

Destabilizing Optimal Trading Strategies in the Stock Market

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Abstract

A simple dynamic general equilibrium model of trading strategies is presented, where investors apply optimal investment strategies with stock and bond based on Merton (1971). Our investors employ two dynamic asset allocation strategies: Constant Proportion Portfolio Insurance (CPPI), and Constant Mix (CM) strategy, being demand and supply strategies of portfolio insurance in the underlying assets markets. We show that it is optimal for the portfolio insurer to buy shares, and for the CM investor to sell shares as their portfolios increase in value due to liquidity shocks, (and vice versa) enabling us to define a Walrassian clearing mechanism for shares periodically. The resulting equilibrium price dynamics is a generalized diffusion that may oscillate at high volatilities if portfolio insurance dominates the market, but the price path can obtain low variability if the contrarian investors dominate. The aggregate effect of the low risk-averse, portfolio insurers increase toward their planning horizon, resulting in an increasing price path, trade volume, and return volatility as the horizon gets closer. In spite of their destabilizing effect on equilibrium prices, regulatory agencies cannot prohibit such strategies since they stem from optimal asset allocation rules between stocks and bonds that each individual investor can apply, while the aggregate, unobservable values matters.

Key words: Destabilizing Trading Strategies, Portfolio Insurance.

Introduction

In a seminal paper, Merton (1971) showed that the optimal asset allocation strategy for a price-taking investor with Hyperbolic Absolute Risk Aversion (HARA) utility function is linear in wealth. Since in his model, the stock price process is an exogenous diffusion and the investor is atomistic, Merton solves for the optimal investment strategy made by the investor. In such an economy, strategies cannot affect prices; prices are determined by the assumption of lognormal return distribution and HARA utility. However, Kedar-Levy (2002) proves that under the HARA utility function used by Merton, the level of the investor's Relative Risk Aversion (RRA) determines whether the investor's next-period demand for units of shares is positive or negative. Investors with RRA lower than the market price for variance risk will demand units of shares when the stock price increases in order to make their optimal allocation. Investors whose RRA is greater than the market price for variance risk must sell units of shares in order to make the optimal asset allocation. Assume, for example, that all investors have an RRA lower than the market price for variance risk and they anticipate a price increase over the next period. All investors want to buy shares, but there is none for sale; hence, the price will increase to reflect the value of the information, but there will be no trade. Alternatively, assume that some of the investors' RRA is higher than the market price for variance risk and they want to sell units of shares given the same information (this is optimal for them since the anticipated price increase makes the value of their existing shares higher than their optimal holdings should be, thus they must sell units of shares.) In this case, the stock price will change and there will be trade. Yet, we argue that investors cannot map the quality of the information to a unique price, and definitely not to agree, instantaneously that this is the "fair" price. Rather, investors will submit to the market a vector of desired trades in units of shares, conditional on the market-clearing price, such that the value of stock they will have in their portfolio will satisfy optimal asset allocation whatever the equilibrium price will be. This basic idea allows for trade and an explicit price revelation within a dynamic asset-pricing model. The present paper is aimed at asking: how will volatility and trade volume vary under different assumptions on investors' utility parameters.

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Kedar-Levy (2002) showed that investors with low RRA (i.e., lower than the market price for variance risk) actually implement a Constant Proportion Portfolio Insurance (CPPI) strategy, while those with high RRA implement a Constant Mix (CM) strategy. CPPI is a portfolio insurance strategy that investors implement with stocks and bonds, rather than options, aimed to assure that their total wealth will not decline under a pre-specified value, the Floor. Since it implies trade in the same direction of the price change, it is referred to as a “momentum” or “trend-chasing” strategy. The CM strategy can be considered the opposite of CPPI in terms of periodic trade: it indicates trading in an opposite direction to price changes, hence denoted “Contrarian.”

In this paper we demonstrate that the lower the RRA parameter of both investors, but particularly of the CCPI investors will be, the equilibrium price volatility will increase to levels typically associated with unstable episodes in the stock market, such as financial bubbles. Since the asset-allocation, and trade decisions are made independently by each individual investor, the relative wealth managed by CCPI and CM strategies and the RRA applied by each investor are implicit in the overall marketplace. This lack of complete structural knowledge might result in episodes of apparent instability and high volatility in the stock market due to over-implementation of portfolio insurance, as indeed has been documented after the October 1987 market crash in the US. (A partial list may include Brennan and Schwartz, 1989, Black and Perold, 1992, Donaldson and Uhlig, 1993, Grossman and Zhou, 1996 and many others.)

If portfolio insurance is applied by institutions, one can empirically reveal the strategy exposed. Tracking individual investors' trades however, is more difficult because of the lack of trade data at the individual investor level. If one assumes that institutions implement investment strategies that reflect the preferences of the individual investors who deposited their funds with the institutions, then the institutions' strategies are informative. Empirical evidence indeed suggests that most institutional investment patterns may be classified as either momentum or contrarian (e.g., Badrinath and Wahal, 1999, Grinblatt, Titman and Wermers, 1995, Lakonishok, Shleifer and Vishny, 1992.) The aggregate value of implemented portfolio insurance strategies cannot be determined since much of it has been executed through derivatives, but there appears to be an agreement among researchers that the popularity of portfolio insurance increased dramatically before the October 1987 market crash. Many researches and two investigation committees concluded that portfolio insurance has probably played an important role in cascading stock price decline¹. No external political or economic event occurred between Friday and Monday capable of justifying such a collapse. Grossman (1988a, 1988b) argues that program trading techniques, designed for market index arbitrage by trading futures, generally should not increase the underlying asset's volatility. However, investors attempting to execute portfolio insurance strategies in the stock and bond markets do increase share price volatility. If the derivative assets market had been more developed, argues Grossman, the underlying assets markets would not have been affected.

