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European and American put options on insurance products

Abstract

We consider a European and American put option defined on pure endowment insurance and risk insurance contracts, respectively. These options give the holder of the option or the beneficiary of said holder the opportunity to exercise the options and earn the difference between the future value of the insurance benefit discounted by a fixed interest rate – the strike price of the option, and the future value of the insurance benefit discounted by the real interest rates which the option writer achieves on the investments through the exercise date. The randomness of the interest rate is modulated by two stochastic processes: the Ornstein-Uhlenbeck (OU) process and the Vasicek process. In each case considered, an explicit expression of the value of the option contract is given, as are numerical examples.

Keywords: American put option, European put option, exotic option, Ornstein-Uhlenbeck process, pure endowment insurance, risk insurance, Vasicek process.
JEL Classification: G21, G23, G24.

Introduction

We propose an option defined on life insurance contracts, whereby insured parties can buy European and American put options on their insurance benefit. The kind of options proposed here is a type of gamble between the option writer and the insured parties who purchase the options, on the interest rate the option writer will achieve by the exercise date of the options. We first consider a European put option defined on a pure endowment insurance contract.

In a pure endowment insurance contract, the insured will have the insurance benefit only by surviving through the maturity date of the policy contract. In a European option contract, the option can be exercised only at the exercise date of the option contract. We suggest a combination of these two types of contracts to form a European option, meaning that option holders can exercise the option only if they survive through the exercise date. In this option, an investor interested in buying this contract has a subjective view on the interest rate that he thinks the option writer will achieve on his investments, and he is willing to gamble on it. On the other hand, the option writer thinks that he could achieve a higher interest rate than is written in the option contract, so he sells a put option in which he has a commitment to pay the option holder at the exercise date (if the option holder survives until this date) an amount of money B, which is the benefit insurance defined in the option contract, discounted by the difference between the fixed interest rates (the subjective interest rate of the option buyer) and the real interest rates achieved on the investment. If the option writer achieves a higher interest rate than the one written in the option contract, the value of the option contract is zero. Thus, option holders gain from holding this option only if two conditions are met: they need to survive through the exercise date, plus the option writer has to achieve a lower interest rate than the one defined in the option contract. Note that this type of option is a put option, because when exercising the option contract the option holder can "sell" to the option writer for a strike price – namely that the future value of the defined insurance benefit discounted by a fixed (subjective) interest rate – and gain the difference between the strike price of the option and the future value of the insurance benefit discounted by the real interest rate achieved by the option writer at the exercise date. Also, note that it is not particularly difficult to monitor the interest rates the option writer achieves by the exercise date of the option: in Israel. For example, insurance companies are obligated to report each month to the government insurance supervisor the interest rate achieved for each type of life insurance contract. The result is full transparency of the interest rates achieved. Moreover, note that the fixed interest rate defined in the option contract should be higher than the interest rate of the long-term government or corporate bonds; otherwise no one will purchase this type of option contract. Given the aforementioned circumstances, the option writers need to invest in the stock market or in other derivatives, in order to achieve a higher interest rate than the one defined in the option contract, to ensure that it is not exercised.

Next, we consider an American put option defined on a risk insurance contract. In a risk insurance contract, if the insured does not survive through the maturity date of the policy contract, the beneficiary receives the sum assured from the insurance company as defined in the insurance contract after death occurrence. In an American put option, the option can be exercised at any point during the life of the option. We suggest that a combination of these two contracts constitutes an exotic option. This means that if the option holder dies prior to the maturity date, the beneficiary could exercise the option contract and receive the difference between the commitment of the option writer, which is the future value of the benefit insurance discounted by a fixed interest rate – the strike price of the option contract, and the future value of the insurance benefit discounted by the real interest rates which the option writer achieves on his investments. Thus, the beneficiary of the option holder only gains from exercising this option if the option writer achieves a lower interest rate than the one defined in the option contract. Note that this type of American put option is not a typical one in the sense that the owner of the option contract does not choose when to exercise the option, since the exercise date depends on the death of the option holder which is supposed to be random (unless we allowed suicide – which in our case, we do not). But such an American option can be viewed in the sense that the exercise date could be any day until the end of the term of the option contract.

