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USING THE THEORY OF MARTINGALES TO PROVE THE SOUNDNESS OF THE ESTIMATES OF THE PARAMETERS OF LINEAR DYNAMIC SYSTEMS

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***Abstract*—*The article is devoted to analytical methods of research of filtration algorithms in conditions of a priori uncertainty of information on statistical characteristics of state noise and measurement in linear dynamic systems. For simple objects it is possible to apply simple evaluation algorithms. In this case, the estimate of the matrix of the dynamics coincides with probability 1 to the true value, and the Kalman filter constructed on such an algorithm gives an estimate which also coincides with the probability of 1 to the estimation of the true Kalman filter. To prove the validity of the estimates, the theory of martingales was applied. Martingales and semimartingales form an important class of processes, which generalizes a class of processes with independent increments. There is a special method for the study of random processes. But in practice, the condition that all components of the matrix of the dynamics are those that can be observed gives the limit to the use of this method. The proposed technique will allow to extend the method of obtaining estimates of the parameters of linear dynamic systems in the case of an arbitrary dynamics matrix.*

Index Terms—Algorithm; filtration; matrix; probability; martingale; linear dynamical systems.

I. INTRODUCTION. THE PROBLEM STATEMENT

For simple objects, it is possible to apply simple evaluation algorithms [1]. In this case, the estimate of the matrix of the dynamics coincides with the probability of 1 to the true value, and Kalman's filter constructed on such an algorithm gives an estimate which also coincides with the probability of 1 to the estimation of the true Kalman filter. To prove the validity of the estimates, the theory of martingales was applied. Martingales and semimartingales form an important class of processes, which generalizes a class of processes with independent increments. There is a special method for the study of random processes. But in practice, the condition that all components of the matrix of the dynamics are those that can be observed gives the limit to the use of this method.

II. CONVERSION OF MEASUREMENTS WITH A NONIDENTITY OBSERVATION MATRIX

Consider a linear dynamic system, which is described by equations:

$$X_{k+1} = \mathbf{\Phi} X_k + w_k, \tag{1}$$

$$y_k = \mathrm{H} X_k + v_k, \tag{2}$$

where X is n measurable state vector of the system; $\mathbf{\Phi}$ is matrix of dynamics, having dimension $n \times n$; w_k is n dimensional vector of input noise: $M(w_k)=0$; correlation function $M(w_i w_k) = W\delta(i-k)$; y_k is p measurable vector of measurements; $\mathbf{H}$ is matrix of observations, having dimension $p \times n$; v_k is p measurable noise vector of measurements: $M(v_k)=0$; correlation function $M(v_i v_k) = V\delta(i-k)$; $M(v_i w_k^T)=0$.

System (1), (2) is sustainable, i.e. spectral radius of the matrix of dynamics $d(\mathbf{\Phi})<1$, but not all elements of the matrix of the dynamics $\mathbf{\Phi}$ can be calculated.

Apply the procedure of linear transformation of the system to construct a model with a unit matrix of observations. This procedure is described in the paper [2].

$$y_k = \mathbf{H} x_k + v_k,$$
$$y_{k+1} = \mathbf{H\Phi} x_k + \mathbf{H} w_k + v_{k+1},$$
$$\ldots$$
$$y_{k+n-1} = \mathbf{H\Phi}^{n-1} x_k + \mathbf{H\Phi}^{n-2} w_k + \ldots + \mathbf{H} w_{k+n-2} + v_{k+n-1}.$$

Let's introduce the designation:

$$y_k^* = \begin{pmatrix} y_k \\ y_{k+1} \\ \ldots \\ y_{k+n-1} \end{pmatrix}, \quad \mathbf{H}^* = \begin{pmatrix} \mathbf{H} \\ \mathbf{H\Phi} \\ \ldots \\ \mathbf{H\Phi}^{n-1} \end{pmatrix},$$

$$v_k^* = \begin{pmatrix} v_k \\ \mathbf{H}w_k + v_{k+1} \\ \ldots \\ \mathbf{H\Phi}^{n-2}w_k + \ldots + \mathbf{H}w_{k+n-2} + v_{k+n-1} \end{pmatrix}.$$

We obtain the equation of measurement: $y_k^* = \mathbf{H}^* x_k + v_k^*$. If you enter a new state vector $x_k^* = \mathbf{H}^* x_k$, then the equation of the state of the system will take the form:

$$x_{k+1}^* = \mathbf{\Phi}^* x_k^* + w_k^*, \text{ where } \mathbf{\Phi}^* = \mathbf{H}^*\mathbf{\Phi}\left(\mathbf{H}^*\right)^{-1}.$$

The model of the meter in this case is given by an equation in which all components of the state vector are observed: $y_k^* = x_k^* + v_k^*$.

