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# STABILITY OF AUTOMATIC CONTROL SYSTEMS

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***Abstract.*** *This article discusses Mikhailov's criterion, a frequency criterion for the stability of automatic control systems.*

**Keywords:** stability; transient functions; root locus; characteristic polynomial.

## I. INTRODUCTION

Resistance is the ability of a technical system to return to its equilibrium position after the termination of the external forces that brought it out of balance.

As the state of equilibrium, we understand state of the system when it is at rest, that is, when its output signal is constant, and all its derivatives vanish. In the field of automatic control systems (ACS), equilibrium position is usually called steady state.

Conclusion about the stability of ACS can be made on the basis of its time characteristics, such as impulse response functions (Fig. 1).

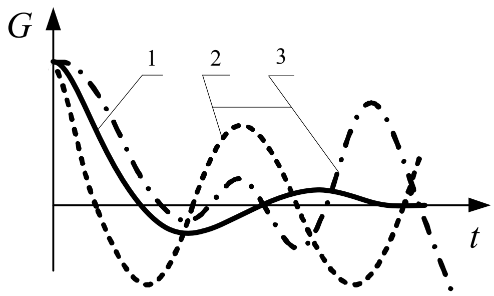


Fig. 1. Impulse response functions

Automatic control system is called stable if its transient response, caused by a short-time external input, will decay (curve 1). If undamped harmonic oscillations (curve 2) emerge in the system, it is on the stability boundary. An unstable system has a divergent transient response (curve 3).

In the case when a constant external input is applied to automatic control system, such as the unit step signal, then its stability is accessed by investigating transient functions (Fig. 2).

Stable ACS after the decay of the transient (curve 1) proceeds from the initial steady state $y_{st1}(t)$ to the new steady state $y_{st2}(t)$.

Automatic control system, which does have steady regimes, i.e., transients do not decay (curves 2, 3), practically is not workable. This should be considered during the operation of ACS and during its configuration and adjustment of parameter values that would ensure the required quality of the transition process and the stability of the system.

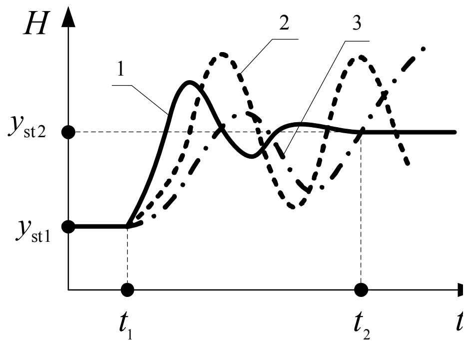


Fig. 2. Transient functions

The basic method of determining the stability of an automatic control system is the method based on solving its linear differential equation of motion. However, it is not always convenient to calculate the roots of the characteristic equation. Therefore, automatic control theory has developed special techniques to assess the stability of a system using so-called stability criteria.

Algebraic criteria allow assessing the stability of ACS based on an analysis of its parameters determined by the coefficients of its characteristic equation.

Frequency criteria have certain advantages over algebraic ones. First, we do not need to solve systems of differential equations, especially of higher orders. Second, it is their clarity. Third, it is the possibility of using the frequency characteristics of systems determined experimentally.

With the help of frequency criteria, it is enough to decide on measures to ensure stability of the system, if the result of investigation suggests that it is unstable.

Nyquist frequency stability criterion was widely advertised lately, especially in translated literature. This criterion is included in Simulink visual simulation software.

In our view, insufficient attention is given to the Mikhailov frequency criterion, which allows to evaluate both closed and open loop systems. And the practice of study of ACS itself dictates the need to include this criterion in simulation packages.

## II. Solution of the problem

Back in the 1936, A.V. Mikhailov showed that signs of the real parts of the roots of a polynomial

$$A(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0 . \quad (1)$$

can be determined from the plot of function

$$A(j\omega) = X(\omega) + jY(\omega),$$

which is obtained after substituting imaginary argument $j\omega$ into the polynomial $A(s)$ instead of $s$.

