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NON-CLASSICAL CONJUGATE VECTORS AND POLAR DIFFERENTIAL EQUATIONS OF ROTATION IN ORIENTATION PROBLEM

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Abstract. *Kinematic and dynamic differential equations of rotation of a rigid body for the «non-classical» conjugate three-dimensional vectors of rotation are considered in this article. The vectors modules contain tangent and cotangent of one-fourth of the rotation angle. This paper also delivers applications of the conjugate equations in the problems of orientation of a rigid body.*

Keywords: rotation vectors; the equation of rotation; orientation.

Introduction

Three-dimensional vectors of rotation with minimal possible quantity of generalized coordinate lining with three freedoms of rigid body (RB) [1]) with fixed point are of the most interest in the problem of dimensional orientation (such as a task of orientation order parameter identification and directing of rotation (spherical) motion of RB.

Classical Rodrig's vector [1] has a module with tangent (tg(φ/2)) of half-angle φ of finite rotation of RB. This vector can not be used in the condition of $\varphi \geq \pi$ in the problems of RB dimensional orientation. Considered in the works [2–4] conjugate rotation vectors $\vec{\boldsymbol{\tau}}=\tau\vec{\mathbf{k}}$, $\vec{\boldsymbol{\rho}}=\rho\vec{\mathbf{k}}$, where $\tau = k_\tau \mathrm{tg}(\varphi/4)$, $\rho = k_\rho \mathrm{c\,tg}(\varphi/4)$ (k_τ, k_ρ. – arbitrary constant coefficients), $\vec{\mathbf{k}}$ — unit vector of Euler axis of RB finite rotation are of the most interest (by $0 < \varphi < 2\pi$)) because of its conjugate features [4].

Problem statement

The article deals with *conjugative polar* non-linear vector kinematic differential equations of RB rotation in the transformation of lineal vector $\vec{\omega}(t)$ of RB angular velocity [3; 4]

$$\vec{\boldsymbol{\tau}}^* = \tau'\theta_{(\tau)}\vec{\boldsymbol{\omega}},\ \vec{\boldsymbol{\rho}}^* = \rho'\theta^{\mathrm{T}}_{(\rho)}\vec{\boldsymbol{\omega}}, \tag{1}$$

where $\vec{\boldsymbol{\tau}}^* = d\vec{\boldsymbol{\tau}}/dt$, $\vec{\boldsymbol{\rho}}^* = d\vec{\boldsymbol{\rho}}/dt$ – local derivative of vector in the time t (derivative towards some connected with RB coordinate base); $\tau' = \partial\tau/\partial\varphi$, $\rho' = \partial\rho/\partial\varphi$ – particular derived modules τ, ρ of vector of angle φ, defined as function $\tau' = \left(k_\tau + \tau^2/k_\tau\right)/4$, $\rho' = -\left(k_\rho + \rho^2/k_\rho\right)/4$; $\theta_{(\tau)}$, $\theta^{\mathrm{T}}_{(\rho)}$ – orthogonal operators [3]:

$$\theta_{(\tau)} = \mathrm{E} + 2\left(k_\tau \mathrm{T} + \mathrm{T}^2\right)/\left(k_\tau^2 + \tau^2\right),$$

$$\theta^T_{(\rho)} = \mathrm{E} + 2\left(-k_\rho R + R^2\right)/\left(k_\rho^2 + \rho^2\right),$$ «T» – transposing.

Therein T, R – skew-symmetric operators of vector multiplication [2, 4], satisfies the identities $\mathrm{T}\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\tau}}\times\vec{\boldsymbol{\tau}} = \vec{0}$, $\mathrm{R}\vec{\boldsymbol{\rho}} = \vec{\boldsymbol{\rho}}\times\vec{\boldsymbol{\rho}} = \vec{0}$, $\vec{0}$ – zero vector, E – unit operator. Operators $\theta_{(\tau)}$, $\theta_{(\rho)}$ satisfy the identity $(\theta_{(\tau)}\theta_{(\rho)})^2 = \mathrm{E}$, determined conjugacy of vectors $\vec{\boldsymbol{\tau}}$, $\vec{\boldsymbol{\rho}}$ and equations (1) [4] as features of 'duality' and isomorphic correspondence.

