A NON-PARAMETRIC COPULA ANALYSIS ON ESTIMATING RETURN DISTRIBUTION FOR PORTFOLIO MANAGEMENT: AN APPLICATION WITH THE US AND BRAZILIAN STOCK MARKETS[1]

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Abstract

Parametric models on modeling stock price behavior have strict assumptions on the return distributions. However, emerging markets show non-linear risk-return relationship because of their chaotic and dynamic characteristics arising from high volatility, low volume and thin trading. From that perspective, financial modeling in the emerging markets requires flexible and non-parametric algorithms to minimize fails and variations in estimations. Additionally, the returns in emerging equity markets might be affected by those of advanced markets due to global risk appetite of the investors. In this research paper, we propose non-parametric copula models based on both Kendall's To and Spearman's Ro to model the distributional dependency between the S&P 500 and Bovespa. The empirical evidence shows that there is a strong dependency between the return distribution of the two markets and the significance of the dependency varies on the copula method applied.

Key words: Copula, Market dependency, Emerging markets, Non-parametric models.
JEL classification: C14, C51, G1.

1. Motivation

Recent increase in free flows of information and capital might create dependencies among the financial markets. To find out those dependencies is crucial for portfolio diversification aims. If there is not a high correlation between two markets, a portfolio comprising financial instruments from those markets might be used to create a diversified portfolio. Butler and Joaquin (2002) for example, show how the change in co-movement of financial markets influences the performance of a diversified portfolio without dynamic rebalancing.

In that perspective, capturing dependencies among financial markets requires a careful econometric analysis. Due to complexity of the markets, the time series of the financial instruments should be carefully examined with the proper methodology being valid for the time period and markets under consideration.

Especially in emerging markets, high financial volatility and political instability might require flexible econometric models to analyze the behaviors of financial markets. Empirical works point out that asset returns are more highly correlated during volatile markets and during market downturns. Deviation from normality might lead to inadequate profit/loss estimations (Longin and Solnik, 2001; Ang and Chen, 2002). In this research paper, we use alternative copula methods to analyze the dependency between S&P and Brazilian equity markets. Sklar (1959) describes the copula as an indicator of the dependencies among the variables. Since it does not have any assumptions on the joint distributions among the financial assets it presents a flexible methodology for econometric analysis of financial time series especially in the emerging markets. Copula creates N marginal distribution for the joint distribution with N dynamics.

The crucial indicator in a joint Gaussian distribution is the correlation coefficient. A low value between two financial markets presents a chance for portfolio diversification. However, it

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should be kept in mind that not only the degree of dependence but also the structure of dependence matters if multivariate analysis is used. Pairs of markets with the same correlation coefficient might have different dependence structures, which could lead changes in the diversification benefit compared to the Gaussian assumption (Hu, 2004).

By using daily returns of S&P 500 and Bovespa from 02.01.2002 to 08.05.2007, we test non-parametric copula methods to model the dependency between the return distributions of the two indexes. The empirical research uses those two markets, however, the aproach and methodology used in the article draw a framework for similar analysis for other financial markets. Though there are parametric copula models, we especially use nonparametric copula methods in order to create flexible metodological platform for the analysis. The empirical findings state that nonparametric copula methodologies might be used for financial market interactions but which copula methodology should be employed is a crucial point to find statistically significant results.

2. Literature Review

Dependencies among financial markets have been pointed out with different methodologies in financial econometrics. Schicsh (2004) examines dependencies between European stock markets when returns are extreme, using daily data on stock market indices for Germany, the UK, France, the Netherlands and Italy from 1973 to 2001. With multivariate extreme value theory, he finds out that dependencies between markets in situations of extreme returns have become closer over time and are generally higher for large negative returns than for large positive ones.

In terms of dependencies between advanced and emerging markets, the 1987 Asia crisis might be seen as crucial change point. Arshanapalli and Doukas (1993) state that the dependencies between the US and emerging equity markets have increased mainly because of trade and capital inflows. King and Wadhwani (1990) test the financial effects of the 1987 crash in the US stock market on the correlations among the markets in the US, the UK and Japan. Lee and Kim (1993) examine its effect on twelve equity markets. Calvo and Reinhart (1996) study the effects of Mexican peso crisis in 1994 on the advanced financial markets. Ozun (2007a) also tests the effects of US interest rates on the major advanced and developing financial markets and shows that interest rates yield curve in US has also effective in returns of world equity indexes. Common empirical result of the research papers is that the interrelations among the advanced and developing markets are strong during the turbulences and the degree of the relations varies in accordance with the econometric model employed.

Berument et al. (2006) argue that a kind of center-periphery relation has been emerging between the advanced and developing stock markets. By using the VAR model with block exogeneity, they show that S&P 500 returns, representing the center, affect the stock returns in the emerging markets either instantaneously or with a time lag depending on their geographical location.