The real part of the complex function $A(j\omega)$ is

$$X(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \ldots,$$

and the imaginary is

$$Y(\omega) = a_1\omega - a_3\omega^3 + a_5\omega^5 - \ldots .$$

At the constant frequency $\omega_i = \text{const}$, a complex number $A(j\omega_i)$ represents a vector in the plane $X(\omega) \to jY(\omega)$.

If we change the frequency $\omega$ in the range from 0 to $+\infty$, then the end of the vector will draw a curve on the plane, which is called *Mikhailov hodograph*. By the form of the hodograph (Fig. 3) we can assess the stability of the system under study.

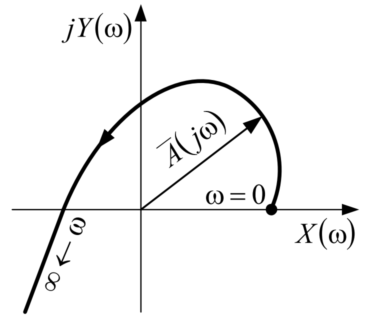


Fig. 3. Mikhailov hodograph

Let us denote the roots of characteristic polynomial (1) of the system as

$$A(s) = 0 \;\Rightarrow\; \lambda_i = \pm\alpha_i \pm j\beta_i \quad (i = 1,2,3, \ldots, n).$$

Then, expanding the polynomial $A(s)$ by factors, we obtain

$$A(s) = a_n(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_{n-1})(s-\lambda_n),$$

and the expression of Mikhailov hodograph will be in the form

$$A(j\omega) = a_n(j\omega-\lambda_1)(j\omega-\lambda_2)\ldots(j\omega-\lambda_n).$$

In the last expression, each of the brackets is a some vector on the complex S-plane of the roots $\alpha \to j\beta$, equal to the difference of vectors $j\omega$ and $\lambda_i$. The whole polynomial $A(j\omega)$ is also a vector, which is equal to the product of $n$ vectors of type $(j\omega - \lambda_i)$ by a constant factor $a_n$.

Taking into account the above remarks, we rewrite the expression in a vector form

$$\bar{A}(j\omega) = a_n(\bar{j}\omega-\bar{\lambda}_1)(\bar{j}\omega-\bar{\lambda}_2)\ldots(\bar{j}\omega-\bar{\lambda}_n).$$

In accordance with the rule of multiplication of complex numbers, the module of the vector $\bar{A}(j\omega)$ will be equal to the product of the magnitudes of vectors $(\bar{j}\omega - \bar{\lambda}_i)$, which are its co-products

$$\left|\bar{A}(j\omega)\right| = \sqrt{X^2(\omega)+Y^2(\omega)} = \prod_{i=1}^{n}\left|(\bar{j}\omega-\bar{\lambda}_i)\right|, \quad (2)$$

and the total rotation angle (argument) of the vector $\bar{A}(j\omega)$ – to the algebraic sum of the angles of rotation (arguments) vectors $(\bar{j}\omega - \bar{\lambda}_i)$

$$\varphi_\Sigma(\omega) = \text{arctg}\frac{Y(\omega)}{X(\omega)} = \sum_{i=1}^{n}\varphi_i(\omega). \quad (3)$$

Consider the arrangement of the vectors $(\bar{j}\omega - \bar{\lambda}_i)$ on the complex $S$-plane of roots $\alpha \to j\beta$.

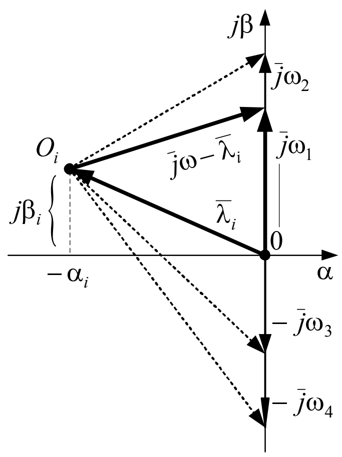


Fig. 4. Roots and factors of the characteristic polynomial

In the case of negative real part, the root $\lambda_i = -\alpha_i \pm j\beta_i$ (Fig. 4) is placed to the left of the imaginary axis and can be represented with a vector $\bar{\lambda}_i$ drawn from the origin to the point $O_i$ with coordinates $(-\alpha_i; \pm j\beta_i)$. Term $\bar{j}\omega$ is represented by the vector, directed along the imaginary axis up or down, depending on the sign of $\omega$. The difference of vectors $(\bar{j}\omega - \bar{\lambda}_i)$ is also a vector, which beginning lies in the point $O_i(-\alpha_i; \pm j\beta_i)$, and the end with change of the frequency $\omega$ will slide along the imaginary axis.