For both types of option contracts considered here, we use two kinds of stochastic processes to modulate the randomness of the interest rates: the Ornstein-Uhlenbeck (OU) process and the Vasicek process. In each one of these stochastic processes, we evaluate the prices of these options.

Actuaries and finance researchers have long been aware of the random nature of interest rates, particularly when dealing with long-term contracts. Recent studies also integrate the mathematics of finance as a part of the mathematics of insurance. Starting with unit-linked life insurance, Bernnan and Schwartz (1976) recognized the option structure of a unit-linked life insurance contract with a guarantee. Briys and de Varenne (1994) deal with the bonus option of the policy-holder and the bankruptcy option of the (owners of the) insurance company in terms of contingent claims analysis. Other recent studies dealing with the bonus option are Miltersen and Persson (1998) or Grosen and Jørgensen (2000).

Other contexts in which two or more stochastic processes govern the life of a put option that have been studied in the literature are the pricing of put options on defaultable bonds or swaps, and the pricing of Asian exchange rate options under stochastic interest rates. The study of options in other contexts, in which two or more stochastic processes govern the life of defaultable bonds or swaps, has a long history, but the seminal paper in this field is most likely the one written by Duffy and Singleton (1997). There, the riskless, instantaneous interest rate is adjusted by the firm issuing the bond or swap default hazard, to yield a model that formally resembles the default-free case, and that can be resolved in a similar manner. The adjustment, however, involves the sum of two hazard-like terms that imply independence, despite the fact that some type of relationship probably exists between the default hazard and instantaneous interest rate. Similarly, Asian options are written on the exchange rate in a two-currency economy. In valuing these options, both the stochastic nature of the foreign and domestic zero-coupon bond prices and the exchange rate process are modeled. A recent treatment of the problem is given by Nielsen and Sandmann (2001), in which the two countries' zero-coupon bond price processes are assumed to be independent geometric-Brownian motions, but the exchange rate process is modeled by a stochastic differential equation that is a geometric Brownian motion based on the difference of the short-term interest rate processes in the two countries.

Both discrete and continuous-time stochastic models for interest rate processes have been presented in the actuarial literature, primarily Gaussian autoregressive processes. Panjer and Bellhouse (1980) provide a thorough review of autoregressive processes of order 1, $AR(1)$, and of order 2, $AR(2)$, with constant volatility (variance). They show how the force of interest may be modeled according to an $AR(1)$ or $AR(2)$ process, leading to formulae for the moments of the cumulative force of interest and the annuity certain function, which is the present value of \$1, n years hence. (1994), and references therein, discusses modeling the force of interest, versus modeling the accumulated force of interest, using a continuous-time autoregressive process of order 1: the OU process with a superimposed linear trend, and the Weiner process with linear trend. More recently, Milevsky and Promislow (2001) modeled the short-rate process itself as a Cox-Ingersoll-Ross (CIR) process. The CIR process is an $AR(1)$ process in continuous time, with random volatility that is proportional to the square root of the instantaneous interest rate just prior to time t. Additionally, the actuarial literature has also considered put options defined on pension insurance. Historically, the study of put options on pension plans could be regarded as an extension of the "pension put option" approach of Sharpe (1976) and Bicksler and Chen (1985) – to a "pension call" model that describes the general phenomenon of the unwillingness of fund sponsors to terminate over-funded plans. Note that in our case of the European put option, defined on pure endowment insurance, instead of receiving the sum assured as a lump sum, it can be received as an annuity. In this case, it can be considered as a European put option defined on a pension annuity. A pension put option, as described by Sharpe (1976), is the sponsor's right to abandon an under-funded pension plan. If the sponsor exercises the pension put option, it leaves the responsibility of the shortfall to either the beneficiaries, or to the PBGC (Pension Benefit Guarantee Corporation) in the short term. Sharpe (1976) argues that to preserve the value of the pension put option; the sponsor would not be motivated to terminate an under-funded plan. Consequently, the value of early termination of a defined benefit plan to the sponsor is similar to the exercise value of a call option on the pension asset portfolio to the insured. The exercise price of the call option is equal to the vested benefit at the time plan termination. Other recent studies combining call options on pension annuity insurance plans, were conducted by Ballotta and Haberman (2003), Yosef, Benzion, and Gross (2004) and Yosef (2006).