Ozun (2007b) tests the effects of volatility in the advanced stock markets onto the stock returns in emerging markets. By using daily values of Bovespa, ISE-100 and Nikkei-225, FTSE-100, Xetra Dax, CAC-40, S&P-500 and Nasdaq from 03.01.2002 to 06.11.2006, he shows that volatilities in the US and European stock markets have strong impact on Bovespa in positive direction. The first lags of the EU markets have also explanatory power due to time difference among the markets. Japanese stock market has also statistically significant positive effect on the Bovespa but at a lower level.

Though they have a long history in mathematics and statistics, copula methods have been employed in finance recently. However, an intensive research in finance has been conducted in recent years by using copula methods. Due to its flexibility in return distribution assumptions, it has been seen a useful methodology in financial econometrics. Since financial markets are so dynamic and chaotic, it is preferred to use limited restrictions on methodological modeling without tolerance the strength of the econometric discipline.

Frey and McNeil (2003) and Hamerle and Raosch (2005) employ the copula method for the option valuation. Junker et al. (2006) use it for modeling the term structures of the interest rates. Li (2000) uses copulas to estimate the default correlations in credit risk models. Giesecke

(2004) and Meneguzzo and Vecchiato (2004) also use different copula methods in the credit risk analysis. Bennett and Kennedy (2003) use copula for pricing foreign exchange rate, and Patton (2006a, 2006b) employs it for modeling time-varying dependence in exchange rates. Ozun and Cifter (2007a, 2007b) use time-varying copula for portfolio risk estimations in the Latin American markets and Asian markets. Other valuable copula applications in finance can be found in the researches of Cherubini et al. (2004), Fortin and Kuzmics (2002), Embrechts et al. (2003), Chen and Fan (2002), Fermanian and Scaillet (2003) and Rosenberg (2003).

Copula methods have been used to examine co-movements among financial markets, as well. Longin and Solnik (2001) employ a Gumbel copula to estimate the extreme correlations across international equity markets. Mashal and Zeevi (2002) predict the degree of freedom in a t copula and translate the model to test the Gaussian assumption in financial markets. Ozbakis (2006) uses different copula methods to test the correlations between Bovespa and Istanbul Stock Exchange. In this research paper, we follow a similar methodology in detecting distributional dependency between Bovespa and S&P 500. Therefore, a more methodological discussion can be found in Ozbakis (2006). Bartram et al. (2004) use a time-varying copula model to investigate the impact of the Euro on the dependence between seventeen European stock markets during the period of 1994-2003.

In this paper, we try to model the return distribution dependency between the US stock markets and Brazilian stock markets by using alternative copula methods with nonparametric parameters. As much as we know, it is the first paper on examining the common distributional characteristics of the two markets with non-parametric copula methods and presenting comparative empirical results of the different copulas.

3. Methodology

A copula is a function connecting the marginal distributions to restore the joint distribution. Different copula functions represent different dependence structures among the variables. For that reason, in a copula analysis, the crucial methodological issue is to find a proper copula function and a related estimation process.

3.a. Mathematics of The Copula Functions

It might be useful to give mathematical descriptions of bivariate and multivariate copula functions in this section. The methodological roots of copula functions go to the Sklar (1959) theorem. When the function of C: $[0, 1]^n \rightarrow [0, 1]$ satisfies the following conditions, it is called as copula with n dimension (Nelsen, 1999):

i) for $i \in \{1, 2, ..., n\}$ and $u_i \in [0, 1]$; at least one of u_i is equal to zero; $C(u_1, u_2, ..., u_n) = 0$;

ii) if all the coordinates except from u_k is 1.; $C(1, 1, ..., u_k, ..., 1) = u_k$;

iii) for all $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \in [0,1]^n$, $a_i \leq b_i$ and for each

$j \in \{1, 2, ..., n\}$ $u_{j1} = a_j$ and $u_{j2} = b_j$; $\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+i_2+...+i_n} . C(u_{1i_1}, ..., u_{ni_n}) \geq 0$.

In a similar way, when the function of C: $[0, 1]^2 \rightarrow [0, 1]$ satisfies the following conditions, it becomes a copula with two dimensions. The notations are from Nelsen (1999) where more detailed methodology on pure copula functions can be found.

i) $C(u_1, u_2, ..., u_n) = 0$ $C(u, 0) = 0$; $u, v \in [0, 1]$;

ii) $C(1, v) = v$, $C(u, 1) = u$; $u, v \in [0, 1]$;

iii) when $a_1 \leq a_2$, $b_1 \leq b_2$ and $a_1, a_2, b_1, b_2 \in [0, 1]$; $C(a_1, b_1) + C(a_2, b_2) - C(a_1, b_2) - C(a_2, b_1) \geq 0$.