If $p > 1$, then when creating a matrix $\mathbf{H}^*$ we will choose only the first ones n linearly independent lines of this matrix. In this case, for building a new measurement vector only m_0 sequential measurements $y_k, y_{k+1}, \ldots, y_{k+m_0-1}$, where m_0 is minimum filter memory, which is determined by the smallest number of measurements needed to construct an estimator of a state with a limited dispersion. Thus, the task of identifying the matrix of the dynamics of the system (1), (2) is reduced to the problem of identifying the matrix of the dynamics of a system with components of the state vector, which can all be observed.

$$X_{k+1}^* = \mathbf{\Phi}^* X_k^* + w_k^*, \quad (3)$$

$$y_k^* = X_k^* + v_k^*. \quad (4)$$

In many cases, the matrix $\mathbf{H}^*$ is unknown, since it is necessary to define matrices for its construction $\mathbf{H\Phi}^i$, $i = 1, 2, \ldots, m_0 - 1$, where m_0 is filter memory. In this case, to evaluate the elements of the output matrix $\mathbf{\Phi}$ you need to have additional relationships that connect the desired items. In the absence of any a priori information about the matrix $\mathbf{\Phi}$ identification $\mathbf{\Phi}^*$ causes the possibility of filtering and extrapolating only the component of the state vector that can be measured. If all the elements are known $\mathbf{H\Phi}^i$, then identification $\mathbf{\Phi}$ allows you to fully evaluate the state vector. When performing the above conditions, the identification problem can be simplified and its dimension reduced, since there is a possibility of unambiguous comparison (for systems with parameters that can all be observed) p measurable vector of measurement n measurable state evaluation vector. To do this, as a model of the meter with a single matrix of observations, one can choose an equation that associates a vector of a state evaluation with its true value.

$$y_k^* = \hat{x}_{k,k}^* = X_k + v_k^*. \quad (5)$$

If memory m_0 auxiliary filter is fixed and minimal, then the sequence v_k^* will be stationary. As a vector of measurements it is suggested to choose a smoothed estimate $\left(k + m_0 - 1\right)$ state of free dynamic system for the latest m_0 measurements y_k, because to determine the score $\hat{x}_{k,k}$ you need to know the back of the matrix dynamics $\mathbf{\Phi}^{i-k}$, $i < k$. So, k measurement y_k^* we put in accordance with the assessment of the state vector $\hat{x}_{k,k+m_0-1}^{(m_0)}$, which is based on the sequence $y_i \left(i = k, \ldots, k + m_0 - 1\right)$, and the new model of the meter will look like:

$$y_k^* = \hat{x}_{k,k}^{(m_0)} = X_k + v_k^*,$$

where v_k^* is estimation error $\hat{x}_{k,k+m_0-1}^{(m_0)}$, which got on the basis of m_0 measurements y_k and such that $M\left(v_k^*\right) = 0$.

The algorithm discussed above guarantees the convergence of the probability of estimates of the matrix $\mathbf{\Phi}$ to its true values. The identification of all elements of the matrix $\mathbf{\Phi}$ can't be made by converting the model of the meter if the state vector is not completely observed.

Consequently, the necessary condition for the identification of the matrix $\mathbf{\Phi}$ is the prior knowledge of some of its elements.

III. The Solidity of the Matrix of Dynamics Estimates

The estimation of the matrix of dynamics $\mathbf{\Phi}$ is calculated by the formula:

$$\hat{\mathbf{\Phi}}_k = \left(\sum_{i=2+m_0}^{k} y_i^* y_{i-m_0-1}^{*T}\right)\left(\sum_{i=2+m_0}^{k} y_{i-1}^* y_{i-m_0-1}^{*T}\right)^+,$$

where $^+$ is pseudo-inversion, m_0 is Kalman filter memory is the dimension of the vector of measurements. We will use the method of demonstrating the strong ability of this estimate, which is proposed in [1] for a system with components that can all be observed. Let

$$z_k^* = v_k^* + w_{k-1} - \Phi v_{k-1}^* = y_k^* - \Phi y_{k-1}^*,$$

$$R_k = \left(\sum_{i=2+m_0}^{k} z_i^* y_{i-m_0-1}^{*T}\right)\left(\sum_{i=2+m_0}^{k} y_{i-1}^* y_{i-m_0-1}^{*T}\right)^+.$$

