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# PROPERTIES OF SIGNAL SWITCH-ON FUNCTIONS AND THEIR USE

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***Abstract*****—***The article is focused on properties of switch-on functions which allow performing operational and spectral calculus of multiple signals without Laplace and Fourier transform.*

**Index Terms**—Signal theory; symbolic representations of signals.

## I. INTRODUCTION & PROBLEM STATEMENT

The switch-on function (SF) is deemed to be the Heaviside step function, $h(t)$, [1], [2]. Henceforth the SF is denoted by a product $v(t)=u(t)h(t)$, where $u(t)$ is any function within an interval $t\in(-\infty,\infty)$. The Laplace operator of $u(t)$, (the unilateral transform) [3], is in fact the operator of SF:

$$L[u]\equiv L[v]. \qquad (1)$$

The spectral density $\dot S[u]$ of signal is generally calculated by Fourier transform [3]. The expressions $\dot S[u]$ of many functions $u(t)$ are given, for instance, in [4], [5]. For many years the spectral density $\dot S[u]$ of signals $u(t)$, $t\ge 0$ has been proposed to define as:

$$\dot S[u]=\lim_{p\to j\omega} L[u]. \qquad (2)$$

However, the formula (2) is proved to be right only for damping, integrated by Fourier signals. For the *n*-power derivative of the signal $u(t)$ it is known [1], [2] that:

$$L[u^{(n)}]=p^nL[u]-\sum_{k=1}^{n}p^{n-k}u^{(k-1)}(0). \qquad (3)$$

This expression is the operator of SF as $u^{(n)}(t)h(t)$, but is not same of derivative $v^{(n)}(t)$.

It is of practical interest to consider the properties of the SF as $v(t)$, their derivatives $v^{(n)}(t)$, their integrals $v^{(-n)}(t)$, and also the expressions $L[v^{(n)}], L[v^{(-n)}]$.

It turned out that the SF use makes easy substantially operational and spectral calculus of signals.

## II. PROPERTIES OF SWITCH-ON FUNCTIONS

Let us deduce the following properties of the SF:

**Property 1.** Unlike any original function $u(t)$ the SF can be differentiated infinitely.

*Example.* Let $u(t)=1t,\ v(t)=uh$, where $h=h_+(t)$, [3]. Then $u^{(1)}=1, u^{(n)}=0$ at $n>1$, but $v^{(1)}=h, v^{(2)}=\delta(t), v^{(n)}=\delta^{(n-2)}(t), n>1$, $\delta$ is the Dirac pulse [6], where $\delta(t)=\delta_+(t)$, [3].

**Property 2.** The formula $v^{(n)}$ contains information of all initial conditions: $u(0),\ u^{(1)}(0),\ \ldots,\ u^{(n-1)}(0)$.

*Example.* Let $u=1\cos\omega_0 t, v=uh$. Then $v^{(2)}=-\omega_0^2\cos\omega_0 th(t)+1\delta^{(1)}(t)$. The coefficients at $\delta^{(k)}(t)$ are the initial conditions $u^{(n-k-1)}(0)$. Herein: $u^{(0)}(0)=1, u^{(1)}(0)=0$.

**Property 3.** The formula $v^{(n)}$ allows determining the only possible antiderivatives $v^{(n-1)}, v^{(n-2)},..v$, using repeated integration as:

$$v=\int_0^t v^{(1)}dt+0=\underbrace{\int_0^t\int_0^t\ldots\int_0^t}_{n} v^{(n)}\underbrace{dtdt..dt}_{n}+0=[v^{(n)}]^{(-n)}.$$

*Example.* Let $u=1+1\sin\omega_0 t, v=uh.$ Then $u^{(1)}=\omega_0\cos\omega_0 t,$ but $v^{(1)}=\delta(t)+\omega_0\cos\omega_0 th(t).$ Hence $[u^{(1)}]^{(-1)}=1\sin\omega_0 t+C$, where C is an unknown additive. But $[v^{(1)}]^{(-1)}=(1+1\sin\omega_0 t)h(t).$

**Property 4.** Operational calculus of $v^{(n)}$ is much easier than same of derivative of the original function (3):

$$L[v^{(n)}]=p^nL[v]=p^nL[u].$$

(This property has been proved in [7]).

Example. Let $u(t)=1\cos\omega_0 t, v=uh,\ v^{(2)}=-\omega_0^2\cdot$ $\cdot\cos\omega_0 th(t)+\delta^{(1)}(t)$. Then $L[v^{(2)}]=p^3/(\omega_0^2+p^2)$, as $L[v]=L[u]=p/(\omega_0^2+p^2), L[\delta^{(1)}]=p.$

**Property 5.** Operation calculus of the integral $v^{(-n)}$ is obtained by multiplying $L[v]$ by the factor $p^{-n}$.

*Example.* Let $L[v]=p^3/(\omega_0^2+p^2)$. Then $L[v^{(-2)}]=p/(\omega_0^2+p^2)$. (Here $v$ is the SF of second derivative of the function $1\cos\omega_0 t$ and $v^{(-2)}$ is the SF as $1\cos\omega_0 th(t)$.