3.b. Dependency Parameters

We assume that there is a random sample of $(X_1, Y_1),...,(X_n, Y_n)$, each of them has continuous distribution function. Our aim is to find a distribution function with two variables, $H(x, y)$ presenting the common behavior of that sample pairs.

When $F(x)$ and $G(y)$ are continuous, there is a copula satisfying that $H(x, y) = C(F(x), G(y)); \ x, y \in R$. Accordingly, common dependency between X and Y can be characterized with a copula function. For the sample copula indicating the common dependency between the variables, C_n will be used as a notation. From that perspective, the dependency can be measured both for sample and theoretically with C_n and C, respectively. In this research paper, we combine two nonparametric dependency measures, namely Spearman's Ro and Kendall's To with alternative copula functions. Before explaining these parameters, it might be helpful to remind the Pearson correlation parameter, which provides a basis for Spearman's Ro and Kendall's To.

By assuming that the random vector of $(X, Y)^T$ has a finite variance and its components are different from zero, we can define Pearson correlation parameter r as,

$$r = \frac{Cov(X, Y)}{\sqrt{Var(X) \ Var(Y)}} . \tag{1}$$

In copula nominations, it can be written as in Equation (2) (Schweizer and Wolff, 1981).

$$r = \frac{1}{\sqrt{Var(X)Var(Y)}} \int_0^1 \int_0^1 [C(u, v) - uv] dF_X^{-1}(u) dF_Y^{-1}(v) . \tag{2}$$

Pearson correlation measures the linear dependency between the variables. The variance of the linear combination is totally determined with the covariances among the variables. Correlation might be proper as a natural numerical measure in ecliptic distributions. However, mostly random variables do not have common ecliptic distributions and using linear correlation might be false in such situations (Acmaz, 2005).

Copula Definition of The Spearman's Ro

By imitating Pearson's approach, we can calculate correlations between the number pairs of (R_i, S_i) or $(\frac{R_i}{n+1}, \frac{S_i}{n+1})$ which is the support sample of C_n. From that imitation, by assuming that $\bar{R} = \frac{1}{n}\sum_{i=1}^n R_i = \frac{n+1}{2} = \frac{1}{n}\sum_{i=1}^n S_i = \bar{S}$; we can reach Spearman's Ro as given in Equation (3).

$$\rho_n = \frac{\sum_{i=1}^n (R_i - \bar{R})(S_i - \bar{S})}{\sqrt{\sum_{i=1}^n (R_i - \bar{R})^2 \sum_{i=1}^n (S_i - \bar{S})^2}} \in [-1, 1] . \tag{3}$$

A more easy presentation of Spearman's Ro can be followed in Equation (4).

$$\rho_n = \frac{12}{n.(n+1).(n-1)} . \sum_{i=1}^n R_i . S_i - 3.\frac{n+1}{n-1} . \tag{4}$$

The Spearman's Ro shares the same form with the sample correlation parameter r_n. Like r_n, ρ_n is zero, if the variables are independent. However, it has some strength over the correla-

tion parameter in terms of econometric modeling. For example, r_n can not be used with fat-tails distributions. In addition, the equation of $E(\rho_n) = \pm 1$ is satisfied only if there is a functional dependency between X and Y, in other words, the copula of (X,Y) is equal to the two limits, M and W of Fréchet-Hoeffding. On the other hand, the equation of $E(r_n) = \pm 1$ is satisfied only if X and Y is a linear function of the other, which is a very important restrictions in terms of financial modeling (Genest and Favre, 2007).

Copula Definition of The Kendall's To

For a random sample of (X_i, Y_i); $i = 1,2,...,n$, Kendall's To is calculated as

$$\tau_n = \frac{P_n - Q_n}{\binom{n}{2}} = \frac{4}{n(n-1)} P_n - 1, \tag{5}$$

where P_n is the proper pairs, Q_n is the improper pairs. The (X_i, Y_i) and (X_j, Y_j) are proper if $(X_i - X_j).(Y_i - Y_j) > 0$. If $(X_i - X_j).(Y_i - Y_j) < 0$, the pairs are labeled as improper. Since X and Y are continuous, we never observe $(X_i - X_j).(Y_i - Y_j) = 0$. Since the condition of $(X_i - X_j).(Y_i - Y_j) > 0$ only holds if $(R_i - R_j).(S_i - S_j) > 0$; we can say that τ_n is a function of ordinal numbers. In addition, τ_n is a function of sample copula of C_n. As proposed by Hoeffding (1948), since $C_n \to C$, the following result described in Equation (6) can be received.

$$\tau_n = 4\frac{n}{n-1}\overline{W} - \frac{n+3}{n-1} \to \tau = 4 \int_{[0,1]^2} C(u,v)dC(u,v) - 1. \tag{6}$$

From the Equation (6), we conclude that τ_n is the asymptotic unbiased estimator for the density parameter, τ (Genest and Favre, 2007).