Brennan and Schwartz (1989), present a single period model of an economy in which financial assets are traded continuously by some “normal” agents and some portfolio insurers. They conclude that return volatility and risk premium increase as the proportion of portfolio insurers increases. In multi-period models, Black and Perold (1992), Grossman and Zhou (1996), Donaldson and Uhlig (1993), Basak (1995) and others found similar qualitative results. In all these models, portfolio insurance is implemented either through options or is justified under peculiar utility functions. In Black and Perold (1992) the utility function is linear up to the minimum consumption level and concave above it; Grossman and Zhou (1996), who use options in their model, write “... *in the presence of portfolio insurers, it is futile to try to construct the price equilibrium as if there is a representative agent with a smooth utility function*”. (p. 1397). In Basak (1995), the portfolio insurer acts as such until his insurance horizon is met, but he acts as a “normal agent” between the insurance horizon and his subsequent consumption horizon. Benninga and Blume (1986) analyze the optimality of multi-period portfolio insurance strategies with options and conclude that: “*In complete markets with continuous rebalancing of portfolios, the characteristics of the implied utility functions are so peculiar that it is doubtful that any investor would want to follow a two-date insurance strategy.*” (p. 1352). The unique feature of our model, when compared to

¹ See Brady and S.E.C. Committees, both in 1988, Kyle (1988), Rubinstein (1988), Shiller (1988).

the above, is that in our model portfolio insurance is an optimal strategy adopted by rational investors who have a smooth utility function.

The rest of this paper is made of Section 1, which describes CPPI and CM; Section 2 where we construct the model and Section 3 where we present simulations. Section 4 concludes.

1. The Nature of CM and CPPI

In both strategies discussed here, the portfolio is solely comprised of stock and bond. CPPI resembles a long position of portfolio insurance and CM is equivalent to a short position of portfolio insurance. CPPI was first documented in Perold (1986) and Black and Jones (1987), who gave it the attractive $e = mc$ formulation. According to that strategy, at each point in time ($t = 0, 1, 2, 3, \dots, T$), the portfolio insurer keeps the exposure (e_t) to stock equal to the product of a multiplier (m) and a cushion (c_t). The cushion is the difference between total assets held by the investor (W_t) and a floor (F_t), which resembles the value under which total assets should not fall. Formally,

$$e_t = mc_t = m(W_t - F_t). \quad (1)$$

Since the investor holds shares (S_t) and bonds (B_t), (1) takes the form

$$e_t = m(S_t + B_t - F_t). \quad (2)$$

An important feature of CPPI results from the fact that $m > 1$. Since the "exposure" at t is the amount held in shares, equation (2) becomes $S_t = m(S_t + B_t - F_t)$. Rearranging it we get $\frac{S_t(1-m)}{m} = B_t - F_t$. Since $m > 1$ the left-hand side must be negative for long positions of stock and bond, which implies $B_t < F_t$.

The Constant Mix strategy (CM) (Perold and Sharpe, 1988) can be described as a CPPI with a multiplier $0 < m < 1$ and a floor of zero, thus (2) becomes $S_t = m(S_t + B_t)$. According to this strategy, the investor invests a constant proportion of wealth in the risky asset, which implies that an increase in the stock value increases its proportion beyond its optimal level, thus the investor must sell some shares in order to restore the optimum proportion.

We argue that the CPPI and CM strategies are optimal for risk averse investors with HARA preferences such as

$$V(C) = \frac{(1-\varphi)}{\varphi} \left(\frac{\beta C}{1-\varphi} + \eta \right)^\varphi, \\ \text{s.t. : } \varphi \neq 1; \beta > 0; \left(\frac{\beta C}{1-\varphi} + \eta \right) > 0; \eta = 1, \text{ if } \varphi = -\infty.$$

Based on Merton's result whereby the optimal equity position is given by

$$\omega_t^* W_t = \frac{\alpha - r}{\kappa \sigma^2} \left(W_t + \frac{\eta \kappa}{\beta} \left(\frac{1 - e^{r(t-T)}}{r} \right) \right), \quad (3)$$

where $\omega_t^* W_t$ is the optimal proportion out of wealth invested in the risky asset, i.e., S_t ; the ratio between equity premium and the variance of the risky asset, divided by $\kappa = 1 - \varphi$, serves as the multiplier, $m = \frac{\alpha - r}{\kappa \sigma^2}$; and the floor is given by $-F_t = \frac{\eta \kappa}{\beta} \left(\frac{1 - e^{r(t-T)}}{r} \right)$. Hence, the

multiplier will be greater than unity when $\kappa < \frac{\alpha - r}{\sigma^2}$, and the floor will be negative if $\eta < 0$, satisfying DRRA preferences for the CPPI strategy. Otherwise, a multiplier less than unity will hold when $\kappa > \frac{\alpha - r}{\sigma^2}$ and the floor will be zero if $\eta = 0$, satisfying CRRA preferences for the CM strategy. Now that we have demonstrated the optimality of CPPI and CM, we turn to develop an equilibrium between both strategists and show how can they destabilize stock prices.

2. The Model

A. Model Formulation

Assume two groups of individual investors manage their portfolios through one of the two dynamic asset allocation rules, CPPI and CM. Investors face household consumption liquidity shocks that add/withdraw from their managed portfolio. We assume that the aggregate amount of wealth managed in the economy through each of the strategies can be represented by standardized, group-wise diffusion processes

$$\Delta \tilde{W}_{D,t} = W_{D,t-1} (\delta \Delta t + \sigma_D Z_D \sqrt{\Delta t}) \quad (4)$$

and

$$\Delta \tilde{W}_{C,t} = W_{C,t-1} (\gamma \Delta t + \sigma_C Z_C \sqrt{\Delta t}), \quad (5)$$

where $W_{K,t-1}$ are wealth as of $t-1$ of group $K=\{C,D\}$; δ and γ are the respective drift terms for groups C and D, respectively (either may be positive, negative or zero); σ_C and σ_D are the standard deviations of the respective Wiener processes; and Z_K is a normally distributed random variable that represents the aggregate shocks of group K . It is assumed that these processes are not correlated.