The remainder of this paper is structured as follows: In Section 2, we present a European put option defined on a pure endowment insurance contract and find an explicit expression for the value of this option contract in case of the OU and the Vasicek processes, which modulate the randomness of the interest rate process. Furthermore, some important features of these processes are provided. In Section 3, we solve the case of an American option contract, defined on risk insurance contracts in the two cases of the stochastic processes presented above. Numerics and conclusions are given in Section 4. Note that we make no attempt to factor in expenses, profits and other administrative charges, but rather assume that everything is presented on a net basis.

1. European put option on pure endowment insurance

The main purpose of this section is to evaluate the European put option defined on pure endowment insurance, as presented above, under the stochastic structure of the interest rates. As aforementioned in this option, an investor interested in buying this contract has a subjective view on the interest rate that he thinks the option writer will achieve on his investments, and he is willing to gamble on it. On the other hand, the option writer thinks that he could achieve a higher interest rate than is written in the option contract, so he sells a put option in which he has a commitment to pay to the option holder at the exercise date (in case that the option holder survives until this date) an amount of money B, which is the benefit insurance defined in the option contract, discounted by the difference between the fixed interest rates and the real interest rates achieved on the investment. If the option writer will achieve a higher interest rate than the one written in the option contract the value of the option contract is zero.

We can write this European put option contract where the mortality and the interest rate are stochastic by:

$$P_{pe}(0) = E\left[e^{-\beta t^0} - e^{-\delta t^0 - \theta X(t^0)}\right]^+ \Pr(T > t^0)B, \qquad (1)$$

where t^0 – the time from 0 to the end of the policy contract; T – random variable that describes the total lifetime of an individual; δ – constant risk-free interest intensity; β – the fixed (subjective) interest rate defined in the option; θ – constant factor; $X(t)_{t\geq 0}$ – the random interest process; B – benefit insurance defined in the option contract; $P_{pe}(0)$ – denotes the present value of this put option defined on pure endowment insurance – at time 0.

Note that formula (1) based on the assumption that the time-at-death random variable is stochastically independent of market rates some measure. Also note that the strike price of this option contract is the future value of the benefit insurance discounted by the fixed interest rate: $Be^{\beta t^0}$. In contrast is the real interest rate that the option writer achieves on his market investments through the exercise date. This interest rate changes randomly according to the stochastic structure of the interest rate process.

As aforementioned, we assume that the stochastic structure of the interest rate follows two types of stochastic processes: the OU process and the Vasicek process. These two processes have interesting behaviors. The OU process has an advantage in that its sample functions tend to revert to the initial position, a property that seems appropriate for many interest rate scenarios. The finite dimensional distributions are normal, and the process has a Markovian property (see Beekaman and Fuelling, 1990, 1991). The Vasicek model has a tendency to fluctuate around a fixed interest rate, $\delta > 0$, with an eventually stabilizing volatility. The connection between these two processes and more about OU and the Vasicek processes will be described in the following subsections.

1.1. The Ornstein-Uhlenbeck process. Let $B(t)$ be a standard Brownian motion, and let $X(t)$ be the unique solution of the stochastic differential equation:

$$dX(t) = -\alpha X(t) + \sigma dB(t), \quad X(0) = x, \qquad (2)$$

where $\alpha > 0, \sigma > 0$. $X(t)$ is termed the Ornstein-Uhlenbeck (OU) process. It is well known that the solution of (2) is a Markov process with continuous sample paths and Gaussian increments. By Karlin and Taylor (1981, p. 332), $X(t) \sim N(xe^{-\alpha t}, h(t))$ where $h(t) = (\sigma^2/2)[1 - e^{-2\alpha t}]$. Denote by $\Gamma_{X(t)}^{(\theta)} = E\left(e^{-\theta X(t)}\right)$ the Laplace Transform (LT) of $X(t)$; it can then be written by

$$\Gamma_{X(t)}^{(\theta)} = e^{-\theta x e^{-\alpha t} + \frac{\theta^2}{2}h(t)}, \quad \theta \in \mathbb{R}. \qquad (3)$$

