APPLICATIONS TO PORTFOLIO THEORY OF MARKET STOCHASTIC BOUNDS

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Abstract

This paper examines the market bounds in order to describe the evolution of investor's optimal choices. Thus, first we describe the distributions of market bounds when limited short sales are allowed. Then, we introduce a linear approximation model that describes the investors' optimal choices as a function of the upper market bound. Finally, we propose an empirical comparison between optimal strategies based on the expected optimal portfolios related to the upper market bound.

Key words: stochastic bounds, stochastic dominance, safety first optimal portfolio.
JEL Classification: G11, G14.

1. Introduction

The theory of portfolio choice is based on the assumption that investors allocate their wealth across the available assets in order to maximize their expected utility. One of the first rigorous approximating results to the portfolio selection problem was given in terms of the mean and the variance by Markowitz and Tobin. Mean variance theory finds its justification in arbitrage theory and in stochastic dominance analysis. In particular, stochastic dominance analysis justifies the partial consistency of mean-variance framework with expected utility maximization when the portfolios are elliptically distributed.

Almost in contradiction to maximization of expected utility approach, there has been the gradual development of a theory of decision making which has focused on agents who seek to attain some aspiration or target level of outcome through their actions. However, it is possible to prove that the two approaches are related and in some cases equivalent even if the economic reasons and justifications are different (see Bordley, LiCalzi (2000) Castagnoli, LiCalzi (1996), Ortobelli and Rachev (2001)). In portfolio literature, Roy (1952), Tesler (1955/6), Bawa (1976, 1978), suggest the safety first rules as a criterion for decision making under uncertainty. In such models, a subsistence or disaster level of returns is identified. The objective is taken to be the maximization of the probability that the returns are above the disaster level (or some other variation on this theme). Further extensions and developments have followed these primary works (see, among others, the discrete time extensions of Roy (1995), Li, Chan, Ng (2000) or the continuous time extensions of Majumdar and Radner (1991) and Dutta (1995)).

This paper describes market stochastic bounds in order to study the evolution of investors' optimal choices. In particular we introduce some management tools to forecast the behavior of future choices. Thus, using the upper stochastic bound in an approximating linear model, we can examine the market trend and we can give a first naive interpretation to safety first analysis in markets with short sale restrictions. This metodology suggests a way to determine an average portfolio of optimal choices. Thus with an ex-post empirical analysis we compare the forecasting power of the expected optimal portfolios and of the portfolio that maximizes the Sharpe ratio.

In the second section we define the stochastic bounds of the market and introduce the analysis of market trend. The third section proposes an empirical comparison of the dominating portfolios and the portfolio that maximizes the Sharpe ratio. Finally we briefly summarize the results.

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2. Stochastic Bounds and Market Trend

Suppose we have a frictionless market with n risky securities where all investors have the same temporal horizon and they act as price takers. The random vector of the gross returns[1] $Z = [Z_1,...,Z_n]'$ is defined on the Polish probability space $(\Omega, \Im, P)$.

In a market with limited short selling opportunities every admissible vector of portfolio weights $x = [x_1,...,x_n]'$ belongs to a compact convex set T. In this case all the admissible portfolios are stochastically bounded, i.e., there exist two random variables Y_U and Y_L such that for every admissible vector of portfolio weights $x = [x_1,...,x_n]'$ belonging to the set T, every non-satiable investor prefers Y_U to $x'Z$ and $x'Z$ to Y_L i.e., if and only if $F_{Y_U}(t) \leq F_{x'Z}(t) \leq F_{Y_L}(t)$ for every real t (see Ortobelli and Rachev (2001), Ortobelli and Pellerey (2007) for further details)[2]. Under these assumptions, we can express analytically the distribution functions of the optimal bounds. As a matter of fact,

$$F_U(\lambda) = \inf_{x \in T} P(x'Z \leq \lambda)$$

is the "smallest" cumulative distribution (in FSD sense) which FSD dominates all portfolios while

$$F_L(\lambda) = \lim_{s \downarrow \lambda^+} \sup_{x \in T} P(x'Z \leq s)$$

is the "greatest" cumulative distribution (in FSD sense) which is FSD dominated by all portfolios. F_U has support $[a,b]$ and F_L has support $[\widehat{a},\widehat{b}]$, where

$$a = \sup_{x \in T} c(x), \quad b = \sup_{x \in T} d(x), \quad \widehat{a} = \inf_{x \in T} c(x), \quad \widehat{b} = \inf_{x \in T} d(x),$$

$$c(x) = \sup\{c \in \Re / P(x'Z \leq c) = 0\} \text{ and } d(x) = \inf\{d \in \Re / P(x'Z > d) = 0\}.$$