Graphic kinematic interpretation – polar precession-nutational model

Equations (1) have the first («trigonometric») integral in the form of scalar product $(\vec{\boldsymbol{\tau}}\cdot\vec{\boldsymbol{\rho}}) = \tau\rho = k_\tau k_\rho = C$ (arbitrary constant) and admit simple and graphic kinematic interpretation – polar precession-nutational model [4] with arbitrary vector $\vec{\omega}_\tau$ and for any RB. In the first equation vector $\vec{\boldsymbol{\omega}}$ is transformed (at every moment of the travelling time of RB) by operator $\theta_{(\tau)}$ into vector $\vec{\boldsymbol{\omega}}_\tau$ as a result of precession в $\vec{\boldsymbol{\omega}}$ on the angle $\psi_{\theta\tau} = \varphi/2$ (rotation vector $\vec{\boldsymbol{\omega}}$ with module $\omega(t)$ about the surface of some circular cone of «velocity» precession cone angle $2\upsilon_\omega$) around Euler axis with unit vector $\vec{\mathbf{k}}$. Angle υ_ω – nutation angle (vector deviation $\vec{\boldsymbol{\omega}}$ from the unit vector $\vec{\mathbf{k}}$). And then a vector $\vec{\boldsymbol{\omega}}_\tau$ is multiplied by scalar operator τ'E. At the second equation vector $\vec{\boldsymbol{\omega}}$ precesses (rotates n the surface of that) at the angle $\psi_{\theta\rho} = (\pi+\varphi/2)$ and is multiplied by scalar operator ρ'E. Angle υ_ω of nutation is determined from scalar product $(\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{\tau}})$, $(\vec{\boldsymbol{\tau}}^*\cdot\vec{\boldsymbol{\tau}})$ or $(\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{\rho}})$, $(\vec{\boldsymbol{\rho}}^*\cdot\vec{\boldsymbol{\rho}})$.

On the basis of such interpretation – equation models (1) come out, for example, kinematic anholonomic equality:

$$\frac{\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{a}\times\vec{b}\right)\right)}{\left(\vec{a}\cdot\vec{a}\right)\left(\vec{b}\cdot\vec{b}\right)} = 1, \tag{2}$$

where $\vec{a} = \vec{\omega}\times\vec{\tau}$, $\vec{b} = \left(\theta_{(\tau)}\vec{\omega}\right)\times\vec{\tau}$. The equality with arbitrary constant $C_\tau = k_\tau$, comes out from equality (2) for example:

$$\left|\frac{C_\tau^2+\tau^2}{C_\tau^2\tau ab}\right|^2((\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b}))=4. \qquad (3)$$

Equalities, analogous to equalities (2), (3), come out of vector change $\vec{\boldsymbol{\tau}}$ to vector $\vec{\boldsymbol{\rho}}$.

Under the numerical integration of equations (1) [1; 4] such values are to exist – $k_{\tau*}$ and $k_{\rho*}$ of coefficients k_τ, k_ρ, where equality is provided $\tau'=1$ and $\rho'=1$ [3; 4]. Then, for example, in [4] $k_{\tau*}=2+\sqrt{4-\tau^2}$, , $k_{\rho*}=-2-\sqrt{4-\rho^2}$, ($\tau \le 2$, $\rho \le 2$) vector transformation $\vec{\boldsymbol{\omega}}$ into derivatives $\vec{\boldsymbol{\tau}}^*$, $\vec{\boldsymbol{\rho}}^*$ will be exercised by orthogonal operators $\theta_{\tau*}$, $\theta^{T}_{\rho*}$, where at every step of calculation new values of "adaptive" coefficients $k_{\tau*}$, $k_{\rho*}$ (instead of constant coefficints k_τ , k_ρ) are used. But in fact equations integrate (1) in the form of orthogonal transformation $\bar{\tau}^*_* = \theta_{\tau*}\bar{\omega}$, $\bar{\rho}^*_* = \theta^{T}_{\rho}$ $\vec{\boldsymbol{\omega}}$ when known (approximately "measured" by integrated gyroscopes [1; 4]) of kinematic integrals (vector modules $\bar{\tau}_*$, $\bar{\rho}_*$) $\tau_* \approx \rho_* \approx q+c_q \approx \varphi$ from scalar functions $\tau^{\bullet}_*(t)$, $\rho^{\bullet}_*(t)$, $\omega(t)$, $\varphi^{\bullet}(t)$ (c_q – arbitrary constant, $\omega(t)$ – module of vector).

Equations (1) have general decisions in the form of Coshi [4; 5]. General decision of the first equation in (1) determines the form of group operations of multiplication and division in associative group of Lee vectors $\vec{\boldsymbol{\tau}}$ [6]. In this group inverse element $\vec{\boldsymbol{\tau}}^{-1}$ is equal to adverse vector, i. e. $\vec{\boldsymbol{\tau}}^{-1}=-\vec{\boldsymbol{\tau}}$, but unit element is equal to zero vector $\bar{0}$. That's why associative group vector algebra with element $\vec{\boldsymbol{\tau}}$ has special ("group") zero details by existence of single-valued operations of division (in contrast to classical non-associative vector lineal algebra, that generally has no division).