Since the distribution of τ_n reaches to a normal distribution with zero mean and $2(2n+5)/\{9n(n-1)\}$ variance, the dependency test of $H_0 : C = \Pi$ can be conducted based on τ_n. For example, Genest and Favre (2007) state that since $\sqrt{\frac{9n(n-1)}{2(2n+5)}}|\tau_n| > 1,96$; the H_0 hypothesis is rejected for 95 % level.

3.c. Selected Copula Families

In this step, we explain how to estimate nonparametric dependency parameters with copula families. Frees and Valdez (1998) give a well-prepared guide for the relationship between the algorithms of copula families and dependency parameters as given in Table 1.

Table 1

Parameters and Their Intervals For Selected Copula Families

Copula Family	Parameter Interval	Kendall's To	Spearman's Ro
Ali-Mikhail-Haq	$-1 \leq \theta \leq 1$	$\left(\dfrac{3\theta-2}{\theta}\right) - \dfrac{2}{3}\left(1-\dfrac{1}{\theta}\right)^2 \ln(1-\theta)$	-
Clayton	$\theta \geq 0$	$\dfrac{\theta}{\theta+2}$	-
Farlie-Gumbel-Morgenstern	$-1 \leq \theta \leq 1$	$\dfrac{2}{9}\theta$	$\dfrac{1}{3}\theta$
Frank	$\theta \neq 0$	$1 - \dfrac{4}{\theta}[D_1(-\theta)-1]$	$1 - \dfrac{12}{\theta}[D_2(-\theta) - D_1(-\theta)]$
Gumbel-Hougaard	$\theta \geq 1$	$1 - \theta^{-1}$	-
Normal (Gaussian)	$-1 \leq \theta \leq 1$	$\dfrac{2}{\pi}\arcsin(\theta)$	$\dfrac{6}{\pi}\arcsin\left(\dfrac{\theta}{2}\right)$
Plackett	$\theta \geq 0$	-	$\dfrac{(\theta+1)}{(\theta-1)} - \dfrac{2\theta\ln\theta}{(\theta-1)^2}$

D_1 and D_2 are the Debye functions and expressed as in Equation (7).

$$D_k(x) = \frac{k}{x}\int_0^x \frac{t^k}{e^t - 1}dt \,. \tag{7}$$

To explain the copula methods based on Kendall's To and Spearman's Ro, it might be proper to choose a copula family, Farlie-Gumbel-Morgenstern for example, and give examples for the usage of Kendall's To and Spearman's Ro with it.

Kendall's To and Copula Algorithms
Let's take the following copula function.

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v)\,, \ u,v \in [0,1]\,, \ \theta \in [-1,1] \,. \tag{8}$$

The theoretical value of Kendall's To is found $\tau = 2\theta/9$, and θ is a real number. Since we aim at estimating θ, we can say that $\widetilde{\theta}_n = \dfrac{9}{2}\tau_n$. Genest and Rivest (1993) show that as

$$\widetilde{W}_i = \frac{1}{n}\sum_{j=1}^n I_{ji} = \frac{1}{n}\#\{j: X_i \leq X_j, Y_i \leq Y_j\} \text{ and } S^2 = \frac{1}{n}\sum_{i=1}^n \left(W_i + \widetilde{W}_i - 2\overline{W}\right)^2 ;$$

$$\sqrt{n}\,\frac{\tau_n - \tau}{4S} \approx N(0,1)\,.$$

By using Delta method, we can argue that if $n \to \infty$, then, $\widetilde{\theta}_n \approx N(\theta, \dfrac{1}{n}\{4S.g'(\tau_n)\}^2)$. In this framework, for θ, the confidence interval of $1-\alpha$ confidence parameter is taken a form described in Equation (9).

$$\widetilde{\theta}_n \pm z_{\alpha/2} \frac{1}{\sqrt{n}} 4S . \left| g'(\tau_n) \right| . \tag{9}$$

For the Archimedean copula families, Genest ve Mackay(1986) show that

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi^{/}(t)} dt . \tag{10}$$

Spearman's Ro and Copula Algorithms

We can examine that method again by using Farlie-Gumbel-Morgenstern copula described as $C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$, $u,v \in [0,1]$, $\theta \in [-1,1]$. For this family, the theoretical value of Spearman's Ro is found as $\rho = \theta/3$. θ is a real number for this copula family. To estimate θ, we reach into $\widetilde{\theta}_n = 3\rho_n$. Since ρ_n, like τ_n is a statistics based on ordinal numbers, the methodology used can be seen as a non-parametric one. In general, if the process has a function like $\theta = h(\rho)$, $\widetilde{\theta}_n$ described as $\widetilde{\theta}_n = g(\rho_n)$ is seen as an estimator based on Spearman's Ro. Gaenssler and Stute (1987) conclude that $\rho_n \approx N(\rho, \frac{\sigma^2}{n})$. Borkowf (2002) shows that asymptotic variance σ^2 here is dependent on the related C copula. As we use again Delta method, we reached into the Equation (11)