Generally, in the market it does not exist a portfolio $x'Z$ that FSD dominates (or is FSD dominated by) all the others. Thus, it does not exist a portfolio $x'Z$ that admits as distribution function F_U (or F_L). When only limited short sales are allowed, we call F_U the *stochastically dominating distribution* of all the admissible portfolios and F_L the *stochastically dominated distribution*. We call *stochastic bounds of all admissible portfolios* two random variables Y_U and Y_L (unique in distribution) defined on $(\Omega, \Im, P)$, with respectively the stochastically dominating distribution F_U and the stochastically dominated distribution F_L. We call Y_U *preferential market growth*, since it represents the maximum factor of future wealth growth. In some sense, Y_U is the maximum price that we can give at the future risky wealth for a unity of wealth invested today.

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[1] We indicate with r_i the rate of return of the ith security and $Z_i = 1 + r_i$ the respective gross return.

[2] Recall that X first stochastically dominates Y (X *FSD* Y) if and only if $\mathsf{E}(\phi(X)) \geq \mathsf{E}(\phi(Y))$ for every non-decreasing function ϕ such that the two expectations exist, i.e., if and only if $F_X(t) \leq F_Y(t)$ for every real t. More generally, we say that X dominates Y in the sense of the α ($\alpha \geq 1$) stochastic dominance order ($X \underset{\alpha}{\geq} Y$) if

$$F_X^{(\alpha)}(t) := \frac{\mathsf{E}\left((t-X)_+^{\alpha-1}\right)}{\Gamma(\alpha)} \leq F_Y^{(\alpha)}(t) \text{ for every real } t \text{ (see Fishburn (1980)).}$$

The above definition of stochastic bounds can be easily extended to a given category of investors. In particular, when limited short selling opportunities are allowed, we can consider the non-dominated choices in the sense of a given $\alpha-$ stochastic order (see Fishburn (1980)) belonging to $\overline{T}^{\alpha}$, closure of the set

$$T^{\alpha} = \left\{ x_{(\alpha)}(t) \in T \;/\; t \in \mathsf{R}, \; x_{(\alpha)}(t) \in \arg\left(\inf_{x \in T} \mathsf{E}\left((t - x'Z)_{+}^{\alpha-1} \right) \right) \right\}.$$

Note that $\overline{T}^{\alpha}$ is a compact subspace of T. As for the stochastic bounds of all admissible portfolios, we consider two random variables $Y_{U,(\alpha)}$ and $Y_{L,(\alpha)}$ (unique in distribution) with respectively the stochastically dominating distribution $F_{U,(\alpha)} = \inf_{x \in \overline{T}^{\alpha}} P(x'Z \leq \lambda)$ and the stochastically dominated distribution $F_{L,(a)} = \lim_{s \to \lambda^{+}} \sup_{x \in \overline{T}^{\alpha}} P(x'Z \leq s)$. Therefore, for every random variable $D_{1} \overset{d}{\neq} Y_{U,(\alpha)}$ which FSD dominates every portfolio $x'Z$ in $\overline{T}^{\alpha}$, then D_{1} FSD $Y_{U,(\alpha)}$, and for every random variable $D_{2} \overset{d}{\neq} Y_{L,(\alpha)}$ which FSD is dominated by every portfolio $x'Z$ in $\overline{T}^{\alpha}$, then $Y_{L,(\alpha)}$ FSD D_{2}. Similarly, we can extend these considerations to the portfolios not dominated in the sense of any given order of preferences. For example we could compute stochastic bounds and optimal portfolios weights:

a) of non satiable risk lover investors, belonging to $\overline{T}^{RL}$, closure of the set

$$T^{RL} = \left\{ x_{(RL)}(t) \in T \;/\; t \in \mathsf{R}, \; x_{(RL)}(t) \in \arg\left(\sup_{x \in T} \mathsf{E}\left((x'Z - t)_{+} \right) \right) \right\},$$

b) of investors that maximize their performance ratio, belonging to $\overline{T}^{PR}$, closure of the set