From Fig. 4, it follows that by change of $\omega$ from $-\infty$ to $+\infty$, the vector $(\bar{j}\omega - \bar{\lambda}_i)$ will rotate by angle $\pi$ in the *positive direction*, that is, counterclockwise.

In the case of positive real part, the root $\lambda_i = +\alpha_i \pm j\beta_i$ (Fig. 5) is placed to the right of the imaginary axis and can be represented with a vector $\bar{\lambda}_i$ drawn from the origin to the point $O_i$ with coordinates $(+\alpha_i; \pm j\beta_i)$. Term $\bar{j}\omega$ is represented by the vector, directed along the imaginary axis up or down, depending on the sign of $\omega$. The difference of vectors $(\bar{j}\omega - \bar{\lambda}_i)$ is also a vector, which beginning lies in the point $O_i(+\alpha_i; \pm j\beta_i)$, and its end with change of the frequency $\omega$ will slide along the imaginary axis.

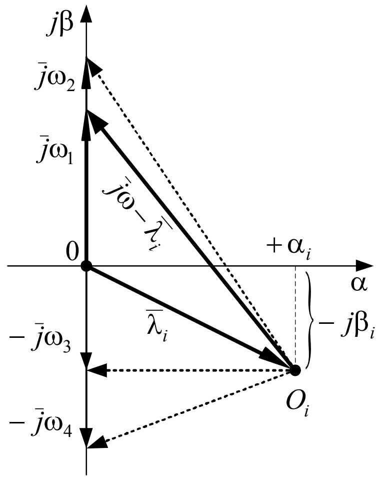


Fig. 5. Roots and factors of the characteristic polynomial

From Fig. 4, it follows that by change of $\omega$ from $-\infty$ to $+\infty$, the vector $(\bar{j}\omega - \bar{\lambda}_i)$will rotate by angle $\pi$ in the *negative direction*, that is, counterclockwise.

Thus, if a characteristic polynomial $A(s)$ of $n$ th order will have $m$ roots with positive real part, and remaining $(n-m)$ roots with negative real part, then by change of frequency in the range from $-\infty$ to $+\infty$, the total rotation of the vector $\bar{A}(j\omega)$ according to (3) will be equal

$$\varphi_\Sigma(\omega) = (n-m)\pi - m\pi = (n-2m)\pi. \quad (4)$$

In other words, the total angle of rotation of the vector $\bar{A}(j\omega)$ depends only on the signs of the real parts of the roots of the characteristic polynomial $A(s)$.

In real automatic control devices, frequency $\omega$ can only be changed in the range from 0 to $+\infty$ (cannot be negative), so the study of ACS is always includes only a positive branch of Mikhailov hodograph. The total angle of rotation of the vector $\bar{A}(j\omega)$ in this case is equal

$$\varphi_\Sigma(\omega) = (n-2m)\frac{\pi}{2}.$$

For stability of $n$ th order system it is necessary that all the roots of the characteristic equation have negative real parts, i.e., to the number $m$ of roots with positive real part is zero.

$$\varphi_\Sigma(\omega) = n\frac{\pi}{2}.$$

Hence follows Mikhailov stability criterion, according to which the linear system of automatic control of $n$ th order is stable, if the hodograph of the vector $\bar{A}(j\omega)$, corresponding to the characteristic polynomial of the system $A(s)$, as with change of the frequency from $\omega = 0$ to $\omega = \infty$ it passes counterclockwise consistently $n$ quadrants on the complex plane $X(\omega) \to jY(\omega)$.