$$\widetilde{\theta}_n \approx N(\theta, \frac{1}{n}\{\sigma_n . h'(\rho_n)\}^2), \tag{11}$$

where σ_n^2 is the estimator being proper for σ^2. In this way, θ in $1-\alpha$ confidence level is

$$\widetilde{\theta}_n \pm z_{\alpha/2} \frac{1}{\sqrt{n}} \sigma_n . \left| h'(\rho_n) \right| . \tag{12}$$

Borkowf (1992) shows the equation for σ_n^2 if we use C_n for random sample, instead of theoretical C as described in Equation (13) (Genest and Favre, 2007)

$$\sigma_n^2 = 144 \left(-9 A_n^2 + B_n + 2C_n + 2D_n + 2E_n \right) . \tag{13}$$

In the equation,

$$A_n = \frac{1}{n} \sum_{i=1}^{n} \frac{R_i}{n+1} \frac{S_i}{n+1} ; \quad B_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{R_i}{n+1} \right)^2 \left(\frac{S_i}{n+1} \right)^2 ;$$

$$C_n = \frac{1}{n^3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{R_i}{n+1} \frac{S_i}{n+1} I(R_k \le R_i, S_k \le S_i) + \frac{1}{4} - A_n$$

$$D_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{S_i}{n+1} \frac{S_j}{n+1} \max\left(\frac{R_i}{n+1} \frac{R_j}{n+1} \right) ;$$

$$E_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{R_i}{n+1} \frac{R_j}{n+1} \max\left(\frac{S_i}{n+1} \frac{S_j}{n+1} \right) .$$

In the empirical part, we use Chi-square test, which is based on the difference between observed and estimated frequencies with degree of freedom of
d.f. $= (I-1)(J-1) - p - (q-1)$ \hfill (14)

where p is the number of estimated parameters, q is the number of combined cells, I is the number of the rows, and J is the number of the columns. In that way, we can examine the proper copula family for the sample data (Genest and Rivest, 1993).

4. Data

Daily values of the S&P 500 and Bovespa indexes from 02.01.2002 to 08.05.2007 are used for the empirical analysis. The logarithmic values of the first difference (return) are calculated. Graphical presentations of daily values of Bovespa and S&P 500 are in Figure 1 and Figure 2 respectively. Logarithmic daily differences of the two financial time series, on the other hand, are given in Figures 3 and 4.
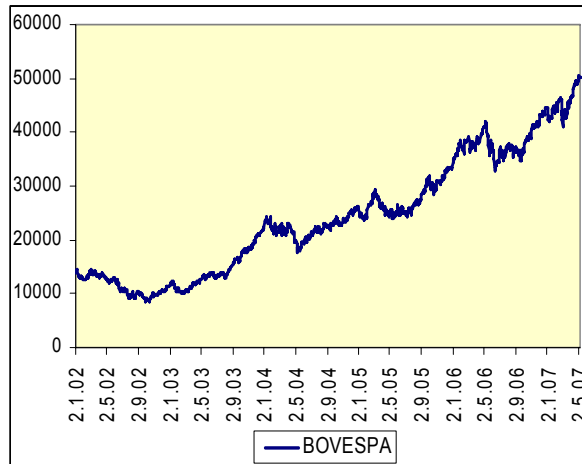
Fig. 1. Daily Values of Bovespa
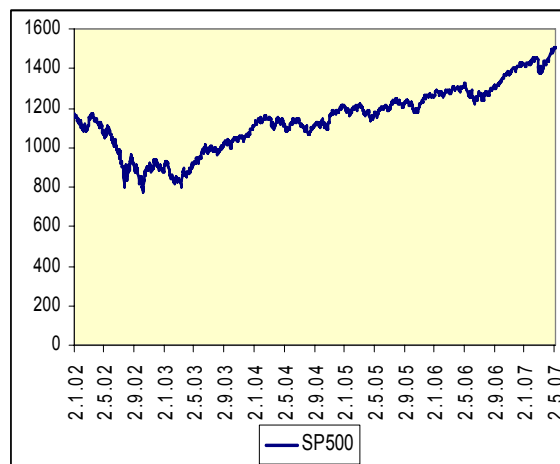
Fig. 2. Daily Values of S&P 500
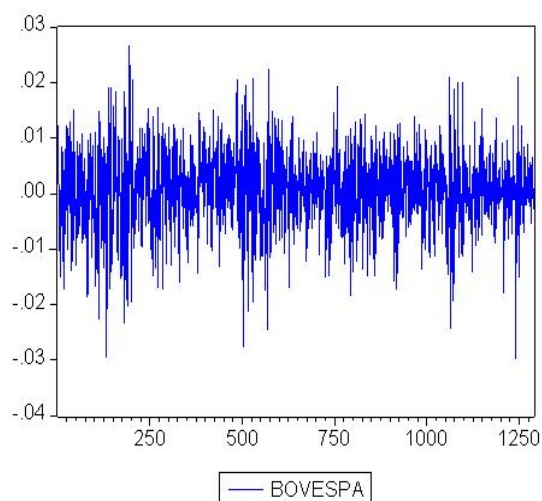
Fig. 3. Daily Log-Return Series of the Bovespa
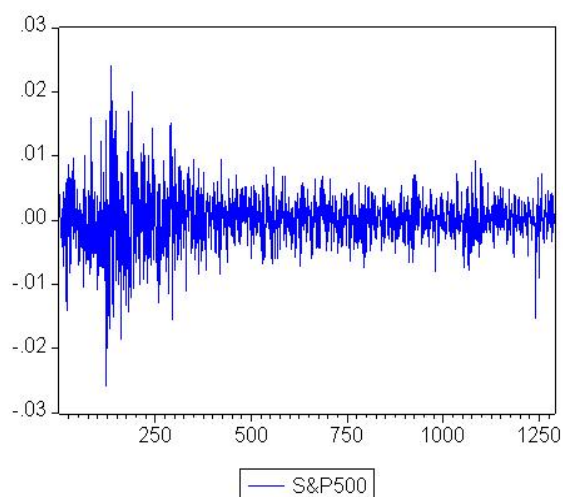
Fig. 4. Daily Log-Return Series of S&P 500

