPs are divided into transient (1 – 4 Hz) and steady-state VEP (5 – 30 Hz).

Figure 1 illustrates examples of transient and steady-state VEPs realizations, which are recorded from surface of human scalp over visual cortex projections.

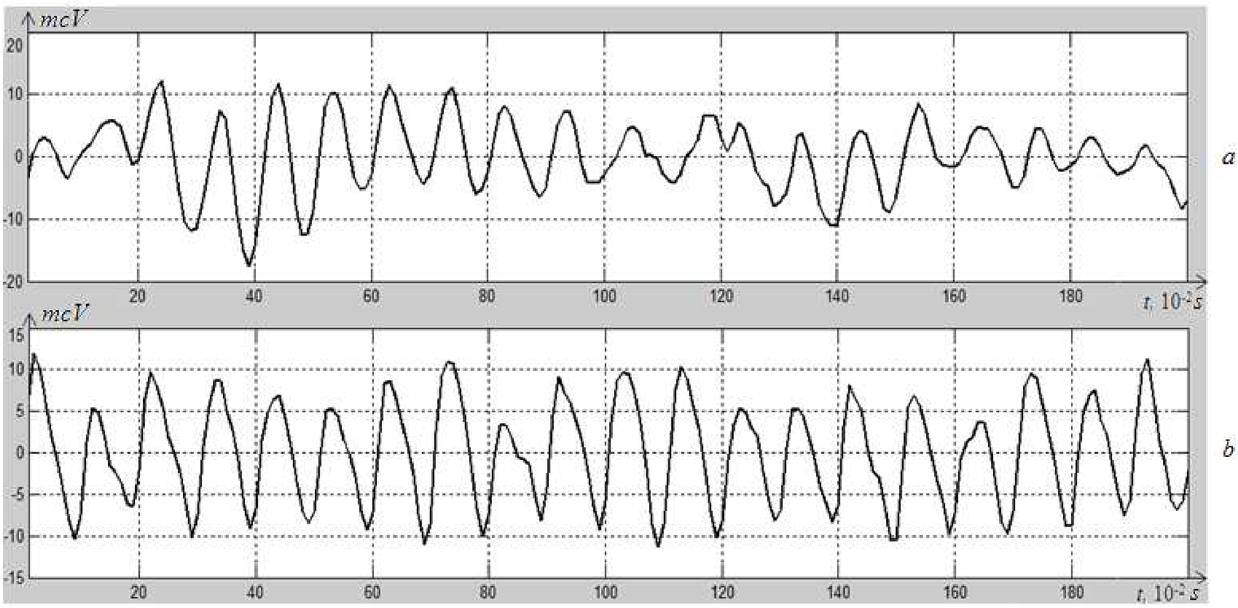

Fig. 1. Realization of transient (*a*, stimulation frequency 2 Hz) and steady-state VEPs (*b*, stimulation frequency 10 Hz)

Modern information-measuring systems (DX-NT, OKYLYAR, Neuron-Spectr, Sierra, Neurofax) mainly record transient VEPs and use amplitudes of its positive and negative components, time of its occurrence (latency). However, steady-state VEPs can allow us to estimate not only the state of visual analyzer, but also the ability of visual departments to long-term action, possibility for respond to high-frequency stimuli [1].

Taking into account the above-mentioned considerations, actual scientific-technical problem is construction of information-measuring system for ophthalmic diagnostics by steady-state VEP. It leads to solve the following tasks: the creation of new software modules for recording steady-state VEPs, the determination of the signal processing methods for this class of signals, finding correlation between visual pathology and informative-diagnostic parameters, the construction of the decision rules. However, the first and fundamental step is adequate mathematical model validation. On the one hand, this model must be based on the biophysical nature of VEP's generating process; on the other hand, informative features for measuring tasks will be defined by this mathematical model.

Analysis of the last publications and research. Additive model is often used to describe the VEP signal. It includes deterministic function representing the VEP, and noise is represented by the second component – centered weakly stationary random process (background EEG). This model is used to perform averaging on the set of post-stimulation signal realizations [2 – 4]. Additive model allows us to define the first moment function as a diagnostic characteristics, but it is insufficient for a detailed description of the signal needs in modern diagnostics.