Note that the OU process has the following properties, assuming $X(0)$ is a random variable:

$$E(X(t)) = E(x(0))\, e^{-\alpha t}$$

$$Var(X(t)) = \frac{\sigma^2}{2\alpha} + \left(Var(X(0)) - \frac{\sigma^2}{2\alpha} \right) e^{-2\alpha t}$$

$$Cov(X(s), X(t)) =$$

$$= \left[Var(X(0)) + \frac{\sigma^2}{2\alpha}\left(e^{2\alpha(\min(t,s)} - 1 \right) \right] e^{-\alpha(t+s)}.$$

If the initial random variable $X(0)$ has a normal distribution with mean zero and variance $\frac{\sigma^2}{2\alpha}$, then $X(t)$ is a stationary, zero-mean Gaussian process with covariance function

$$\rho(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|},$$

see Beekman and Fuelling (1990).

Now for the evaluation of (1), we first prove the following lemma:

Lemma 1. *Let $X(t)$ follow the OU process as described in (2), then for $\theta > 0$ the size*

$$E\left[e^{-\beta t^0} - e^{-\delta t^0 - \theta X(t^0)} \right]^+$$

can be written by:

$$e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)\, t^0}{\theta} - xe^{-\alpha t^0}}{\sqrt{h(t^0)}} \right) \right]$$

$$- e^{-\delta t^0} \Gamma^{(\theta)}_{X(t^0)} \left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - (xe^{-\alpha t^0} - h(t^0)\theta)}{\sqrt{h(t^0)}} \right) \right],$$

where $\Phi(\cdot)$ is the cumulative distribution function of the normal distribution, $\Gamma^{(\theta)}_{X(t^0)}$ is the LT of the OU process which is given in (3) at t^0 and where $h(t^0) = (\sigma^2/2)\left[1 - e^{-2\alpha t^0}\right]$.

Proof. Denote by $dF^{(y)}_{X(t^0)}$ the cumulative distribution function of the OU process at t^0 at point y, then for $\theta > 0$

$$E\left[e^{-\beta t^0} - e^{-\delta t^0 - \theta X(t^0)} \right]^+ =$$

$$= \int_{1\left\{ X(t^0) > \frac{(\beta-\delta)t^0}{\theta} \right\}} \left(e^{-\beta t^0} - e^{-\delta t^0 - \theta y} \right) dF^{(y)}_{X(t^0)}$$

$$= \int_{-\infty}^{\infty} \left(e^{-\beta t^0} - e^{-\delta t^0 - \theta y} \right) dF^{(y)}_{X(t^0)} \quad .$$

$$- \int_{1\left\{ X(t^0) < \frac{(\beta-\delta)t^0}{\theta} \right\}} \left(e^{-\beta t^0} - e^{-\delta t^0 - \theta y} \right) dF^{(y)}_{X(t^0)}$$

Now since $X(t^0) \sim N(xe^{-\alpha t^0}, h(t^0))$, we can solve these two integrals and get

$$= e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - xe^{-\alpha t^0}}{\sqrt{h(t^0)}} \right) \right]$$

$$- \left[e^{-\delta t^0} \Gamma^{(\theta)}_{X(t^0)} - e^{-\delta t^0} \int_{-\infty}^{\frac{(\beta-\delta)t^0}{\theta}} \left(e^{-\theta y} \right) dF^{(y)}_{X(t^0)} \right]$$

$$= e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - xe^{-\alpha t^0}}{\sqrt{h(t^0)}} \right) \right]$$

$$- e^{-\delta t^0} \Gamma^{(\theta)}_{X(t^0)} \left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - (xe^{-\alpha t^0} + h(t^0)\theta)}{\sqrt{h(t^0)}} \right) \right] \quad ,$$

where $h(t^0) = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t^0}\right)$.