The descriptive statistics and Jarque-Bera test results of the log-returns of two variables are presented in Table 2.

Table 2

Descriptive Statistics and Jargue-Bera Test Results

Variable	Maximum	Minimum	Std. Dev.	Skewness	Kurtosis	Jarque-Bera
S&P 500	0.024209	-0.025876	0.004439	0.054494	6.734983	749.8745
Bovespa	0.026710	-0.029777	0.007525	-0.284168	3.737095	46.5830

The Jarque-Bera test statistics show that the time series satisfy the normality distribution assumption. The unit root for serial correlation and stationary tests are performed with Augmented

Dickey-Fuller (ADF) and Phillips and Perron (P-P) unit root tests. For ADF test, Schwarz Information Criterion; for P-P test, Newey-West Bandwidth are used. The results presented in Table 3 show that there does not exist any unit root problem and the series are stationary.

Table 3

The ADF and P-P Tests Results

Variables	ADF Test	P-P Test
	t-statistics	t-statistics
S&P 500	-37.71148	-37.91735
Bovespa	-34.92754	-34.9364

5. Empirical Evidence

In order to conduct empirical tests with copula methods, we firstly create matrixes for observed frequencies of logarithmic returns of S&P500 and Bovespa. Table 4 shows the 8x8 matrix of the two variables. As the cell limits for the variables; $j = 1,2,...,8$; the values matched with approximately $[1289 \times j / 8]$ order number are taken.

Table 4

Observed Frequencies of Log-returns for S&P 500 and BOVESPA

Variables*	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	72	35	15	13	8	6	11	1	161
$X_{(322)}$	27	35	28	19	23	14	7	8	161
$X_{(483)}$	22	20	33	30	21	19	12	4	161
$X_{(644)}$	18	20	23	28	22	23	12	15	161
$X_{(805)}$	8	22	17	23	37	28	15	11	161
$X_{(966)}$	5	15	27	20	21	27	23	23	161
$X_{(1127)}$	6	6	12	18	14	27	42	36	161
$X_{(1289)}$	3	8	6	10	15	17	39	64	162
Total	161	161	161	161	161	161	161	162	1289

* X denotes log-returns for Bovespa, Y does S&P500.

For the data set, we estimate Spearman's Ro as 0,520137; and Kendall's To as 0,36817. From that point, the main task is to find the best appropriate copula family based on Spearman's Ro and Kendall's To and its parameter for estimating the distributional dependencies between two markets. In order to reach that aim, the selected copula families are statistically tested and best model for our matrix is tried to find by comparing the empirical results.

5.a. Dependency Estimation With Kendall's To and Ali-Mikhail-Haq Copula

The relationship between Kendall's To and Ali-Mikhail-Haq copula family is given in equation (15)

$$\tau = \left(\frac{3\theta - 2}{\theta} \right) - \frac{2}{3}\left(1 - \frac{1}{\theta} \right)^2 \ln(1 - \theta). \tag{15}$$

From equation (15), the parameter ($\hat{\theta}$) of Ali-Mikhail-Haq copula family that fits Kendall's To as 0,36817 is estimated as 0,7260187. Table 5 shows the expected frequencies of the $\hat{\theta}$ parameter (0,7260187) for Ali-Mikhail-Haq copula family.