There is a single risky asset (possibly the market portfolio) of which N shares are traded and held by either of the two investor groups, thus $N = N_{D,t} + N_{C,t} \quad \forall t$. There is a single riskless asset, which earns a given riskless return r .

At the beginning of each period, all investors must reallocate their portfolio according to their strategies between the risky and riskless assets, given the individual liquidity shock. Therefore, the aggregate wealth at t can be given by

$$\tilde{W}_{K,t} = W_{K,t-1} + \Delta \tilde{W}_{K,t} = \tilde{S}_{K,t} + \tilde{B}_{K,t}. \quad (6)$$

Portfolio rebalancing for the CPPI strategy is made according to the following formulation,

$$P_t \tilde{N}_{D,t} = m_D (P_t \tilde{N}_{D,t} + \tilde{B}_{D,t} - F_{D,t}), \quad (7)$$

where $P_t N_{D,t} = \omega_{D,t}^* W_{D,t}$, $m_D = \frac{\alpha - r}{\kappa_D \sigma^2}$, $\tilde{W}_{D,t} = P_t \tilde{N}_{D,t} + \tilde{B}_{D,t}$, and $\tilde{N}_{D,t} = N_{D,t-1} + \Delta \tilde{N}_{D,t}$.

Notice that $\Delta \tilde{N}_{K,t}$ is the number of shares investor K must trade in order to bring her portfolio to

its optimal weights. For now, we let the floor as $-F_{D,t} = \frac{\eta_D \kappa_D}{\beta_D} \left(\frac{1 - e^{r(t-T)}}{r} \right)$.

Following the CM strategy, investors of type C will rebalance their portfolio each period according to their given preferences, which in aggregate we denote as,

$$P_t \tilde{N}_{C,t} = m_C (P_t \tilde{N}_{C,t} + \tilde{B}_{C,t}) \quad (8)$$

with an equivalent notation, and specifically, $\tilde{N}_{C,t} = N_{C,t-1} + \Delta\tilde{N}_{C,t}$.

Since the period t equilibrium price, P_t , and the number of shares each investor would optimally trade ($\Delta\tilde{N}_{D,t}, \Delta\tilde{N}_{C,t}$) such that market clearing holds are unknown at the beginning of t , they must be solved together in equilibrium. We turn now to formalize the periodic demand/supply schedules for units of shares by each strategy, starting with CPPI. Extracting P_t from (7) we get

$$P_t^s = \frac{m_D(\tilde{B}_{D,t} - F_t)}{(N_{D,t-1} + \Delta\tilde{N}_{D,t})(1 - m_D)}. \quad (9)$$

In the Price-Quantity plane this function has a positive relationship between P_t and $\Delta\tilde{N}_{D,t}$ since $m_D > 1$. We therefore refer to (9) as the "supply function" and denoted s . Taking P_t from the CM strategy (8) yields,

$$P_t^d = \frac{m_C\tilde{B}_{C,t}}{(N_{C,t-1} + \Delta\tilde{N}_{C,t})(1 - m_C)}. \quad (10)$$

In this equation there is a negative relationship between price and quantity of shares $\Delta\tilde{N}_{C,t}$ in every t , since $m_C < 1$, thus (10) could be referred to as the "demand function," and be assigned the symbol d .

Isolating the marginal changes in traded shares as a function of price

$$\Delta N_{D,t} = \frac{m_D(\tilde{B}_{D,t} - F_t)}{P_t^s(1 - m_D)} - N_{D,t-1}, \quad (11)$$

$$\Delta N_{C,t} = \frac{m_C\tilde{B}_{C,t}}{P_t^d(1 - m_C)} - N_{C,t-1}, \quad (12)$$

and impose market clearing conditions $\left\{ \begin{array}{l} \Delta\tilde{N}_{C,t} = -\Delta\tilde{N}_{D,t} \\ P_t^s = P_t^d \end{array} \right\}$. By equating (11) and (12) we get

$$\frac{m_D(\tilde{B}_{D,t} - F_t)}{P_t^s(1 - m_D)} + \frac{m_C\tilde{B}_{C,t}}{P_t^d(1 - m_C)} = N_{D,t-1} + N_{C,t-1} = N. \quad (13)$$

In equilibrium $P_t^* \equiv P_t^s = P_t^d$, which we replaced in (13) and solve for it

$$P_t^* = \frac{1}{N} \left[\frac{m_D(\tilde{B}_{D,t} - F_t)}{(1 - m_D)} + \frac{m_C\tilde{B}_{C,t}}{(1 - m_C)} \right]. \quad (14)$$

Rearranging the variables in (14) based on their definitions the dynamic equilibrium price for the shares reduces to

$$P_t^* = \frac{m_D(\tilde{W}_{D,t} - F_t) + m_C\tilde{W}_{C,t}}{N}. \quad (15)$$

Therefore, at every point in time, P_t^* is the equilibrium price for shares in a stock market with the CM and CPPI strategies. It is straightforward to show that the resulting equilibrium price path will be a generalized diffusion process with a weighted average drift term

$$\Delta P_t^* = (m_D(W_{D,t-1}\delta - F_t') + m_C W_{C,t-1}\gamma) \Delta t. \quad (16)$$

B. Possible Floor Paths

The literature assigns the CPPI floor increasing paths in time (Perold and Sharpe, 1988), constant (Black and Jones, 1987, 1988, Zhu and Kavee, 1987) or decreasing as in Merton, (1971). In our setting, the relative weight of the CPPI vs. CM strategists may change, and hence change the proportion of the floor value vs. total wealth under management, and the aggregate allocation between CPPI and CM. Additionally, changes in demographic parameters such as age cohorts, life expectancy and standard of living may change the coefficient $\eta_D \kappa_D / \beta_D$ and the horizon T over time. Since the aggregate floor path affects the equilibrium stock price-path we shall explore the increasing, constant and decreasing path alternatives for the floor.