$$T^{PR} = \left\{ x_{(PR)}(t) \in T \;/\; t \in \mathsf{R}, \; x_{(PR)}(t) \in \arg\left(\sup_{x \in T} \frac{v(x'Z)}{\rho(x'Z)} \right) \right\}.$$

where $v(x'Z)$, $\rho(x'Z)$ are respectively a positive reward measure and a positive risk measure of the portfolio (see Biglova et al. (2004), and Rachev et al. (2007));
c) of non dominated choices in a multiparameter framework (see Ortobelli (2001)).
In the following we will consider portfolio choice among n risky assets when no short sales are allowed, i.e. portfolio weights $x \in T = \left\{ y \in R^{n} \;:\; y_{i} \geq 0; \sum_{i=1}^{n} y_{i} = 1 \right\}$. We also assume that it does not exist a portfolio $x'Z$ in the market that FSD dominates (or is FSD dominated by) all the others, and we examine stochastic bounds for all admissible portfolios belonging to T. First of all we discuss the uniqueness, for every λ belonging to $[a,b]$, of the portfolio weight

$$x_{U}(\lambda) \in \arg\left(\inf_{x \in T} P(x'Z \leq \lambda) \right)$$

when limited short sales are allowed. Actually, the uniqueness is not always satisfied in presence of riskless assets, as shown in the following counterexample.

Counterexample: Let z_0 be the return of a riskless asset, and suppose that the vector Z of the other assets is jointly Gaussian distributed. Under this assumptions, we generally share the risky components from the riskless one for notation, i.e., for every $y \in T = \left\{ y \in R^{n+1} \ : \ y_i \geq 0; \sum_{i=1}^{n+1} y_i = 1 \right\}$ we consider $y = (x, 1 - x'e); \quad x \in \Re^n$, $e = [1, ..., 1]'$. Then, every portfolio $(1 - x'e)z_0 + x'Z$, is Gaussian distributed with mean $(1 - x'e)z_0 + x'\mathsf{E}(Z)$ and variance $x'Qx$, where Q is the variance covariance matrix of Z and $1 - x'e \geq 0, \ x_i \geq 0$ for every $i = 1, ..., n$. When there exists a risky portfolio with mean greater than z_0, non-satiable risk averse investors maximize the Sharpe ratio

$$\frac{(1 - x'e)z_0 + x'\mathsf{E}(Z) - z_0}{\sqrt{x'Qx}} = \frac{x'\left(\mathsf{E}(Z) - ez_0\right)}{\sqrt{x'Qx}}$$

(see Sharpe (2004)), and all optimal choices belong to a linear combination between the riskless return z_0 and the market portfolio $\tilde{x}'Z$, where $\tilde{x}$ is the solution of the optimization problem

$$\sup_{x \ : \ x_i \geq 0; \sum_{i=1}^{n} x_i = 1} \frac{x'E(Z) - z_0}{\sqrt{x'Qx}} \ .$$

Thus, in this case, when $\lambda = z_0$, we get that

$$\alpha\tilde{x} \in \arg\left(\inf_{x \ : \ x_i \geq 0; \sum_{i=1}^{n} x_i \leq 1} P(x'Z - (1 - x'e)z_0 \leq \lambda) \right) =$$

$$= \arg\left(\sup_{x \ : \ x_i \geq 0; \sum_{i=1}^{n} x_i \leq 1} \frac{x'E(Z) - z_0}{\sqrt{x'Qx}} \right)$$

for every $\alpha \in (0, 1]$, i.e., $x_U(\lambda)$ is not unique.

The above counterexample can be further extended to more general scalar and translation invariant families of distribution functions (see Ortobelli (2001)). On the other hand we can guarantee the uniqueness when the risk-free asset is not allowed and the risky returns are elliptically distributed (see Ortobelli and Rachev (2001)). For this reason in all the following considerations we assume that risk-free assets are not allowed.

Since the stochastic bounds are the market limits, then the market trend is implicitly described by them and it has sense to study the evolution of investor's choices in relation to the market trend. Assume now that $x_U(\lambda) \in \arg\left(\inf_{x \in T} P(x'Z \leq \lambda) \right)$ is a measurable vectorial function. Then, by the definition of the stochastically dominating distribution it follows that $F_U(\lambda) = \inf_{x \in T} P(x'_U(\lambda)Z \leq \lambda)$.