Denote by $P^{OU}_{pe}(0)$ the price of this European put option defined on pure endowment insurance under the OU process. Thus, we can rewrite the price of the European put option presented in (1) by:

$$P^{OU}_{pe}(0) = \left\{ \begin{array}{l} e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - xe^{-\alpha t^0}}{\sqrt{h(t^0)}} \right) \right] \\ - e^{-\delta t^0} \Gamma^{(\theta)}_{X(t^0)} \left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - (xe^{-\alpha t^0} - h(t^0)\theta)}{\sqrt{h(t^0)}} \right) \right] \end{array} \right\} \Pr(T > t^0) B, \quad (4)$$

where $\Gamma^{(\theta)}_{X(t^0)}$ is the LT of the OU process given in (3) at point t^0, $\theta > 0$ and where $h(t^0) = (\sigma^2/2)\left[1 - e^{-2\alpha t^0}\right]$.

1.2. The Vasicek model. We are now interested in valuating (1), where the interest rate is modulated by the Vasicek process. Denote by $\widetilde{X}(t)$ the Vasicek process that is defined as a diffusion process satisfying the stochastic differential equation:

$$d(\widetilde{X}(t)) = \alpha(\gamma - \widetilde{X}(t))dt + \sigma dB(t), \quad (5)$$

where $(\alpha, \gamma, \sigma) > 0$ and $(B(t))_{t \geq 0}$ is the standard Brownian motion with drift 0 and variance 1 per unit time (see Baxter and Rennie, 1996, p. 15). In terms of a stochastic integral, the solution of (5) is given by

$$\widetilde{X}(t) = \gamma + (\widetilde{X}(0) - \gamma)\ e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB(s). \qquad (6)$$

According to (6), $\widetilde{X}$ has a drift towards γ of (state-dependent) size $\alpha(\gamma - \widetilde{X}(t))$, which is thus proportional to the distance from γ. Note that we can represent the Vasicek process by

$$\widetilde{X}(t) = \gamma(1 - e^{-\alpha t}) + X(t), \qquad (7)$$

where $X(t)$ is the OU process given in (2). Thus from (7) and (3), we can write the LT of the Vasicek process, $\Gamma^{(\theta)}_{\widetilde{X}(t)}$, by:

$$\Gamma^{(\theta)}_{\widetilde{X}(t)} = E\left(e^{-\theta \widetilde{X}(t)}\right) = e^{-\theta x e^{-\alpha t} - \gamma(1 - e^{-\alpha t})\theta + \frac{\theta^2}{2} h(t)}, \quad \theta \in R. \qquad (8)$$

Lemma 2. *Let* $\widetilde{X}(t)$ *follow the Vasicek process as described in* (5). *We can then write the size*

$$E\left[e^{-\beta t^0} - e^{-\delta t^0 - \theta \widetilde{X}(t^0)}\right]^+$$

by:

$$e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - (xe^{-\alpha t^0} + \gamma(1 - e^{-\alpha t^0}))}{\sqrt{h(t^0)}}\right)\right] -$$

$$-e^{-\delta t^0}\Gamma^{(\theta)}_{\widetilde{X}(t^0)}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - [xe^{-\alpha t^0} + \gamma(1 - e^{-\alpha t^0}) - h(t^0)\theta]}{\sqrt{h(t^0)}}\right)\right],$$

where $\Phi(\cdot)$ *is the cumulative distribution function of the normal distribution,* $\Gamma^{(\theta)}_{\widetilde{X}(t^0)}$ *is the LT of the Vasicek process given in* (8) *at* t^0, *and where* $h(t^0) = (\sigma^2/2)\left[1 - e^{-2\alpha t^0}\right]$.

Proof. Since $X(t^0) \sim N(xe^{-\alpha t^0}, h(t^0))$ and using (7), $\widetilde{X}(t^0)$ has a normal distribution with mean $(xe^{-\alpha t^0} + \gamma(1 - e^{-\alpha t^0}))$ and variance $h(t^0)$. So following the proof of Lemma 2, the result is as follows.