B.1 Decreasing Floor

A decreasing floor represents an assumption that the floor is a capitalized value of some minimum periodic consumption level, $\frac{\eta_D \kappa_D}{\beta_D}$. The capitalized floor decreases in time, reaching zero at T . A decreasing floor will force share prices upwards, as given in (16).

B.2 Increasing Floor

In this alternative (subscript A), the floor, $F_{t,A} = X e^{-r(T-t)}$ ($X =$ positive constant, $X < W_{D,t}, \forall t$), will increase in time at the local derivative rate $F'_{t,A} = r X e^{-r(T-t)}$ ¹. The interesting result stemming from that definition of the floor, is that the equilibrium price path is *not* forced up or down, as opposed to the finding that price path is forced upwards when the floor decreases in time. When the floor increases in time, the upper boundary on B_t is released thus allowing share price to move up or down more freely, which may result in an increase in rate of return variability.

B.3 Fixed Floor

A third alternative for the floor definition represents a fixed amount of time-to-horizon T , (subscript B), that is, $F_{t,B} = X e^{r\tau}$, $\tau =$ constant, $X < W_{D,t}, \forall t$. In that case, the effect on bond holdings, and therefore on the stock price path, will be almost similar to the previous case, only the upper boundary on bond holdings will not be released in time since the partial derivative of this floor with respect to time is zero. A flat floor sets a fixed upper limit on bond holdings. Should the limit be effective, due to the combination of multipliers and/or relative holdings of bonds and stocks, it may keep the share prices at a level higher than would otherwise be obtained.

B.4 Demographic Changes

This alternative, where the floor exhibits oscillations resulting from changes in demographic data appears more realistic when modeling economic agents' lifetime planning. Long-term oscillations in demographic variables may affect the economic real variables gradually, but systematically as it affects CPPI through the floor. These effects on the floor are a function of cohort size and minimum consumption per capita, which varies by age. Browsing the vast amount of data, one can find that young (Ages 0-25) and old (55+) cohorts consume about 33%-50% less than prime age cohorts (25-55), depending on the average income bracket the consumer family is in (Welch, 1979). Estimating changes in population age distribution together with immigration and net population growth is not a trivial task, as one can learn from differences across estimating agencies². In order to obtain a reasonable estimation for the floor one must define and estimate minimum consumption, analyze its distribution by age and in time. Finding the proper lifetime capitalization rate over the life cycle is another issue. This task in itself deserves a dedicated study and is beyond the scope of this paper. However, in order to get a feeling of their potential impact,

¹ A similar definition of the floor has been presented in Perold and Sharpe (1988).

² The UN estimate of US population distribution (forecast to 2050) consists of three growth scenarios, while US federal agencies have a different estimate.

pseudo demographic oscillations are presented in simulations N1-N4 from which one can learn that these oscillations can result in long-term cyclical patterns of the stock prices.

3. Simulations

In all 28 simulations presented below the same periodic, random liquidity shocks, were used for each group. Share prices are then calculated in each simulation based on the relevant supply and demand functions parameters in t . This was done in order to allow proper comparison of different market conditions¹. In all simulations, assets at $t = 0$ were set equal between both groups. Additionally, the nominal initial number of shares held by group D is constant in all simulations in order to allow proper trading volume comparisons. Due to the fact that group D's portfolio composition and rebalancing amounts are sensitive to the ratio of the floor to total assets held by the group, we have additionally controlled for this ratio at the beginning of each simulation (F_0/W_D). The risk free interest rate was held fixed at 3% per annum (except where otherwise indicated), leaving changes in the floor to be made via the periodic consumption coefficient $\eta_D \kappa_D / \beta_D$ only. It should be emphasized however that exogenous changes in the risk free rate would result in a change in the floor and in share prices. An increase in the risk free rate will reduce the floor level, which will reduce shares rate of return and volatility, and vice versa.

A. Constant, Infinite and Fixed Floors

The first set of simulations in Figure 1 analyzes the fixed floor vs. the decreasing floor alternatives for different multiplier combinations. We mention that different multipliers result from different RRA parameters. The first 12 simulations (A1-A6 and B1-B6) have a $T = 500$ periods horizon, from which statistics are drawn for the initial 250 periods, since the floor in this range is flat. In all "A" simulations the multiplier m_C is equal to 0.01, representing an extremely high risk-aversion, while in all "B" simulations m_C was set to 0.95, nearly its highest possible value. The multiplier m_D is either at its lowest possible value, 1.01 or 7.50, representing low risk aversion, and in any case it was set similar between all pairs of simulation A1-B1, A2-B2, C1-D1, etc. Finally, the ratio F_0/W_D was set to 0.25 in simulations A1, A2, B1, B2; it was then set to 0.475 in simulations A3, A4, B3, B4 and to 0.95 in simulations A5, A6, B5, B6. This structure allows us to analyze the effect of each multiplier, the ratio F_0/W_D and the horizon on the stability and expected return of equilibrium stock prices.

Figure 1 demonstrates the following:

1. By comparing A1 with B1, A2-B2, A3-B3 etc., one can conclude that the higher multiplier by the CM strategy stabilizes the stock price path. The intuition is that the greater the multiplier m_C is, the greater will be the supply (demand) of shares when there is an increase (decrease) in the flow of funds to the market, resulting in a stabilized price path.
2. By comparing A1-A2, A3-A4 and A5-A6, as well as their equivalent pairs of simulation in series B, one can see that an increase in m_D will destabilize the equilibrium price path.
3. By comparing the four simulations A1, B1, A2, B2 with A3, B3, A4, B4 and with simulations A5, B5, A6, B6 one can see that an increase in the ratio F_0/W_D substantially increases the stock price volatility, potentially destabilizes the market.

¹ Characteristics of the underlying diffusion processes for the groups are: Average periodic change in group D's diffusion process is 0.00031 (0.0775 for the entire 250 periods), with a variance of periodic change 0.00032. Group C's periodic average diffusion process change is 0.00022 (0.055 for 250 periods) with a variance of 0.00025. The covariance between the periodic rate of change for both processes is 0.00013.