Next, we say that it is satisfied the *based target* axiom when we assume non satiable investors minimize the probability that their returns are lower than a given value of the preferential market growth. The based-target axiom can be justified even in terms of the classic von Neumann-Morgenstern approach. As a matter of fact, Castagnoli and LiCalzi (1996) have shown that the based-target approach is a generalization of the von Neumann-Morgenstern (vNM) one and even the vNM investors minimize the probability of being under a given target. Assuming that the based-target axiom holds, optimal choices are well represented by the random vectorial surjective function X_U defined on $(\Omega, \Im, P)$ by

$$X_U(w) = x_U(Y_U(w)) = \arg\left(\inf_{x \in T} P(x'Z \leq Y_U(w))\right) \text{ for every } w \in \Omega.$$

We call the random vectorial function X_U *dominating portfolio of all admissible choices*. The dominating portfolio is a random indicator of the investors' optimal choices, since it represents all the "safety first" choices (see Tesler (1955/6)).

Therefore, it could be important to understand how optimal allocations vary among the *n* risky assets under preferential market growth changes. That is, the implicit question is: How does X_U change when Y_U changes ? or, more suggestively, Where does the market go? These questions have not easy solutions even when returns are elliptically distributed, because if no short sales are allowed, it is still difficult to describe the Markowitz-Tobin efficient frontier. Moreover, the portfolio distributions are not necessarily elliptically distributed. When Y_U admits finite variance $\sigma_{Y_U}^2$, we can consider the best linear approximation between X_U and Y_U with the least square estimator (LSE). Thus, we can assume that the following model holds:

$$X_{U,i} = \mathsf{E}(X_{U,i}) + \frac{\text{cov}(X_{U,i}, Y_U)}{\sigma_{Y_U}^2}(Y_U - \mathsf{E}(Y_U)) + \varepsilon_i, \quad i = 1, \ldots, n, \qquad (1)$$

where $\text{E}(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$, $\text{E}(\varepsilon_j) = 0$ for all $j = 1, \ldots, n$, and $\sum_{i=1}^{n} \varepsilon_i = 0$.

In this model we can distinguish different market indicators that can be interpreted by an economic point of view:

- the vector of the expected values of the dominating portfolio

$$x_D = \mathsf{E}(X_U) = \int_{\bar{\Omega}} x_U \circ Y_U dP = \int_{[a,b]} x_U(\lambda) dF_U(\lambda), \qquad (2)$$

  represents the portfolio weights where the investors' preferences collapse in average.

  In addition, when the set of all optimal safety first portfolios is convex, then x_D belongs to the portfolio weight efficient frontier U, since it is a convex combination of optimal portfolios;

- the average of the upper stochastic bound

$$\tilde{\lambda} = \mathsf{E}(Y_U) = \int_{\bar{\Omega}} Y_U dP = \int_{[a,b]} \lambda dF_U(\lambda), \qquad (3)$$

  is another interesting indicator of the market growth, since Y_U is the first random variable which dominates all portfolios. Then $\tilde{\lambda}$ can be considered as the average of the maximum future price of risky wealth for a unity of wealth invested today;

♦   moreover, we can interpret the portfolio

$$y_D = \frac{\mathsf{E}(X_U Y_U)}{\tilde{\lambda}} = \frac{\int_{[a,b]} \lambda x_U(\lambda) dF_U(\lambda)}{\tilde{\lambda}}, \tag{4}$$

as the value that we give today to the differently allocated unitary wealth capitalized tomorrow with the maximum future price of risky wealth. Thus, y_D represents in some sense how we look today at the evolution of the market portfolio tomorrow.

The model (1) is well defined. As a matter of fact X_U remains a random vector of portfolio allocations since if we consider the vector $\mathrm{cov}(X_U, Y_U)$ then we see that $\mathrm{cov}(X_U, Y_U) = \mathsf{E}(X_U Y_U) - \tilde{\lambda} x_D$ that is $\mathrm{cov}(X_U, Y_U) = \tilde{\lambda}(y_D - x_D)$, and therefore $\sum_{i=1}^{n} X_{U,i} = 1$. We observe that the dependence of the ith component of X_U with Y_U is determined by the sign of $\mathrm{cov}(X_{U,i}, Y_U)$. Therefore, the above relations can be interpreted in the following way:

1. If $y_{D,i} - x_{D,i}$ is greater than zero, then we can think the component $x_{U,i}(Y_U)$ has the same trend as Y_U. Thus, if the market is growing, non satiable investors tend to increase their position on the ith investment.