**Property 6.** The Delay Theorem for the SF is always valid, even upon failure to comply with the required condition [3]: $u(t)=0$ at $t<0$. Then

$$L[v(t-\tau)]=e^{-p\tau}L[v(t)], \qquad (6)$$

even though $u(t)\neq 0$ at $t<0$, $u(t-\tau)\neq 0$ at $t<\tau$. The validity of this property is related to the equalities $h(t)=0$ at $t<0$ and $h(t-\tau)$ at $t<\tau$.

*Example.* Let $u(t)=1\sin\omega_0 t$, $u(t)\neq 0$ at $t<0$. $L[u]=L[v]=\omega_0/(\omega_0^2+p^2)$. For the function $u(t-\tau)=-1\cos\omega_0 t$ at $\omega_0\tau=\pi/2$ we have $L[v(t-\tau)]=e^{-p\tau}\omega_0/\omega_0^2+p^2$.

Let us consider the use of the SF properties for operational calculus of signals, without Laplace and Fourier transform.

## III. USE OF THE SF PROPERTIES

*1. Operational calculus of any harmonic function*

This approach comprises differentiation of the function $v(t)$ and use the relation (4).

*Example.* Let $u=1\cos\omega_0 th(t)$.

Then

$$v^{(2)}={}_{+}\delta^{(1)}(t)-\omega_0^2 v, L[v^{(2)}]=p-\omega_0^2 L[v]=p^2 L[v].$$

Hence, without Laplace transform:

$$L[v]=p/(\omega_0^2+p^2).$$

*2. Spectral calculus of an impulse described by power functions*

This approach comprises an impulse $w(t)$ differentiation that reduces to the primitive time functions with known symbolic representations.

*Example.* Let $w(t)=1t^2[h(t)-h(t-\tau)]$.

Then

$$w^{(1)}=2t[h(t)-h(t-\tau)]-\tau^2\delta(t-\tau),$$

$$w^{(2)}=2[h(t)-h(t-\tau)]-2\tau\delta(t-\tau)-\tau^2\delta^{(1)}(t-\tau),$$

$$w^{(3)}=2[\delta(t)-\delta(t-\tau)]-2\tau\delta^{(1)}(t-\tau)-\tau^2\delta^{(2)}(t-\tau).$$

Hence, without Fourier transform, we have:

$$\dot S[w^{(3)}]=2(1-e^{-j\omega\tau})-2j\omega\tau e^{-j\omega\tau}-(j\omega)^2\tau^2 e^{-j\omega\tau}.$$

Using the property 5, we have: $\dot S[w]=(j\omega)^3\dot S[w^{(3)}]$. In this case we turn the operational calculus (2) into the spectral one, since the impulse $w(t)$ is an integrated signal.

*3. Operational calculus of a sustained power function of time*

This approach comprises integration of the Heaviside step function.

*Example.* $v(t)=1t^2h(t)$. Note that $h^{(-1)}(t)=th(t), h^{(-2)}=0.5t^2h(t)$. Hence $v(t)=2h^{(-2)}$. If $L[h]=p^{-1}$, we can find immediately $L[v]=2p^{-3}$. There exist the spectrum for $h(t)$ [4], [5]: $\dot S[h]=(j\omega)^{-1}+\pi\delta(\omega)$. Then, vain attempting to calculate Fourier integral we can obtain at once:

$$\dot S[v]=(j\omega)^{-2}2\dot S[h]=2(j\omega)^{-3}+2(j\omega)^{-2}\pi\delta(\omega).$$

*4. Operational calculus of an impulse using the Delay Theorem*

Let us explain this approach to the specific example, when

$$w(t)=1\sin\omega_0 t[h(t)-h(t-\tau)], \omega_0\tau=\pi,$$

i. e. we have a positive sinusoidal impulse. Here $\sin\omega_0 th(t)=v(t)$. Let us denote $\sin\omega_0 th(t-\tau)$ by $v(t-\tau)$. Now we can write down:

$$\sin\omega_0 t=\sin[\omega_0(t-\tau)+\omega_0\tau]=-\sin\omega_0(t-\tau).$$

then $w=v(t)+v(t-\tau)$. Since it is known from [1] that $L[v(t)]=\omega_0/\omega_0^2+p^2$, then according to the Delay Theorem we receive

$$L[w]=\omega_0(1+e^{-p\tau})/(\omega_0^2+p^2),$$

which implies $\dot S[w]=\omega_0(1+e^{-j\omega\tau})/(\omega_0^2-\omega^2)$.