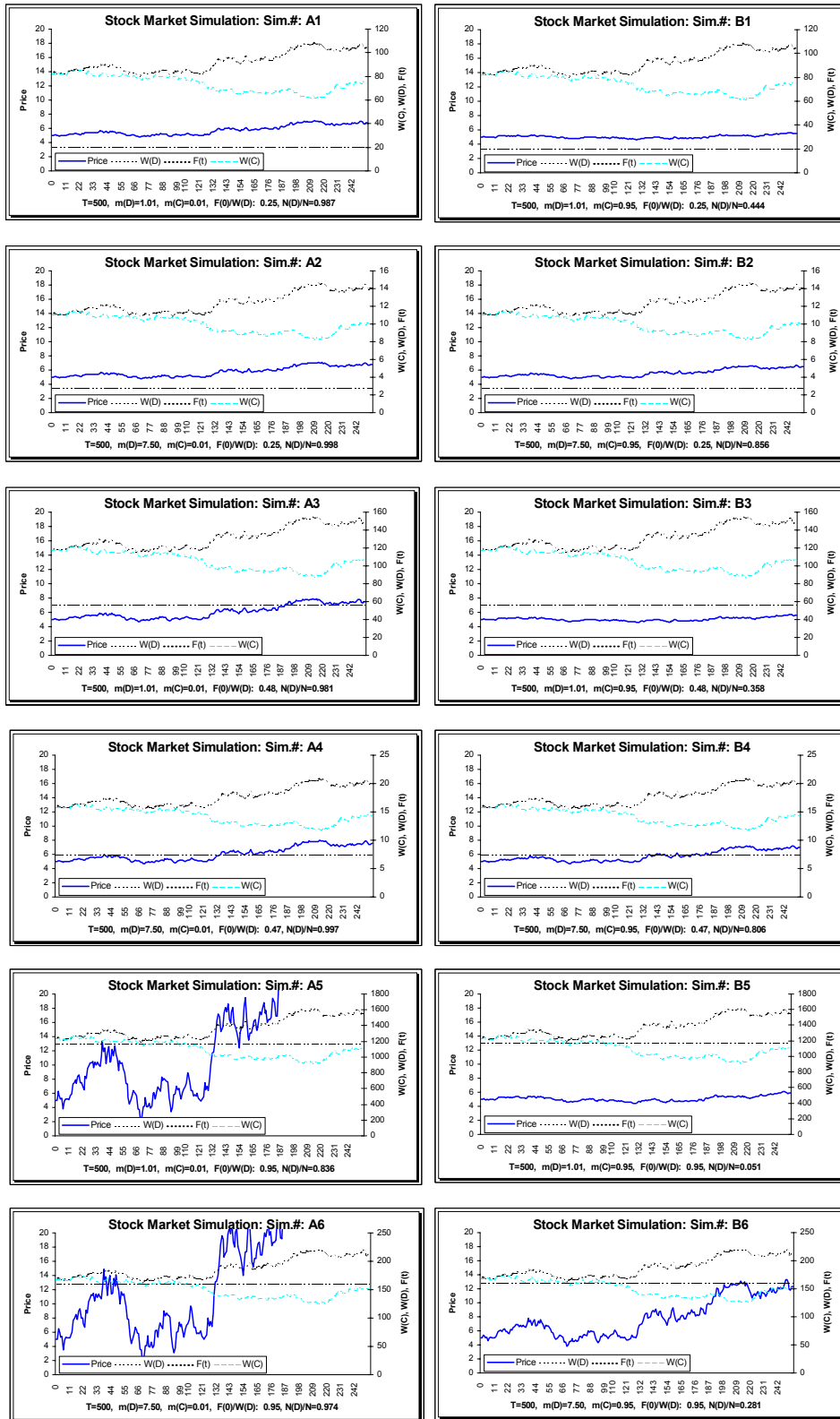


Fig. 1. Simulations A1-A6 and B1-B6: T=500 periods

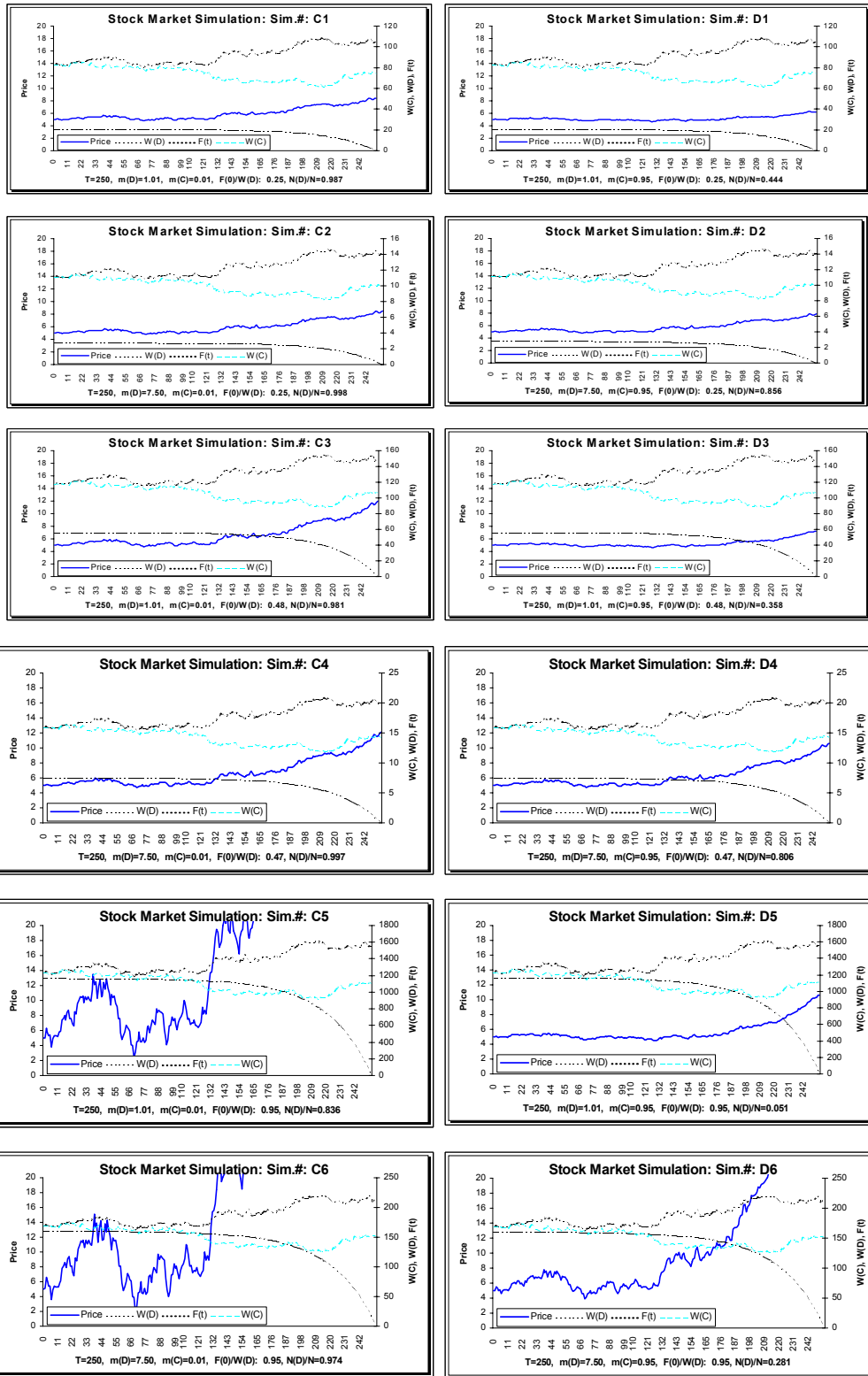


Fig. 2. Simulations C1-C6 and D1-D6: T=250 periods

Figure 2 shows the exact same conditions as in the simulations of Figure 1, only now the horizon is $T=250$, and therefore the floor of the CPPI investors declines and affects the equilibrium price. Essentially, all our conclusions with respect to Figure 1 still hold, with one important addition: as the horizon gets nearer, and the floor declines, the price path increases steadily and turns less stable. The price paths in C5 and C6 appear steeper than those in A5 and A6, and resemble boom and crash patterns. These results suggest not only that CPPI can destabilize the market, but its effect increases both with its popularity, with the decline in its investors relative risk aversion, and with the shortening horizon.

We turn now to compare the variety of market conditions in simulations A1-A6 with the corresponding conditions in C1-C6, and those in simulations B1-B6 with D1-D6, (each pair $x\#-y\#$ differs only with respect to the horizon.) This comparison yields statistics concerning the stock's rate of return (ROR) in Table 1, ROR volatility in Table 2 and absolute trading volume in Table 4.

A.1 ROR Analysis

Table 1

Periodic Average Rate of Return ($r = 3\%$)

Horizon	Floor relative to D's assets	Firm D's multiplier	Firm C's multiplier $m(C)=0.95$	Firm C's multiplier $M(C)=0.01$
T=500	F ₀ /W=25%	m(D)=1.01	0.000	0.001
		m(D)=7.50	0.001	0.001
	F ₀ /W=47.5%	m(D)=1.01	0.001	0.002
		m(D)=7.50	0.001	0.002
	F ₀ /W=95%	m(D)=1.01	0.001	0.014
		m(D)=7.5	0.005	0.017
T=250	F ₀ /W=25%	m(D)=1.01	0.001	0.002
		m(D)=7.50	0.002	0.002
	F ₀ /W=47.5%	m(D)=1.01	0.002	0.004
		m(D)=7.50	0.003	0.004
	F ₀ /W=95%	m(D)=1.01	0.003	0.018
		m(D)=7.5	0.009	0.021

Note: Data should read 0.001 = 0.1% per period. Periodic Average ROR increases with the ratio of the floor to total assets held by the CPPI trader, as well as with his multiplier. ROR decreases with the CM trader multiplier.