Mikhailov hodograph of a stable system, which is described by a differential equation of the third order, is shown in Fig. 6a.

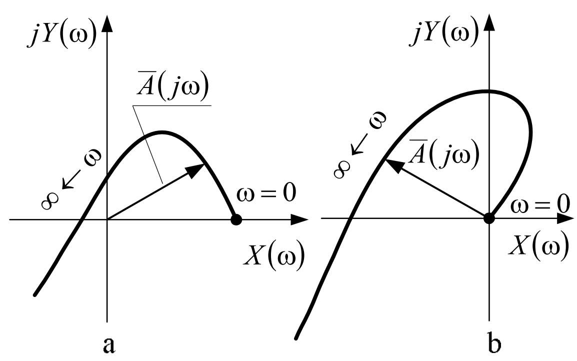


Fig. 6. Mikhailov hodographs of stable third-order systems: a is all roots with negative real part; *b* is there is a zero root

If the characteristic polynomial will have one zero root $\lambda_1 = 0$ except roots with negative real part, then, in accordance with (2), at the frequency $\omega = 0$ the magnitude of the vector $\overline{A}(j\omega)$ vanishes.

$$\left|\overline{A}(j\omega)\right| = \sqrt{X^2(0) + Y^2(0)} = \prod_{i=2}^{n} \omega \left|\overline{\lambda}_i\right|_{\omega=0} = 0.$$

Therefore, the starting point of Mikhailov hodograph will be in the beginning of the coordinate system $X(\omega) \to jY(\omega)$. Hodograph of *aperiodically stable* system of third order is shown in Fig. 6b.

If the characteristic polynomial except roots with negative real part has a pair of purely imaginary roots $\lambda_{1,2} = \pm j\beta_0$, then, according to (2), at the frequency $\omega = \beta_0$ the magnitude of the vector $\overline{A}(j\omega)$ will be zero

$$\left|\overline{A}(j\omega)\right| = \sqrt{X^2(\beta_0) + Y^2(\beta_0)} = \prod_{i=2}^{n} \left|\bar{j}\omega \mp \bar{j}\beta_0\right|\left|\bar{j}\omega - \overline{\lambda}_i\right|_{\omega=\beta_0} = 0. \quad (5)$$

Position of vectors $(\bar{j}\omega - \bar{j}\beta_0)$ on the complex plane for the discussed case is shown in Fig. 7a.

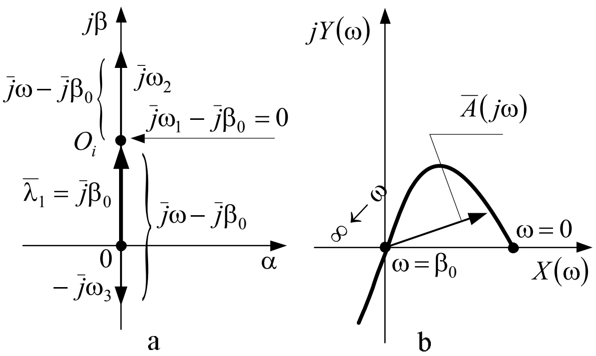


Fig. 7. System of the third order at the boundary of stability: a are vectors $(\bar{j}\omega - \bar{j}\beta_0)$; b is Mikhailov hodograph

Thus, when changing frequencies in the range from $-\infty$ to $+\infty$ the total rotation of the vector $\overline{A}(j\omega)$ according to (4) will be

$$\varphi_\Sigma(\omega) = (n-1)\pi - \frac{\pi}{2} + \frac{\pi}{2} = (n-1)\pi..$$

With regard to real technical systems, when the frequency $\omega$ varies from 0 to $+\infty$, we obtain

$$\varphi_\Sigma(\omega) = (n-1)\frac{\pi}{2}. \quad (6)$$

Fulfillment of conditions (5) and (6) means that Mikhailov hodograph of a system on boundary of stability must pass through the origin of the complex plane $X(\omega) \to jY(\omega)$. Mikhailov hodograph of such third-order system is shown in Fig. 7, b.