2. If $y_{D,i} - x_{D,i}$ is lower than zero, then we can think the component $x_{U,i}(Y_U)$ has opposite trend than Y_U. Thus, if the market is growing, we have that non-satiable investors tend to disinvest their position on the ith investment.

3. Opposite conclusions follow when the market is downing.

This analysis is coherent with the economic interpretation given to the portfolios y_D and x_D, and could be potentially useful to address the future risk management choices. In addition, all the indicators $\tilde{\lambda}$, y_D and x_D can be estimated numerically. Clearly, these approximations can be only used to understand indicatively the trend of the market growth by the point of view of non-satiable investors. Finally, observe that all the above considerations can be extended to the stochastic bounds of portfolios belonging to any compact set of optimal choices contained in $T=S$ (such as $\overline{T}^{\alpha}$, $\overline{T}^{RL}$ and $\overline{T}^{PR}$). For example, let us suppose we have N observations with probability q_k ($k=1,...,N$) of n risky assets. Typically, we can assume $q_k = (1 - g^{1/N}) g^{(N-k)/N} + g/N$, where $g \in (0,1]$, so that when $g=1$ we implicitly assume the observations are independent identically distributed (i.i.d.); otherwise, (i.e., $g \in (0,1)$) we consider an exponential weighted behavior of the historical observations (see, among others, Longerstaey and Zangari (1996)). Then, in order to determine optimal portfolios for non-satiable risk averse investors belonging to $\overline{T}^2$, we have to solve the following linear optimization problem for different values of t:

$$\min_{y} \sum_{k=1}^{N} v_k$$

$$\sum_{i=1}^{n} y_i = 1; \quad y_i \geq 0; \quad i = 1,...,n$$

$$v_k \geq 0; \quad v_k \geq \left(t - \sum_{i=1}^{n} y_i z_{i,k} \right) q_k \quad k = 1,...,N$$

where $z_{i,k}$ is the *k-th* observation of the *i-th* gross return. Then we can estimate the indicators

relative to non-satiable risk averse $F_{U,2}(\lambda) = \inf_{x \in \overline{T}^2} P(x'Z \leq \lambda)$, $\tilde{\lambda}_{(2)} = \mathsf{E}(Y_{U,(2)})$,

$y_{D,(2)} = \dfrac{\mathsf{E}(X_{U,(2)}Y_{U,(2)})}{\tilde{\lambda}_{(2)}}$ and $x_{D,(2)} = \mathsf{E}(X_{U,(2)})$ for the class of non-satiable risk averse investors,

where $X_{U,(2)}(w) = \arg\left(\inf_{x \in \overline{T}^2} P(x'Z \leq Y_{U,(2)}(w)) \right)$ for every $w \in \Omega$ defines the dominating

portfolio of non-satiable and risk averse choices. Analogously, we can estimate optimal choices of non-satiable risk lover investors, solving a similar optimization problem where we have the maxi-

mization instead of minimization and the constraints $v_k \geq q_k \left(\sum_{i=1}^{n} y_i z_{i,k} - t \right); \quad k = 1,...,N$ in-

stead of constraints $v_k \geq q_k \left(t - \sum_{i=1}^{n} y_i z_{i,k} \right); \quad k = 1,...,N$. Similarly, we can also estimate the

indicators $\tilde{\lambda}_{(RL)}$, $y_{D,(RL)}$, $x_{D,(RL)}$ of non-satiable risk lover investors.

3. An Empirical Analysis on the Market Stochastic Bounds

In this section we propose an empirical analysis of the stochastic bounds presented above. Since we do not know the distribution functions of the asset returns, we approximate the indicators

x_D, $\tilde{\lambda}$ and y_D using the empirical distributions of portfolio returns. The simplest way to get an estimation for these quantities is by taking a partition of $[a,b]$, say $[\lambda_1 = a, \lambda_2,..., \lambda_f = b]$, and considering the approximations

$$\tilde{\lambda} = \lambda_1 F_U(\lambda_1) + \lambda_2(F_U(\lambda_2) - F_U(\lambda_1)) + ... \atop ... + \lambda_f(F_U(\lambda_f) - F_U(\lambda_{f-1})), \tag{5}$$

$$x_D = x(\lambda_1)F_U(\lambda_1) + x(\lambda_2)(F_U(\lambda_2) - F_U(\lambda_1)) + ... \atop + x(\lambda_f)(F_U(\lambda_f) - F_U(\lambda_{f-1})), \tag{6}$$

and

$$y_D = \tfrac{1}{\tilde{\lambda}} \big[\lambda_1 x(\lambda_1)F_U(\lambda_1) + \lambda_2 x(\lambda_2)(F_U(\lambda_2) - F_U(\lambda_1)) + \atop ... + \lambda_f x(\lambda_f)(F_U(\lambda_f) - F_U(\lambda_{f-1})) \big]. \tag{7}$$