In paper [5] researchers use GARCH model (generalized autoregressive conditional heteroscedasticity), which is a suitable tool for modeling stochastic processes with rapid changes in spectral properties. It should be also noted that this model does not take into consideration cyclicity of stimuli, and therefore it is inappropriate to use for steady-state VEPs modeling.

The resulting VEP signal can also be presented by a component model [6]. Its components are the processes generated by individual sources of brain electrical activity. Independent component analysis (ICA) and principal component analysis (PCA) are used for the analysis and separation of constituent individual components. When the combined source signals can be assumed to be independent from each other, this concept plays a crucial role in separation and denoising the signals [6]. One of the conditions to use the ICA is a statistical independence of components, unlike PCA which requires absence of nonlinear correlation between them. Component model allows us to retrace operation of the resulting signal forming and to distinguish different sources of generating components. However, correct usage of separate components needs different models and corresponding methods in diagnostics. Taking into account the aforesaid, component model requires significant expenditure of time, and computational complexity will be increase.

In paper [7] the artificial neural network model is used for the analysis and classification tasks to VEP signal. In spite of all its advantages, it disregards the stochastic nature of the VEP signal and generating mechanism of the human brain electrical activity by individual neurons; the learning process can be very long to find the optimal weight coefficient; the distribution of the input data is not known.

Whereas, taking into account some cyclical (rhythmic) properties of the steady-state visual evokes potential realization, the author [8] uses for investigated signal describing stochastic periodic random process, the probabilistic characteristics of that are periodic functions of the time. Such property accounts the cyclicity of photo stimulation, but it has not biophysical background (not taking into accounts the resulting signal generation process by means of individual neurons).

The linear random process (LRP) is also used to describe the VEP signal. In paper [4] the registered signal from the surface of the human scalp is represented as the summation of a linear piecewise stationary random stochastic process (spontaneous EEGs) and deterministic function (VEPs). The authors [4] investigate the received signal realization only on stationary EEG's intervals that are equal to the time of registration VEP signal (250 ms). The process of electrical activity generating is not stationary, and therefore within the considered model, the stationary intervals (its magnitude depends on the conditions of the experiment and the physical state of the person) of the received signal will also be justified and experimentally determined.

Problem statement. Actual and primary goal of this paper is creation and justification an adequate mathematical model of steady VEP signal, which will allow us:

– to describe the mechanism of signal generation by individual cortical neurons and their spreading in the extracellular environment;

– to reflect the stochastic nature of the VEP signal;

– to determine the informative and diagnostic parameters that will be based on the biophysical nature of the investigated process by appropriate methods and procedures;

– to take into account the cyclical properties of the VEPs due rhythmicity of photic stimulation;

– to be suitable for use in medical diagnostics, computer simulation modeling.

Analysis of the generation mechanism of the VEPs. Every neuron consists of dendrites, soma and axon. Within the membrane theory [7] nerve cell is presented as a separate biological environment which contains the ions K^+, Cl^-, Ca^{++} in the cytoplasm, and Na^+ externally. If inside the cell the ions Na^+ come up to certain concentration value and impulse is obtained from neighboring neurons, the membrane will be depolarized (becomes more positive) and generate some short answer-pulse that called the action potential (AP). Electrical activity of the central nervous system (CNS) is produced by synapses (excitatory, inhibitory) between dendrites of neighboring cells or dendrites and axons [7; 9]. The AP is transmitted from the initiation point to the synapse along nerve fibers by changing the potential of the neighboring areas. Depending on the type of synapses in dendrites there are excitatory or inhibitory postsynaptic potentials (EPSP, IPSP). These potentials have a considerable longer duration and lower amplitude than the AP (fig. 2) and also are the reason for the continuous oscillations of intracellular potential. If the last one comes up to certain threshold value the neuron will generate the next new potential.