As an illustration, Fig. 8 shows Mikhailov hodographs of stable systems of various orders.

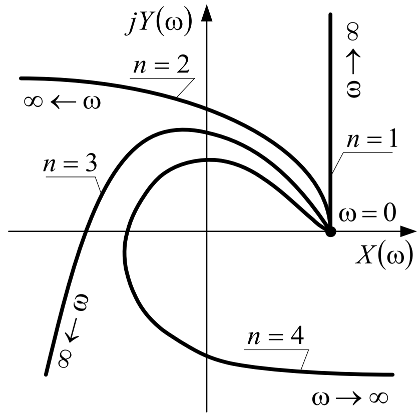


Fig. 8. Mikhailov hodographs for stable systems of different orders

Hodographs of fourth-order systems with the same coefficient $a_0$ are shown in Figure 9.

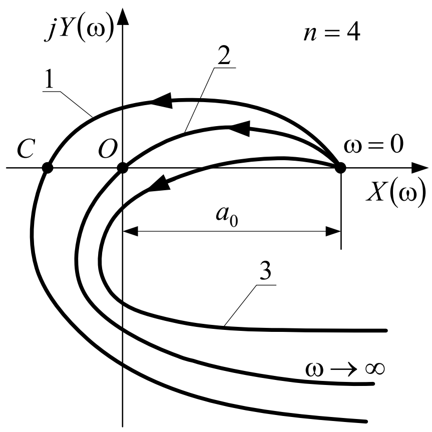


Fig. 9. Mikhailov hodographs of 4th order system of different degrees of stability

Hodograph 1 corresponds to a stable system, hodograph 3, in which vector $\overline{A}(j\omega)$ violates the sequence of passing quadrants on the complex plane – to an unstable system, and hodograph 2, which passes the origin – to a system with boundary stability.

Stability margin of a system can be roughly assessed by the length of a segment *OC* that shows how the system is far from the boundary of stability.

From analysis of the sequence of how Mikhailov hodograph of a stable system passes through the quadrants of the complex plane, when components $X(\omega)$ and $Y(\omega)$ of the vector $\overline{A}(j\omega)$ vanish by turns, important consequence from the Mikhailov criterion follows. It is formulated as: for stability of *n*th order system it is necessary when frequency changes from $\omega = 0$ to $\omega = \infty$ that curves $X(\omega)$ and $Y(\omega)$ cross the *x*-axis $n$ times by turns, including the point $\omega = 0$.

Figure 10 shows the variants how curves $X(\omega)$ and $Y(\omega)$ pass for stable (a) and unstable (b) fourth-order systems.

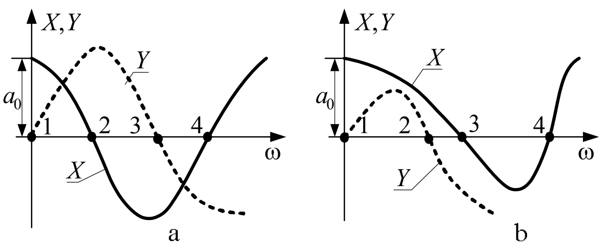


Fig. 10. Characteristics $X(\omega)$ и $Y(\omega)$ of a fourth order system: a is stable; b is unstable

## III. Conclusions

Mikhailov frequency criterion is a significant technique for the research of stability of automatic control systems.

Every engineer dealing with research and design of automatic systems, should master this criterion, along with widely advertised others.

Mikhailov stability criterion, in our opinion, should be included in software for simulation of automatic control systems.

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**О. К. Аблесімов, В. С. Яцківський. Стійкість систем автоматичного керування**

Розглянуто частотний критерій Михайлова оцінки стійкості систем автоматичного керування.

**Ключові слова:** стійкість; перехідні функції; годограф; характеристичний багаточлен.

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**А. К. Аблесимов, В. С. Яцковский. Устойчивость систем автоматического управления**

Рассмотрен частотный критерий Михайлова оценки устойчивости систем автоматического управления.

**Ключевые слова:** устойчивость; переходные функции; годограф; характеристический многочлен.

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