*5. Spectral density calculus of a modulated signal*

Let us consider a modulated damped radio signal as $v_n=e^{-\alpha t}u_n(t)h(t)$, where $u_{1,2}=e^{\pm j\omega_0 t}$, $u_3=\cos\omega_0 t$, $u_4=\sin\omega_0 t$. For $v_1$ we note: $v_1=(-\alpha+j\omega_0)v_1+\delta(t)$. This implies:

$$L[v_1]=pL[v_1]=(-\alpha+j\omega)L[v_1]+1,$$

$$L[v_1]=1/(p+\alpha-j\omega_0).$$

Likewise we have: $L[v_2]=1/(p+\alpha+j\omega_0)$. Hence it follows: $L[v_3]=0.5(L[v_1]+L[v_2])$, $L[v_4]=-j0.5(L[v_1]-L[v_2])$, Then we have the spectrum $\dot S[v_4]$ as

$$\dot{S}[v_4] = -j0.5[1/(\alpha + j(\omega - \omega_0)) + \\ + 1/(\alpha + j(\omega + \omega_0))].$$

Note that an electric modulating signal can be represented as a polynomial: $k = \sum_{n=0}^{k} a_n t^n$. Thus, it is possible to obtain spectrum of an amplitude-modulated damped oscillation or of a sustained oscillation pulse.

*6. Spectral calculus of some non-integrated signals u(t), v(t)*

*Example 1*. Let us assume the signal $u(t)$ as $u(t) = u(t)h(t) + u(t)h(-t)$. For example, $1 = h(t) + +h(-t)$. Supposing that $\dot{S}[h] = (j\omega)^{-1} + \pi\delta(\omega)$, we can write down: $\dot{S}[h(-t)] = (-j\omega)^{-1} + \pi\delta(-\omega)$, where $\delta(-\omega) = \delta(\omega)$. Then we have: $\dot{S}[1] = \dot{S}[h(t)] + \dot{S}[h(-t)] = 2\pi\delta(\omega)$, that has been obtained in [4], [5] using other special approach.

*Example 2*. A signal $u = 1\sin\omega_0 t$ can be represented as: $u = \sin\omega_0 th(t) + \sin\omega_0 th(-t) = = \sin\omega_0 th(t) - \sin\omega_0(-t)h(-t) = v(t) - v(-t)$.

We know that $L[v] = \omega_0/(\omega_0^2 + p^2)$, then $L[v(-t) = L[v(t)]$, i. e. $\dot{S}[u]$ can be described by δ-impulses only. It is provided in [5]:

$$\dot{S}[u] = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0).$$

But substitution *t* by *-t* in the time function implies a sign change at ω in the formula of spectrum. Since here we have $u = v(t) - v(-t)$, then upon turning $v(t)$ into $u(t)$ the spectrum component $\dot{S}[v]$, which comprises δ-impulses, should be doubled. Hence we can rewrite the component $\dot{S}[v]$ as $0.5\dot{S}[u]$. Thereby we can obtain:

$$\dot{S}[v] = 0.5\dot{S}[u] + \omega_0/(\omega_0^2 - \omega^2).$$

Likewise, for $u = 1\cos\omega_0 t, v = uh,$ we have:

$$\dot{S}[v] = \pi/2[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] + j\omega/(\omega_0^2 - \omega^2).$$

## Conclusions

1. The SF $v(t)$ represented as $v = uh$ has a number of properties useful for operational calculus of signals, without Laplace and Fourier transform.
2. The use of the SF properties sufficiently simplifies formulas of Laplace and Fourier transform for derivatives and integrals of time functions.
3. Conforming to the type of the SF derivatives, the operational and spectral calculus of many signals may be defined without Laplace and Fourier transform.
4. The properties of SF allow determining unambiguously the original time functions by their derivatives and symbolic representations of these functions by the SF derivatives.
5. The properties of SF allow excluding the constraint of the Delay Theorem upon spectral calculus of the pulses.

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**Ю. О. Єгоршин, О. Ю. Красноусова. Властивості функцій вмикання сигналів та їх використання**

Розглянуто властивості функцій вмикання сигналів, які дозволяють визначити операторні зображення і частотні спектри багатьох сигналів без обчислення інтегралів Лапласа та Фур'є.

**Ключові слова**: теорія сигналів; символічні зображення сигналів.

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**Ю. А. Егоршин, О. Ю. Красноусова. Свойства функций включения сигналов и их использование**

Рассматриваются свойства функций включения сигналов, которые позволяют определять операторные изображения и частотные спектры множеств сигналов, без вычислений интегралов Лапласа и Фурье.

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