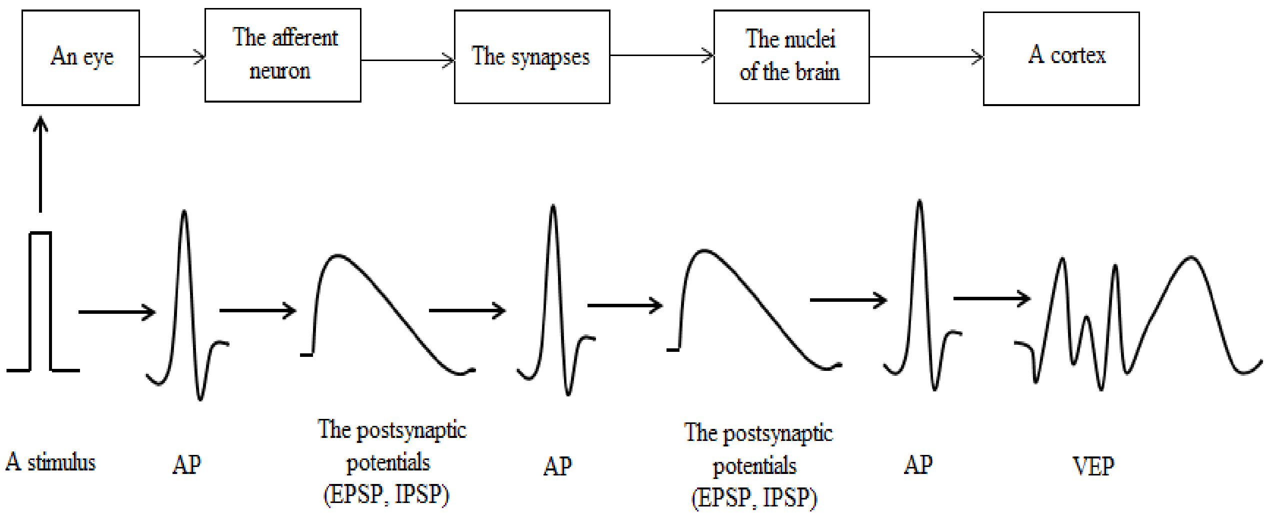

Fig. 2. Scheme of the VEPs generation mechanism by main elements of the CNS [2]

The generation mechanism of VEP signal is presented schematically in fig. 2. After stimulation the afferent neurons produce the action potential that is transmitting into the neighboring neurons and activates IPSP, EPSP [9]. The VEP's origin is the visual parts of the brain that are constructed by large number of neurons that receive stimulation from each other. Based on aforesaid sentences, registered VEP signal can be represented as the summation of a large number of random impulses that happen at random moments. Moreover, the impulse characteristics depend on the time. For example, immediately after the stimulus passing the action potential appears with a small amplitude and there will be EPSP, IPSP with larger amplitude and duration, etc. That is, based on the analysis of biophysical generating mechanism of the VEP signal, we can conclude that the signal recorded from the scalp surface is not stationary.

Mathematical model justification. After stimulation neurons generate and transmit AP, EPP, IPP that had appeared in random sequential time points $\{...,\tau_{-2},\tau_{-1},\tau_0,\tau_1,\tau_2,...,\tau_n,...\}=\{\tau_n, n\in\mathbf{Z}\}$. Experimentally investigated and presented distribution histogram in papers [10; 11] demonstrates that quantities of time interval $\{\Delta\tau_0=\tau_0,...,\Delta\tau_n= =\tau_n-\tau_{n-1},...\}$ between impulse appearances are independent random variables and exponentially distributed with parameter $\lambda(\tau), \tau\in(-\infty,\infty)$. Parameter $\lambda(\tau)$ – is deterministic function that characterizes the intensity of pulses appearance from active neurons.

Based on the above-mentioned considerations we can conclude that the sequence of time moments $\tau_n, n\in\mathbf{Z}$ is non-uniform Poisson flow with parameter $\lambda(\tau)$.

The results of experimental research confirm that statistical correlation between different neurons is insignificant. Especially, presented data in paper [9] asserts that the correlation coefficient between the potentials of different neurons is 0,006-0,01. Taking into account the above-mentioned considerations and modern electro-genesis theories of EEGs and VEPs we assumed that impulses are statistically independent (the weak correlation between them will be rejected in the developed model).