Table 5

Expected Frequencies For Dependency Parameter With Ali-Mikhail-Haq Copula

Variables	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	45	32	23	18	14	12	9	8	161
$X_{(322)}$	32	27	24	20	18	15	14	11	161
$X_{(483)}$	23	24	23	22	19	18	16	16	161
$X_{(644)}$	18	20	22	20	21	21	20	19	161
$X_{(805)}$	14	18	19	21	23	21	23	22	161
$X_{(966)}$	12	15	18	21	21	25	24	25	161
$X_{(1127)}$	9	14	16	20	23	24	26	29	161
$X_{(1289)}$	8	11	16	19	22	25	29	32	162
Total	161	161	161	161	161	161	161	162	1289

The Chi-square value for *observed* log-returns of BOVESPA and S&P 500 in Table 4 and *expected* frequencies for two indexes in Table 5 is found as 177,08. Since the critical value of the Chi-square test with 49 degree of freedom and 0,01 confidences level is 74.92, we reject the hypothesis that the data set is proper for Ali-Mikhail-Haq copula family.

5.b. Dependency Estimation With Kendall's To and Gumbel-Hougaard Copula

The relationship between Kendall's To and Gumbel-Hougaard copula family is given in equation (16)

$$\tau = 1 - \theta^{-1}. \tag{16}$$

From equation (16), the estimated Gumbel-Hougaard copula parameter that gives Kendall's To as 0,36817 is found as 1,582704.

Table 6

Expected Frequencies For Dependency Parameter With Gumbel-Hougaard Copula

Variables	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	51	34	25	19	14	9	6	3	161
$X_{(322)}$	34	31	28	22	18	15	9	4	161
$X_{(483)}$	25	28	26	25	22	17	13	5	161
$X_{(644)}$	19	22	25	25	24	22	15	9	161
$X_{(805)}$	14	18	22	24	26	25	21	11	161
$X_{(966)}$	9	15	17	22	25	27	29	17	161
$X_{(1127)}$	6	9	13	15	21	29	37	31	161
$X_{(1289)}$	3	4	5	9	11	17	31	82	162
Total	161	161	161	161	161	161	161	162	1289

However, since related Chi-square value is 84,18 for the observed and estimated frequencies, statistically we cannot say that the parameter is not proper for our data set.

5.c. Dependency Estimation With Kendall's To and Clayton Copula

The equation (17) gives the relationship between Kendall's To and Clayton copula

$$\tau = \frac{\theta}{\theta + 2}. \tag{17}$$

From that equation the proper parameter for Clayton copula is estimated as 1,165408.

Table 7

Expected Frequencies For Dependency Parameter With Clayton Copula

Variables	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	92	32	15	8	6	3	3	2	161
$X_{(322)}$	32	38	29	21	14	12	8	7	161
$X_{(483)}$	15	29	28	24	21	17	15	12	161
$X_{(644)}$	8	21	24	24	24	22	20	18	161
$X_{(805)}$	6	14	21	24	25	24	24	23	161
$X_{(966)}$	3	12	17	22	24	28	27	28	161
$X_{(1127)}$	3	8	15	20	24	27	31	33	161
$X_{(1289)}$	2	7	12	18	23	28	33	39	162
Total	161	161	161	161	161	161	161	162	1289

The related Chi-square value for observed and expected frequencies for Bovespa and S&P 500 is 148,88. Since it is above the critical value (74,92) we conclude that Clayton copula is also not proper for the variables under investigation.

5.d. Dependency Estimation With Spearman's Ro and Plackett Copula

The relationship between Spearman's Ro and Plackett copula is presented with equation (18)

$$\rho = \frac{(\theta + 1)}{(\theta - 1)} - \frac{2\theta \ln \theta}{(\theta - 1)^2} \ . \tag{18}$$

The dependency parameter ($\hat{\theta}$) for Plackett copula that fixes the Spearman's Ro to 0,520137 is estimated as 5,538032.

Table 8

Expected Frequencies For Dependency Parameter With Plackett Copula

Variables	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	58	36	23	15	11	8	6	4	161
$X_{(322)}$	36	36	29	21	15	10	8	6	161
$X_{(483)}$	23	29	30	26	20	15	10	8	161
$X_{(644)}$	15	21	26	27	25	21	15	11	161
$X_{(805)}$	11	15	20	25	28	25	22	15	161
$X_{(966)}$	8	10	15	21	25	31	28	23	161
$X_{(1127)}$	6	8	10	15	22	28	36	36	161
$X_{(1289)}$	4	6	8	11	15	23	36	59	162
Total	161	161	161	161	161	161	161	162	1289

The observed frequencies and frequencies estimated by Plackett copula with 5,538032 parameter between Bovespa and S&P 500 are compared with Chi-square test. The Chi-square value is found as 70.01, which is below to the critical value for 49 degree of freedom and 99% significance level. From this empirical result, we accept the null hypothesis that the distributional dependency between S&P 500 and Bovespa is statistically modeled with Plackett copula family.