In order to test the forecasting power of these indicators, we propose an ex-post analysis. In particular, we use daily returns quoted on the market from January 1995 to May 2005. Assuming that short selling is not allowed, we examine optimal allocation among 10 stock returns components of the Dow Jones Industrials[1]: Altria Group, Citigroup, General Electric, Home de pot, Intel, Johnson and Johnson, Microsoft, Pfizer, United Technologies, and Wal Mart Stores. We propose a performance comparison with two different assumptions: in the first we assume i.i.d. observations and we make use of a window of 250 daily observations to approximate the optimal portfolios; in the second we use a window of 750 daily observations and we assume the k-th observation has the probability $q_k = (1 - g^{1/N})g^{(N-k)/N} + g/N$, with $g = 10^{-4}$. In particular, we compare the final wealth processes obtained with the portfolios x_D and y_D with the ex-post final wealth sample path obtained investing in the maximum Sharpe ratio portfolio (i.e., in the portfolio weights $v = \arg\left(\max_x \frac{x'E(Z) - z_0}{\sqrt{x'Qx}}\right)$ assuming a null riskless return $z_0 = 0$).
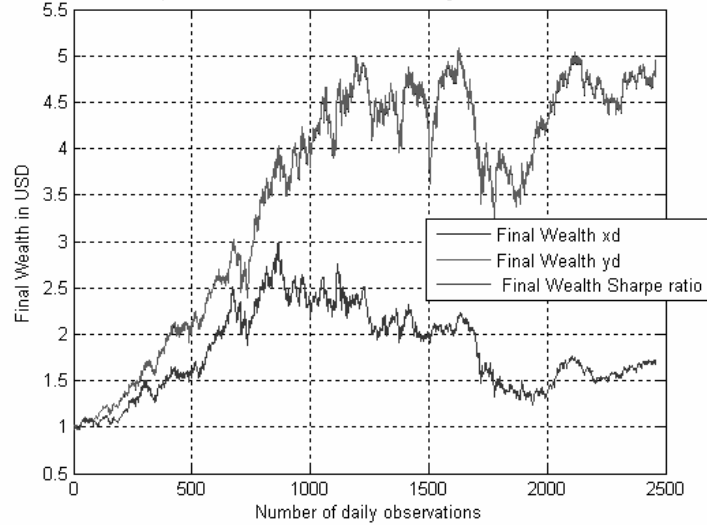

Fig. 1. Ex-post final wealth obtained assuming i.i.d. observations

Note: This figure presents the ex-post final wealth processes (from December 1995 till May 2005) obtained investing daily either in portfolio x_D or in y_D, or in portfolio that maximizes the Sharpe ratio.

We compute the resulting ex-post sample paths of the final wealth obtained investing the wealth every day in the portfolios x_D and y_D. That's why, we approximate x_D and y_D, by means of formulas (5), (6) and (7), computing 100 optimal portfolios $x(\lambda_i)$, with $\lambda_i \in [a,b]$, $i = 1,...,100$, assuming that the investors have an initial wealth $W_{t_0} = 1$. Thus, once we approximated the optimal portfolios $x_{D,(t_k)}$, and $y_{D,(t_k)}$ at each time t_k, we calculate the *ex-post* final wealths $W_{(x)t_{k+1}} := W_{(x)t_k}\left(x_{D,t_k}\right)' Z^{(t_k+1)}$ and $W_{(y)t_{k+1}}$ capitalized at time t_{k+1} with the ex-post observed gross returns $Z^{(t_k+1)}$.

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[1] We take data from DATASTREAM.

In the first four figures we compare the different results from December 18th 1995 till May 24th 2005 assuming i.i.d. observations (i.e., $q_k = 1/250$), while in the last figure we compare the ex-post final wealths when historical observations are exponentially weighted.