Random function that describes the electrical potential changing of separate neuron is labeled by symbol $V_n(\tau_n,t)$, where τ_n – is the random activation moment time of *n*-th neuron, *t* – is the moment of observation. Whereas, recorded signal is a total response for potential changing of all active neurons during the observation time *t*, then the resulting VEP signal can be written in the form of a random process:

$$\xi(t)=\sum_{n=-\infty}^{\infty}V_n(\tau_n,t), t\in(-\infty,\infty). \quad (1)$$

For function concretization $V_n(\tau_n,t)$ we used papers [7; 9] where interested us potentials are presented by decaying impulse with oscillating character. Based on these papers, impulses $V_n(\tau_n,t)$ can be expressed in the next form:

$$V_n(\tau_n,t)=\alpha_n\varphi(\tau_n,t), \quad (2)$$

where $\varphi(\tau,t)$ – is nonrandom function, which can be presented as:

$$\varphi(\tau,t)=e^{-\beta(\tau)(t-\tau)}\sin(\omega(\tau)(t-\tau))U(t-\tau), \quad (3)$$

$\alpha_n, n\in\mathbf{Z}$ – sequence of independent similarly distributed random variables with distribution function $F_\alpha(x;\tau), x\in\mathbf{R}$ and finite variance; $U(s)=\begin{cases}1, s\geq 0\\ 0, s<0\end{cases}$ – is the Heaviside function; $\beta(\tau)>0$ – is nonrandom function that describes the coefficient of impulse damping; $\omega(\tau)>0$ – is nonrandom function that describes the impulse frequency.

Figure 3 shows an example of function $V_n(\tau_n,t)$.

It should be also emphasized that $F_\alpha(x;\tau), \beta(\tau), \omega(\tau)$ (characteristics of impulses) are depend on τ, because for every *n*-th impulse $V_n(\tau_n,t)=\alpha_n e^{-\beta(\tau_n)(t-\tau_n)}\sin(\omega(\tau_n)(t-\tau_n))U(t-\tau_n)$ the distribution function of its amplitude $F_\alpha(x;\tau_n)$, the coefficient of impulse damping $\beta(\tau_n)$ and the impulse frequency $\omega(\tau_n)$ are dependent on the moment time of impulse occurrence.

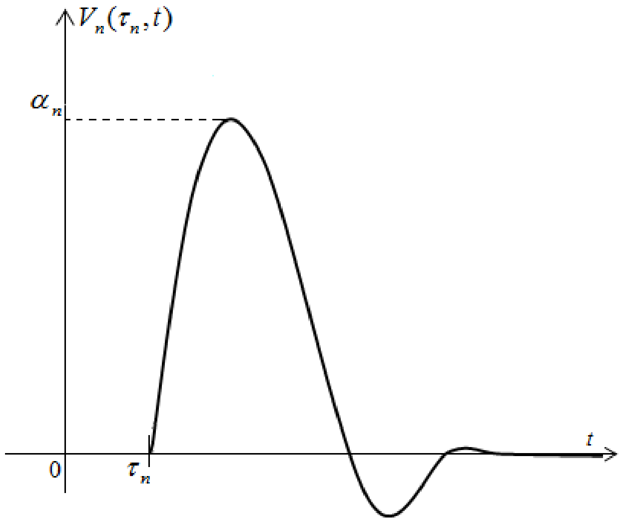

Fig. 3. Graphic representation of changing the electric potential of a neuron

Consequently, random process (1) can be written in the next form:

$$\xi(t)=\sum_{n=-\infty}^{\infty}\alpha_n e^{-\beta(\tau_n)(t-\tau_n)}\sin(\omega(\tau_n)(t-\tau_n))U(t-\tau_n). \tag{4}$$

Let us introduce $\pi_1(\tau),\ \tau\in(-\infty,\infty),\ \mathbf{P}\{\pi_1(0)=0\}=1$– is nonhomogeneous generalized Poisson process, jumps of which happen at the same time moments $\tau_n,\ n\in\mathbf{Z}$ and the value of each jump is equal to the random variable $\alpha_n,\ n\in\mathbf{Z}$.

In this case the process $\xi(t)$ can be written as the stochastic integral

$$\xi(t)=\int_{-\infty}^{\infty}\phi(\tau,t)d\pi_1(\tau)=\int_{-\infty}^{\infty}e^{-\beta(\tau)(t-\tau)}\sin(\omega(\tau)(t-\tau))U(t-\tau)d\pi_1(\tau). \tag{5}$$

According to the definitions given in the papers [12; 13], random process (5) is a linear random process (shortly LRP) with the kernel $\varphi(\tau,t)$ and the generating process $\pi_1(\tau)$. Process $\xi(t)$ is a Hilbert process, that is for kernel (5) the energy $\int_{-\infty}^{\infty}\varphi^2(\tau,t)d\tau<\infty,\forall t$ for each fixed t and variance of increments of generating process $\mathbf{D}(d\pi_1(\tau))<\infty,\forall t$ are finite.