Different new (polar) dynamical differential equations of RB rotation come out on the basis of kinematic equations (1).

In classical Euler occasion, for example, problem of decision of the system of six dynamical differential equations of Euler–Poisson [7] reduces to integration of the system of just three dynamical equations with two independent classical first integrals (of energy and square [7]). These two integrals are enough to consider the system of three equations to be integrated [7].

In this case change is introduced in the equations (1) $\vec{\boldsymbol{\omega}}=S^{-1}\vec{\mathbf{g}}$, where $\vec{\mathbf{g}}$ – constant vector of kinetic moment (constant module and in the line of supporting basis I [7]); S^{-1} – inverse operator S of RB inertia [3; 4] (constant in connected with RB basis J).

Contained of (1), for example, polar matrix differential equation with vector coordinates $\vec{\boldsymbol{\tau}}$ has the form (see also [3])

$$\vec{\boldsymbol{\tau}}^*_J=\tau'\left(\theta_{(\tau)J}S_J^{-1}\theta^{T}_{(\tau)J}\right)\theta^{T}_{(\tau)J}\vec{\mathbf{g}}_I\,, \qquad (4)$$

where $\vec{\boldsymbol{\tau}}^*_J=\left[\dot{\tau}_{j1}\,\dot{\tau}_{j2}\,\dot{\tau}_{j3}\right]^{T}$ – column matrix with derivative coordinates $\vec{\tau}^*$ in basis J; $\boldsymbol{\theta}_{(\tau)J}$ – matrix (3×3) of operator $\theta_{(\tau)}$ in basis J; S_J – diagonal matrix (3×3) with three constant the main moments of RB inertia; $\mathbf{S}_J^{-1}$ – inverse matrix, $g_I = [g_{i1}\ g_{i2}\ g_{i3}]^{T}$ – constant column matrix with vector coordinates $\vec{\mathbf{g}}$ in basis I.

Three vector coordinates $\vec{\tau}^*$ definitely determine RB orientation (in contrast to three direction cosines determined orientation of just one unit vector of supporting basis I in Poisson equations). In the problem of synthesis of RB orientation control law (determination of control moments) applying functions of Lyapunov (see., for example, [8]) equation (4) is used together with classical dynamical equations of Euler [1; 7].

Conjugate vectors $\vec{\boldsymbol{\tau}}$, $\vec{\boldsymbol{\rho}}$ are considering as parts of vectors with non-traditional non-normalized (non-Hamiltonian) rotation quaternion [8]. Such non-normalized quaternions can be effectively used instead of classical normalized quaternions (with parameters of Rodrig–Hamilton) in different "particular" tasks of inertial orientation [1; 4; 9; 10], including RB orientation control problem [8].

Conclusions

Sets of coordinates of conjugate vectors $\vec{\boldsymbol{\tau}}$, $\vec{\boldsymbol{\rho}}$, corresponded to them quaternions and parameters of five-dimensional vectors [9] sufficiently broaden (along with parameters of «exclusive» non-normalized quaternions [9]) the circle of new RB orientation parameters. Such parameters can be of interest in the problem of global parameterization [9] of three-dimensional rotation group and Lorentz group (in quantum mechanics), either in problem of building of new vector finite-dimensional (for instance five-dimensional) groups of Lee and associative group algebra of rotation with division (without zero division, as Flobernius theorem of exclusiveness of quaternions algebra demands [9]).

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А. П. Панов. Некласичні спряження векторів і полярні диференціальні рівняння обертання у задачах орієнтації

Розглянуто кінематичні і динамічні диференціальні рівняння обертання твердого тіла для «некласичних» спряжених тривимірних векторів обертання. Модулі векторів містять тангенс і котангенс однієї чверті від кута повороту. Запропоновано застосування спряжених рівнянь у задачах орієнтації твердого тіла.
Ключові слова: вектори обертання; рівняння обертання; орієнтація.

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А. П. Панов. Неклассические сопряжения векторов и полярные дифференциальные уравнения вращения в задачах ориентации

Рассмотрены абстрактные кинематические и динамические дифференциальные уравнения вращения твердого тела для «неклассических» сопряженных трехмерных векторов вращения. Модули векторов содержат тангенс и котангенс одной четверти от угла поворота. Предложено применение сопряженного уравнения в задачах ориентации твердого тела.
Ключевые слова: векторы вращения; уравнения вращения; ориентация.

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