Main Findings:

1. ROR increases with the ratio of the floor to group D's assets. This results from the fact that the higher the ratio F_0/W_D is, the larger will be the proportional change in group D's shares, thus, a larger demand relative to the total fixed number of shares in the market.
2. ROR increases with group D's multiplier for a given ratio of F_0/W_D since an increase in share price is followed by buying more shares by group D when its multiplier is higher, and vice-versa.
3. ROR decreases with group C's multiplier, since an increase in share price is followed by selling more shares by group C when its multiplier is higher, and vice versa. Being a counter-trend strategy, the Constant-Mix PI selling strategy will support a declining market and hold back a bullish one.
4. ROR increases as T is approached (All average RORs are higher for $T = 250$ vs. $T = 500$) since group D replaces bonds with stocks.

5. When group C's multiplier is near zero, increasing m_D for a given F_0/W_D has no effect on ROR, except for very high F_0/W_D ratio. However, when m_C approaches 1.0, increasing m_D results in most cases in an increase in average ROR.

A.2 Standard Deviation of ROR Analysis

Table 2

Periodic Standard Deviation of Rate of Return ($r = 3\%$)

Horizon	Floor relative to D's assets	Firm D's multiplier	Firm C's multiplier $m(C)=0.95$	Firm C's multiplier $m(C)=0.01$
T=500	F ₀ /W=25%	m(D)=1.01	0.010	0.016
		m(D)=7.50	0.014	0.016
	F ₀ /W=47.5%	m(D)=1.01	0.012	0.021
		m(D)=7.50	0.018	0.022
	F ₀ /W=95%	m(D)=1.01	0.017	0.122
		m(D)=7.5	0.054	0.145
T=250	F ₀ /W=25%	m(D)=1.01	0.010	0.015
		m(D)=7.50	0.014	0.016
	F ₀ /W=47.5%	m(D)=1.01	0.011	0.020
		m(D)=7.50	0.018	0.021
	F ₀ /W=95%	m(D)=1.01	0.016	0.109
		m(D)=7.5	0.048	0.126

Note: Data should read 0.001 = 0.1% per period. ROR Variability increases with the ratio of the floor to total assets held by the CPPI traders, as well as with their multiplier. It decreases with the CM trader multiplier and as the CPPI horizon is approached.