Figure 1 presents the comparison among the ex-post final wealth processes. Note that, in practice, there is no difference between the sample path of the final wealth obtained with portfolios y_D and the one obtained considering portfolios x_D even if both portfolios x_D and y_D give a performance better than that obtained considering the Sharpe ratio.

The reason of this equality is explained by Figure 2, that reports the sample path of the differences $y_{D,i} - x_{D,i}$ between the components of x_D and y_D.
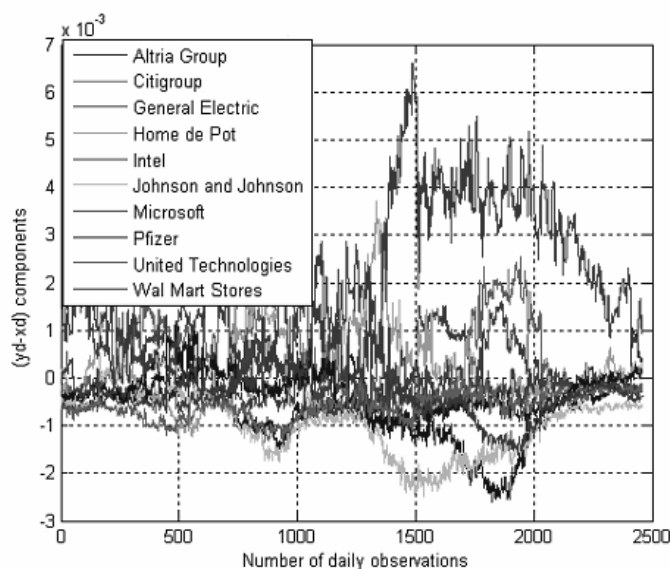

Fig. 2. Component trend analysis

Note: This figure presents the daily difference (from December 1995 till May 2005) between the components of portfolios y_D and x_D.

These differences are of order 10^{-3}, and do not imply a substantial difference in the final wealth. In Figure 3 we compare the behaviors over time of the expected upper and lower bounds. In particular, we observe that during the market crises the distance between the two bounds is higher since the market is much more volatile. The geometric means of the expected upper and lower bounds $\mathsf{E}(Y_U)$ and $\mathsf{E}(Y_L)$ during the period of observation are 1.00612809 and 0.99615493 respectively. Thus, probably, there exist several strategies that could give better performances than that obtained with the expected dominating portfolio for non-satiable investors. For example, let us compute the *ex-post* final wealth sample paths that one can obtain considering the expected dominating portfolios $x_{D,(2)}$ and $x_{D,(RL)}$ of non-satiable and respectively risk averse and risk lover investors (analogous results can be obtained considering respectively the portfolios $y_{D,(2)}$ and $y_{D,(RL)}$). Figure 4 shows the comparison among the ex-post final wealth processes. As we could expect, the strategies of non-satiable risk lover investors sometimes give better performance than the other ones. However, these strategies lose much more than the others during the crises after September 11th 2001. Among all the strategies, the one based on the expected dominating portfolio of all non-satiable investors gives the best performance.

Finally, in Figure 5 we compare the performance of all ex-post final wealth processes from October 17th 2000 till May 24th 2005 when we assume an exponential probability of the historical observations (i.e., we use $g = 0,0001$, N=750). Even in this case we observe the better performance of the expected dominant portfolio of all non-satiable investors.
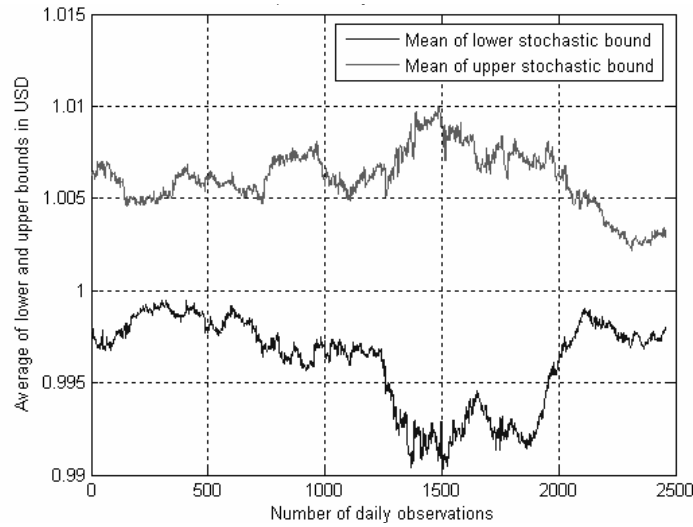