We can conclude that the distributional dependency between the returns of S&P 500 and Bovespa can be modeled with Plackett copula algorithm. From Table 8, we can easily observe that the distribution dependency is symmetric and in the same direction. In that framework, return distributions of two stock markets have symmetric dependency. The return distributions of the two markets are dependent to each other in the same and similar direction.

5.e. Dependency Estimation With Kendall's To and Frank Copula

An alternative copula methodology to estimate dependency between two financial market is to mix Kendall's To and Frank copula family. The relationship between Kendall's To and Frank copula is given in equation (19).

$$\tau = 1 - \frac{4}{\theta}[D_1(-\theta) - 1] \cdot \qquad (19)$$

In equation (19), D_1 is the Debye function as explained in the methodology part. In estimating Frank copula parameter, when calculating the integral in Debye function, the Riemann method is used. The Riemann method nominated in equation (20) is based on the division of [a, b] integral interval into m sub-intervals.

$$\int_a^b f(x)dx = \sum_{k=1}^m f(x^k)\Delta x \cdot \qquad (20)$$

By using that methodology, proper parameter for Frank copula that gives Kendall's To as 0,36817 is estimated as $\hat{\theta} = 2,205$.

Table 9

Expected Frequencies For Dependency Parameter With Frank Copula

Variables	$Y_{(161)}$	$Y_{(322)}$	$Y_{(483)}$	$Y_{(644)}$	$Y_{(805)}$	$Y_{(966)}$	$Y_{(1127)}$	$Y_{(1289)}$	Total
$X_{(161)}$	39	32	26	20	15	12	10	7	161
$X_{(322)}$	32	29	24	22	18	15	12	9	161
$X_{(483)}$	26	24	25	22	20	18	14	12	161
$X_{(644)}$	20	22	22	21	22	19	18	17	161
$X_{(805)}$	15	18	20	22	22	23	21	20	161
$X_{(966)}$	12	15	18	19	23	23	26	25	161
$X_{(1127)}$	10	12	14	18	21	26	28	32	161
$X_{(1289)}$	7	9	12	17	20	25	32	40	162
Total	161	161	161	161	161	161	161	162	1289

However, since Chi-square value for the observed and expected frequency values (147,32) is above the critical value (74,92), we cannot conclude that Frank copula method is proper to estimate the dependency between Bovespa and S&P 500.

From the empirical results, we conclude that the Plackett copula family is appropriate to model the return distribution dependency between Bovespa and S&P 500. The estimated parameter, $\hat{\theta}$, equals 5,538032 fitting the Spearman's Ro to 0,520137. In terms of financial investment point of view, we can statistically argue that a distribution of daily returns of S&P 500 and Bovespa moves in a similar direction with a symmetric way. Thus, the return distributions of the US stock markets and Brazilian stock markets are statistically significant, and strong interaction should be taken into account in creating international investment portfolios.

6. Concluding Remarks

Globalization in the financial markets has two main effects in terms of financial economics. First of all, it creates intensive information and cash flows among the markets; thus requires flexible but complex econometric models for estimation. Secondly, it causes dependencies among the returns and return distributions among financial markets.

From that perspective, we use different copula models with nonparametric parameters, which present flexible algorithms without any distributional assumptions among the financial time series in order to test the distributional dependency between the Brazilian equity markets and the US equity markets. By employing alternative copula models with different nonparametric dependency parameters, it is statistically showed that the distributional characteristics of daily returns of

Bovespa and the S&P 500 have a strong degree of dependency. What is more, the statistically significant copula algorithm that can estimate that dependency is the Plackett copula with Spearman's Ro. In that respect, the dependency is symmetric and moves in the same direction.

The empirical findings have two main conclusions. The first one is that it is empirically showed that the complexity in the financial markets requires advanced and flexible methodologies in financial and statistical modeling. Secondly, it is figure out that the portfolio managers in their risk diversification should be aware of the fact that the return distributions of Brazilian equity market and the US market are affected from each other. Before 1987 Asia crisis, it was the usual trend that the advanced and emerging markets show opposite behaviors giving a chance for portfolio managers to diversify their risk. However, in a global economy in which money and information freely and fast flow, the investors should be aware of the fact that distributional characteristics of stock returns are getting similar to each other. Thus, risk diversification might require derivative instruments rather than geographical or country specific portfolio diversification.

The future research might concentrate on the time-scale of the dependencies or distributional dependencies among the markets. In that respect, wavelets or wavelet networks can be used to identify the time-scale of the statistical characteristics of interactions among the markets.

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