Main Findings:

- ROR variability increases with F_0/W_D . This result is due to the same mechanism which increases average ROR (finding 1, Table 1).
- ROR variability increases with an increase in m_D for a given F_0/W_D . This effect is stronger when m_C approaches 1.0.
- ROR variability decreases with an increase in m_C for the same reason as in finding 3, Table 1.
- ROR variability decreases as T is approached (values for $T = 250$ are lower than for $T = 500$, mainly for high F_0/W_D). Recall that for the $T = 500$ case we only take statistics of the first half of the "lifetime," whereas in the case of $T = 250$ the entire lifetime is considered. Even in those simulations where the conditions enable price acceleration towards T , the local ROR variation tends to be small during the accelerated price increase since not much trade is possible (see below). A good example of that phenomenon is evident in simulations A1 vs. C1, A2 vs. C2 and B2 vs. D2.

A.3 Traded Volume Analysis

For a given change in wealth, with two traders and two trading volume categories per trader ("Large" and "Small" change in share ownership), trading volume analysis could be presented in a simple 2X2 matrix. Four hypotheses (marked "H") are summarized in Table 3 together with findings from the simulations (marked "F"). It appears reasonable to hypothesize that a large proportion of absolute trading volume to total number of shares in the market (hereinafter ATV/N)

will result when both traders intend to trade many shares in response to a given wealth change, as in *H1*. *ATV/N* is expected to be at a lower level compared to *H1* if one of the traders wish to trade a small number of shares in response to the given wealth change, as in *H2* and *H3*. Finally, the lowest *ATV/N* is expected in *H4*, where both traders wish to trade a small number of shares due to the given price change.

Table 3

Hypotheses (H) and Findings (F) Regarding *ATV/N* in Four Market Conditions ($T = 250$)

Change in Group C Share Ownership, $\Delta N_{C,t}$

		Large, $m(C)=0.95$	Small, $m(C)=0.01$
Change in Firm D Share Ownership, $\Delta N_{D,t}$	Large, $F_0/W=95\%$, $m(D)=7.5$	H1: Large <i>ATV/N</i> F1: 4.434	H2: Small <i>ATV/N</i> F2: 0.342
	Small, $F_0/W=25\%$, $m(D)=1.01$	H3: Small <i>ATV/N</i> F3: 1.040	H4: Very small <i>ATV/N</i> F4: 0.042

A higher level of both multipliers and ratio of F_0/W will result in a higher trading volume.

Differences are evidently large enough to conclude that trading volume is indeed a function of both multipliers and the ratio F_0/W_D . Additional simulations with different random paths will not alter these results and we may thus turn to analyze volume in Table 4.

Table 4

ATV/N in Different Market Conditions

Horizon	Floor relative to D's assets	Firm D's multiplier	Firm C's multiplier $m(C)=0.95$	Firm C's multiplier $m(C)=0.01$
T=500	F ₀ /W=25%	m(D)=1.01	1.051	0.043
		m(D)=7.50	0.437	0.006
	F ₀ /W=47.5%	m(D)=1.01	1.260	0.072
		m(D)=7.50	0.672	0.010
	F ₀ /W=95%	m(D)=1.01	2.300	2.311
		m(D)=7.5	5.230	0.439
T=250	F ₀ /W=25%	m(D)=1.01	1.040	0.042
		m(D)=7.50	0.425	0.006
	F ₀ /W=47.5%	m(D)=1.01	1.229	0.068
		m(D)=7.50	0.632	0.009
	F ₀ /W=95%	m(D)=1.01	2.108	1.870
		m(D)=7.5	4.434	0.342

Note: Data represent ratio of *ATV/N*.

Trading volume increases with both multipliers and with the ratio F_0/W . It decreases though as T is approached.

Main Findings:

1. *ATV/N* increases with F_0/W_D as group D has to protect its higher floor.

2. ATV/N increases with m_C . The reason for this finding is that group C's absence from the stock market disables trade thus volume is low for low $m(C)$ and increases with it. (A single exception for $m_D = 1.01$, $m_C = 0.01$, $F_0/W_D = 95\%$ and $T = 500$, due to the non-linearity of a 16 times smaller number of shares in the economy vs. the $m_C = 0.95$ case (2.311 vs. 2.300).)
3. ATV/N decrease with an increase in m_D for a given F_0/W_D . The reason is that a higher multiplier for D indicates to the group that it should offer/demand a larger *value* of stock for a given market condition. That forces larger price change on the account of traded volume (as seen in ROR variability findings).
4. ATV/N decreases as T is approached due to higher price levels, which allows group D to trade the same value of stock with a smaller number of shares.

B. Long-term Cyclical Price Patterns

An economy in a steady demographic state should be characterized by constant aggregate minimum consumption, given a fixed minimum consumption level percapita. However, significant changes in population distribution, such as a baby-boom or immigration waves, may generate not only an impact on aggregate minimum consumption as of the time of change, but also echo effects for generations to come. The relevance of this point to the model described here is made through the aggregate capitalized minimum consumption level $F_{D,t}$. When the total number of consumers in the economy fluctuates over time, $F_{D,t}$ will swing too. As seen earlier, in times of lower than average floor level, if and when the floor poses an effective upper limit on bond holdings, share prices will be forced to increase through the mechanism explained earlier. Equivalently, as the floor increases back, share prices will be free to decline. In the very long-term, share price level will gradually converge to a steady-state drift rate which is the weighted average of both investor types in the economy, weighted by their wealth and multipliers (see eq. (16)).