Characteristic function for process (5) is presented by expression [12; 13]:

$$f_\xi(u_1,u_2,\dots,u_m;t_1,t_2,\dots,t_m)=\mathbf{M}\ e^{i\sum_{n=1}^{m}u_n\xi(t_n)}=\exp\left[i\sum_{n=1}^{m}u_n\int_{-\infty}^{\infty}\varphi(\tau,t_n)da(\tau)+\right.$$

$$\left.+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{ix\sum_{n=1}^{m}u_n\varphi(\tau,t_n)}-1-ix\sum_{n=1}^{m}u_n\varphi(\tau,t_n)\right)\frac{d_x d_\tau K(x;\tau)}{x^2}\right], \tag{6}$$

$$u_n,t_n\in(-\infty,\infty),\ n=\overline{1,m},\ i=\sqrt{-1},$$

where $a(\tau) = \mathbf{M}\,\pi_1(\tau)$; $K(x;\tau),\ x \in (-\infty,\infty)$ – is Poisson spectrum of jumps in the Kolmogorov's form such that $K(-\infty;\tau) = 0,\ K(\infty;\tau) = \int_{-\infty}^{\infty} d_x K(x;\tau) = \mathbf{D}\,\pi_1(\tau) = b(\tau),\ \forall\tau.$

Based on the expression (6) we can find cumulant function of linear random process. Particularly the mathematical expectation $\mathbf{M}\,\xi(t)$ and the correlation function $R_\xi(t_1,t_2)$ of process (5) will be presented in the next forms, respectively [12; 13]:

$$\mathbf{M}\,\xi(t) = \int_{-\infty}^{\infty} \varphi(\tau,t)\,da(\tau), \quad R_\xi(t_1,t_2) = \int_{-\infty}^{\infty} \varphi(\tau,t_1)\,\varphi(\tau,t_2)\,db(\tau). \tag{7}$$

Thereby based on the analysis of the generation mechanism of electrical activity by brain neurons and taking into consideration the requirements the mathematical model was constructed in form of the linear random process. Application of LRP is often used for construction of new mathematical model because the properties of this model allow computing the moment and cumulant functions of any order, using probabilistic analysis of the investigated signal by characteristic functions method. One of the biggest advantages of LRP is peculiarity that this model takes into account the physical mechanisms of production or the generation of the analyzed process.

Conclusion. Taking into account the stimulation of human visual analyzer, the analysis of the generating mechanism of the brain electrical activity by individual neurons was done. The finding results were used for further construction of an adequate model.

Based on the above-described objectives and requirements the mathematical model of steady-state visual evoked potential in a form of a linear random process is justified. This model allows not only identifying information and diagnostic parameters of the investigated process, but also gives them a biophysical explanation.

The expression of proposed mathematical model of steady-state VEPs characteristic function is presented. It allows us to compute moment and cumulant functions of any order. The expressions of mathematical expectation and correlation function are presented too.

Taking into account the cyclic photo stimulation of the visual analyzer, periodic properties of the constructed model will be justified, the properties of linear periodic random process will be explore a too and the adequacy of the constructed model will be experimentally verified in the next papers.

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М. Є. Фриз, М. А. Стадник

Обґрунтування математичної моделі усталених зорових викликаних потенціалів у вигляді лінійного випадкового процесу

Обґрунтовано математичну модель усталеного зорового викликаного потенціалу у вигляді лінійного випадкового процесу, що враховує біофізичні особливості формування потенціалу. Описано механізм його генерації окремими нейронами головного мозку людини. Наведено вираз характеристичної функції запропонованої математичної моделі.

М. Е. Фриз, М. А. Стадник

Обоснование математической модели зрительных вызванных потенциалов устойчивого состояния в виде линейного случайного процесса

Обосновано математическую модель зрительного вызванного потенциала устойчивого состояния в виде линейного случайного процесса, что учитывает биофизические особенности формирования потенциала. Описано также его механизм генерации отдельными нейронами главного мозга человека. Приведено выражение характеристической функции предложенной математической модели.