To prove that with probability 1 at $k \to \infty$ $\hat{\Phi}_k \to \Phi$ ($\hat{\Phi}_k = \Phi + R_k$) let's show that

$$\frac{1}{k}\sum_{i=2+m_0}^{k} z_i^* y_{i-m_0-1}^{*T} \to 0, \qquad (6)$$

$$\frac{1}{k}\sum_{i=2+m_0}^{k} y_{i-1}^* y_{i-m_0-1}^{*T} \to \Psi, \qquad (7)$$

where Ψ is some nondegenerate matrix. The proofs of (6) and (7) will be sufficient to prove convergence with probability 1. For proof (6), it must be shown that there is a boundary between the covariance matrix

$$y_k^* = \Phi^k X_0 + v_k^* + \sum_{j=0}^{k-1}\Phi^j w_{k-j-1}.$$

Really,

$$M\left(y_k^*\right) = M\left(\Phi^k X_0 + v_k^* + \sum_{j=0}^{k-1}\Phi^j w_{k-j-1}\right) = \Phi^k M\left(X_0\right) + 0 + 0 = \Phi^k M\left(X_0\right),$$

$$\operatorname{cov}\left(y_k^*\right) = V^* + \sum_{j=0}^{k-1}\mathbf{\Phi}^j W\left(\mathbf{\Phi}^j\right)^T + \mathbf{\Phi}^k \operatorname{cov}\left(X_0\right)\left(\mathbf{\Phi}^j\right)^T \to V^* + \sum_{j=0}^{k-1}\mathbf{\Phi}^j W\left(\mathbf{\Phi}^j\right)^T$$

if $k \to \infty$, because it is known that $\lim_{k\to\infty}\frac{1}{n}\left\|\mathbf{\Phi}^k\right\| = d\left(\mathbf{\Phi}\right) < 1$. Then, using already proven, we get:

$$\sum_{i=2+m_0}^{k} z_i^* y_{i-m_0-1}^{*T} = \sum_{i=2+m_0}^{k}\left(v_i^* + w_{i-1} - \mathbf{\Phi}v_{i-1}^*\right) y_{i-m_0-1}^{*T} = \sum_{i=2+m_0}^{k} v_i^* y_{i-m_0-1}^{*T} + \sum_{i=2+m_0}^{k}\left(w_{i-1} - \mathbf{\Phi}v_{i-1}^*\right) y_{i-m_0-1}^{*T}.$$

Consider sequences:

$$S_{k,1} = \sum_{i=2+m_0}^{k}\frac{1}{i} v_i^* y_{i-m_0-1}^{*T},$$

$$S_{k,2} = \sum_{i=2+m_0}^{k}\frac{1}{i}\left(w_{i-1} - \mathbf{\Phi}v_{i-1}^*\right) y_{i-m_0-1}^{*T}.$$

Let $\mathbf{R}^p$ is real linear space of vector of dimensional columns p, and $\mathbf{L}^p$ is linear space of dimension matrices $p \times p$ with valid elements. If $x \in \mathbf{R}^p$, then $\|x\| = \sqrt{x^T x}$, and if $G \in \mathbf{L}^p$, then $\|G\| = \sqrt{\operatorname{tr}(G \cdot G^T)}$. F_k is the smallest σ algebra, in relation to which quantity $x_0, v_1^*, \ldots, v_k^*, w_1, \ldots, w_k$ are measurable. We use the theory of martingales for further proof. The set of actual values $x(t)$, $(t \in T)$ is: artingale provided that $M\left(x(t)\right)$, $(t \ge 0)$ exists and is a constant value; supermartingale, if $M\left(x(t)\right)$, $(t \ge 0)$ does not increase.

Consequently, we show that the sets of real variables $\left\{a^T S_{k,1} b \middle| F_k\right\}$ and $\left\{a^T S_{k,2} b \middle| F_k\right\}$ are martingales ($a, b \in \mathbf{R}^p$).

Similarly, we prove that $\left\{a^T S_{k,2} b \middle| F_k\right\}$ is martingale. So if $\left\{a^T S_{k,i} b \middle| F_k\right\}_{i=1,2}$ are martingales for which $M\left(a^T S_{k,2} b \middle| F_k\right) < \infty$, then with probability 1 there is $\lim_{k\to\infty}\left(a^T S_{k,2} b \middle| F_k\right)$. Necessary to find this boundary.

$$\begin{aligned} &a^T\left(\frac{1}{k}\sum_{i=2+m_0}^{k} v_i^* y_{i-m_0-1}^{*T}\right) b \to 0, \\ &a^T\left(\frac{1}{k}\sum_{i=2+m_0}^{k}\left(w_{i-1} - \mathbf{\Phi}v_{i-1}^*\right) y_{i-m_0-1}^{*T}\right) b \to 0, \end{aligned} \qquad (8)$$

with a probability of 1 at $k \to \infty$. Taking into account mutual independence $w_1, \ldots, w_n$ for $y_1^*, \ldots, y_k^*$, we get:

$$M\left(a^T S_{k,1} b\right)^2 = \sum_{i=2+m_0}^{k}\frac{1}{i^2} M\left(a^T\left(v_i^* y_{i-m_0-1}^{*T}\right) b\right)^2 = \sum_{i=2+m_0}^{k}\frac{1}{i^2}\left(a^T V a\right) b^T M\left(y_{i-m_0-1}^* \cdot y_{i-m_0-1}^{*T}\right) b,$$

is the quantity is limited. Similarly, the value

$$M\left(a^T S_{k,2} b\right)^2 = \sum_{i=2+m_0}^{k}\frac{1}{i^2}\left(a^T\left(W + \mathbf{\Phi}V\mathbf{\Phi}^T\right) a\right) b^T M\left(y_{i-m_0-1}^* \cdot y_{i-m_0-1}^{*T}\right) b,$$

is also limited and the relation (5) follows from the fact that the vectors a, b are arbitrarily selected. Consequently, the assertion (6) is proved. In order to prove the statement (7) we use the fact that y_k^* is almost rolling average for v_i^* and w_j. Let $w_{-1}, w_{-2}, \ldots$ is a sequence of independent random variables that have the same distribution as that $w_1, \ldots w_k$, and while the elements of this sequence are independent of $x_0, v_1^*, \ldots, v_k^*, w_1, \ldots w_k$. Such a sequence can always be constructed by expanding the probability space.

Using equality $y_k^* = u_k - q_k$, where

$$u_k = v_k^* + \sum_{j=0}^{k-1} \mathbf{\Phi}^j w_{k-1-j},\ k \ge 1,$$

$$q_k = \mathbf{\Phi}^k \left(\sum_{j=0}^{k-1} \mathbf{\Phi}^j w_{-j-1} - x_0 \right) = \mathbf{\Phi}^k q_0.$$

Using the condition $d(\Phi) < 1$ is showed that u_k, q_k are stationary sequences and u_k is sequence of moving average values v_i^* and w_j, and $|q_k|$ coincides with $M(|q_k|) = 0$ not only in the averaged quadratic, but with a probability of 1 at $k \to \infty$.

IV. Conclusions

The method of equivalent transformation of the correlation model of the object and the meter is proposed. The method allows to generalize the known methods for identifying the parameters of the model, in which it is impossible to observe all components of the state vector. The extension of the method applies to the case of a non-unique matrix of observations. The algorithm of estimation of dispersion of noise of model is considered. It is shown that the algorithm coincides with probability 1. Hence, it is proved that the simulation of linear dynamic systems can be performed also for the case of the matrix of dynamics with parameters that are not all observable. This solution can be used to simulate and predict the trajectories of aircraft with variable values of their parameters.

References

[1] W. Anderson, et al., “Consistent Estimates of the Parameters of a Liner System, *Ann. Math. Statist.,* 40, 1969, pp. 2064–2075.

[2] D. Grop, *Methods for identifying systems*, Moscow: Mir, 1979. (in Russian)

[3] M. Zgurovsky, V. Podladchikov, *Analytic methods of Kalman filtration for systems with a priori uncertainty*, Kiev: Science opinion, 1995. (in Russian)

[4] P. Siminelakis, “Martingales and Stopping Times Use of martingales in obtaining bounds and analyzing algorithms”, http://www.corelab.ece.ntua.gr/courses/rand-alg/slides/Martingales-topping_Times.pdf

[5] “Martingale”, *Encyclopedia of Mathematics.* https://www.encyclo-pediaofmath.org /index.php /Martingale

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В. І. Сущук-Слюсаренко, Н. А. Рибачок., Л. М. Олещенко. Використання теорії мартингалів для доведення ґрунтовності оцінок параметрів лінійних динамічних систем

Статтю присвячено аналітичним методам дослідження алгоритмів фільтрації в умовах апріорної невизначеності інформації про статистичні характеристики шумів стану і вимірювання в лінійних динамічних системах. Для доведення ґрунтовності оцінок було застосовано теорію мартингалів. Запропонована методика дозволить поширити метод отримання оцінок параметрів лінійних динамічних систем на випадок довільної матриці динаміки.

Ключові слова: алгоритм; фільтрація; ймовірність; мартингал; лінійні динамічні системи.

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В. И. Сущук-Слюсаренко, Н. А. Рыбачок., Л. М. Олещенко. Использование теории мартингалов для доказательства основательности оценок параметров линейных динамических систем.

Статья посвящена аналитическим методам исследования алгоритмов фильтрации в условиях априорной неопределенности информации о статистические характеристики шумов состояния и измерения в линейных динамических системах. Для доказательства основательности оценок было применено теорию мартингалов. Предложенная методика позволит распространить метод получения оценок параметров линейных динамических систем на случай произвольной матрицы динамики.

Ключевые слова: алгоритм; фильтрация; вероятность; мартингал; линейные динамические системы.

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