The case of long-term swings in the floor is presented graphically in simulations N1-N4 of Figure 3. It is evident from these simulations that the higher the proportion of floor to group D assets (F_0/W_D) is, the stronger the impact on share prices path will be. While the long-term path is affected by the ratio of F_0/W_D , the path variability is affected by the combination of both multipliers and the changes in floor level. (In simulations N1 and N2 the floor and group D multiplier are similar, but in N1 group C's multiplier is 0.01, while in N2 it is 0.99.) The floor impact on price path could be seen when comparing the floor in simulation N4 ($F_0/W_D = 0.95$) to N1 and N2 ($F_0/W_D = 0.8$) and all three simulations compared to N3 ($F_0/W_D = 0.44$). In the "very long" horizon, the floor will converge to a steady state level, as population distribution will stabilize (simulation N3).

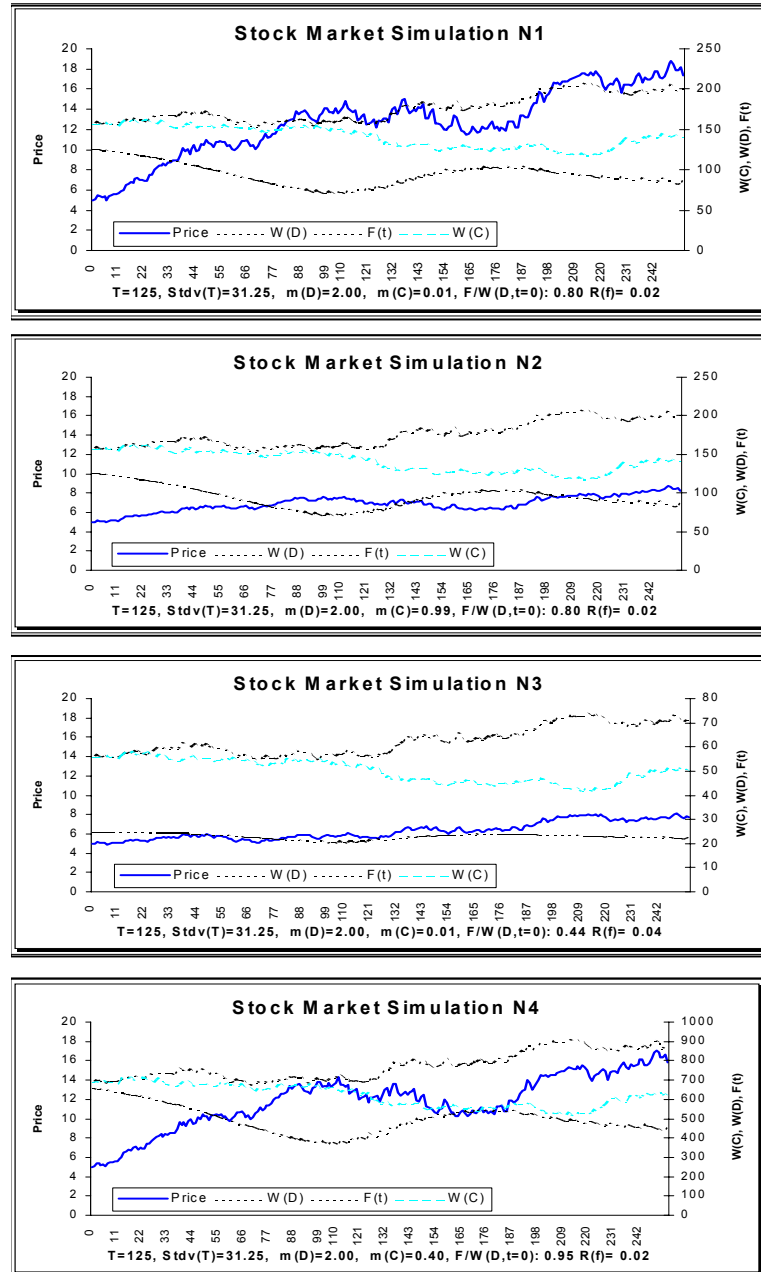


Fig. 3: Simulations N1-N4

4. Summary

This paper constructs a simple general equilibrium model of a stock market with two investor types, both have Hyperbolic Absolute Risk Aversion (HARA) utilities, but differ with respect to relative risk aversion and displacement utility factors. The investors apply individually optimal portfolio strategies that allow us to clear the market and reveal the equilibrium price process endogenously. We map the optimal portfolio rules of investors with Decreasing Relative Risk Aversion (DRRA) preferences to a Constant Proportion Portfolio Insurance (CPPI) strategy, and CRRA preferences to a Constant Mix (CM) strategy.

We show that if the funds managed by CPPI increase vs. overall market value, they will destabilize the stock price path by increased return variability and high trading volume in a fashion that will ex-post be similar to a financial bubble. On the other hand, the CM investor reduces the equilibrium price volatility by selling portfolio insurance. We also find that when the aggregate CPPI strategy approaches a fixed horizon, T , be it a "generation" change or an end of a planning period, equilibrium share prices will increase toward T at increasing volatility.

Destabilization of the equilibrium price will intensify if the CM strategists are absent from the market. This might occur either because their relative risk aversion is extremely high, which implies a very low position in the risky asset, or because a relatively low proportion of funds are managed through this strategy. The opposite holds as well: The greater the proportion of (high risk-averse) CM investors in the market is, the greater will be their stabilizing effect on equilibrium prices.

Finally, we show that demographic oscillations will result in oscillations in the equilibrium stock price-process. Empirical evidence concerning short-term random walk but long-term cyclical patterns in share prices may also be explained by the results of this model. Such will be the case if the insured floor, being a capitalized value of lifetime minimum consumption oscillates in time. Oscillations might result from changes in population age structure or standard of living. The technique used to solve this model is simpler than existing stochastic optimal control techniques yet it is powerful and provides the economic insights without mathematical complexities.

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