Fig. 3. Expected daily stochastic bounds

Note: This figure presents the daily expected upper and lower bounds for all non-satiable investor valued from December 1995 till May 2005.
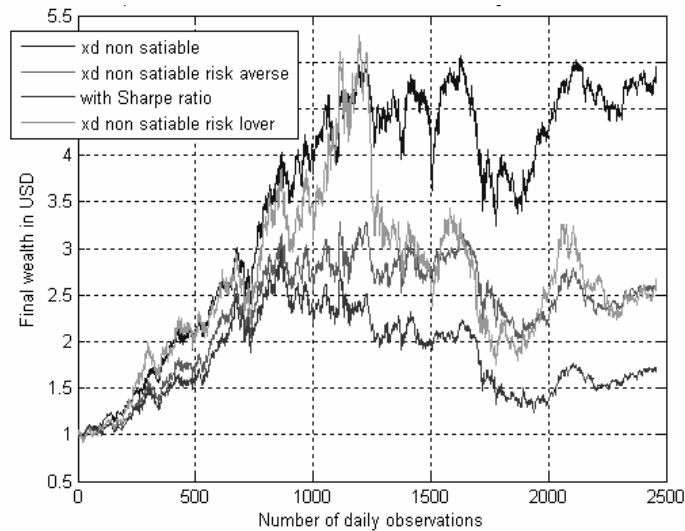

Fig. 4. Ex-post final wealth processes obtained assuming i.i.d. observations

Note: This figure presents the ex-post final wealth sample from December 1995 till May 2005 when we assume i.i.d. observations. We compare the ex-post final wealth process obtained investing daily either in the portfolio x_D or in $x_{D,(2)}$, or in $x_{D,(RL)}$, or in the portfolio that maximizes Sharpe ratio.
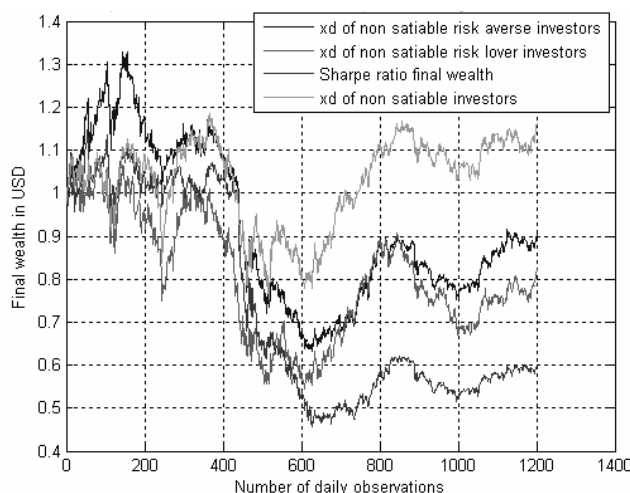
Fig. 5. Ex-post final wealth obtained with exponential weighted observations

Note: This figure presents the ex-post final wealth sample path from October 2000 till May 2005 when we assume exponential weighted observations. We compare the ex post final wealth obtained investing daily either in the portfolio x_D, or $x_{D,(2)}$, or $x_{D,(RL)}$, or in the portfolio that maximizes the Sharpe ratio.

4. Concluding Remarks

In the paper we study the distribution law of the upper bound, and, in the first empirical analysis, we compare the forecasting power of two portfolios related to the upper bound of the market. In particular, we discuss how we can use the market stochastic bounds to analyze investors' optimal choices in markets with short sale restrictions. On the other hand, when distribution functions of returns are elliptical, we can analytically describe how choices change in relation to the upper market stochastic bound (see Ortobelli and Pellerey (2007)). Clearly, the results of this paper can be discussed and studied in a more general context where some decision making agents have to choose a parametric random variable X_θ (with θ being a parameter belonging to a compact convex set $\Theta \subset \Re^n$). Moreover, we believe that further discussions and extensions could rise from this first empirical analysis. For example:

♦   we should consider strategies that minimize a distributional distance from the preferential market growth;

♦   we should consider dynamic strategies taking into account the intertemporal dependence of historical observations;

♦   we should examine non dominating strategies with respect to behavioral orderings (see Levy and Levy (2002)).

However, a more general empirical analysis with further studies and comparisons of the above model should be an object of future researches.

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