The price at time zero of these European put options defined on pure endowment insurance under the Vasicek process, $P^{Vasicek}_{pe}(0)$, can be written by:

$$P^{\text{Vasicek}}_{pe}(0) = \left\{ \begin{aligned} & e^{-\beta t^0}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - (xe^{-\alpha t^0} + \gamma(1 - e^{-\alpha t^0}))}{\sqrt{h(t^0)}}\right)\right] \\ & -e^{-\delta t^0}\Gamma^{(\theta)}_{\widetilde{X}(t^0)}\left[1 - \Phi\left(\frac{\frac{(\beta-\delta)t^0}{\theta} - [xe^{-\alpha t^0} + \gamma(1 - e^{-\alpha t^0}) - h(t^0)\theta]}{\sqrt{h(t^0)}}\right)\right] \end{aligned} \right\} \Pr(T > t^0) B,$$

where $\Gamma^{(\theta)}_{\widetilde{X}(t^0)}$ is the LT of the Vasicek process given in (8) at point t^0, $\theta > 0$ and where $h(t^0) = (\sigma^2/2)\left[1 - e^{-2\alpha t^0}\right]$.

2. American put option on risk insurance

This section examines pricing the value of a put option defined on risk insurance under the stochastic structure of the interest rates. As previously mentioned, this type of option gives the beneficiary of the option holder the opportunity to exercise this option for a strike price defined in the option contract, only in case of death of the option holder prior to the exercise date of the option. Should the option holder survive through the exercise date, the worth of this option is zero. Note that the death of the option holder can occur at any time prior to the exercise date, meaning that the option could be exercised by the beneficiaries of the option holder any time prior to the exercise date.

Now the value of this American put option can be written by:

$$P_{ri}(0) = E\left[e^{-\beta(T \wedge t^0)} - e^{-\delta(T \wedge t^0) - \theta X(T \wedge t^0)}\right]^+ B, \qquad (10)$$

where t^0, T, δ, β, θ, B, $X(t)$ as defined in the above subsection and where $P_{ri}(0)$ denotes the present value of this put option at time 0.

We now turn our attention to calculating the value of the option contract presented in (10) under the OU process, denoted by $P^{OU}_{ri}(0)$, and under the Vasicek process, denoted by $P^{Vasicek}_{ri}(0)$. We begin with the Vasicek process.

Lemma 3. *Let* $\widetilde{X}(t)$ *follow the Vasicek process as described in* (5). *Then for* $\theta > 0$, *we can write* (*10*) *by:*

$$P_{ri}^{Vasicek}(0) = B\int_{t=0}^{t^0} e^{-\beta t}\left(1-\Phi\left(\frac{\frac{(\beta-\delta)t}{\theta}-(xe^{-at}+\gamma(1-e^{-at}))}{\sqrt{h(t)}}\right)\right)dF_T^{(t)} -$$
$$-B\int_{t=0}^{t^0} e^{-\delta t}\Gamma_{\tilde{X}(t^0)}^{(\theta)}\left[1-\Phi\left(\frac{\frac{(\beta-\delta)t}{\theta}-\left[xe^{-at}+\gamma(1-e^{-at})-h(t)\theta\right]}{\sqrt{h(t)}}\right)\right]dF_T^{(t)} \quad (11)$$

where $dF_T^{(t)}$ *is the cumulative distribution function of T,* $\Phi(\cdot)$ *is the cumulative distribution function of the normal distribution, and* $\Gamma_{\tilde{X}(t)}^{(\theta)}$ *is the LT of the Vasicek process given in* (5).

Proof. Let $\tau = \min[T, t_0]$. Then for $\theta > 0$,

$$P_{ri}^{\text{Vasicek}}(0) = BE\left[e^{-\beta\tau} - e^{-\delta\tau-\theta\tilde{X}(\tau)}\right]^+$$
$$= BE_\tau\left(E_{\tilde{X}(\tau)}\left[e^{-\beta\tau} - e^{-\delta\tau-\theta\tilde{X}(\tau)}1_{\left\{\tilde{X}(\tau)>\frac{(\beta-\delta)\tau}{\theta}\right\}} \mid \tau\right]\right).$$

Now the size of the conditional expectation with respect to $\tilde{X}(\tau)$, could be written by:

$$E_{\tilde{X}(\tau)}\left[e^{-\beta\tau} - e^{-\delta\tau-\theta\tilde{X}(\tau)}1_{\left\{\tilde{X}(\tau)>\frac{(\beta-\delta)\tau}{\theta}\right\}} \mid \tau\right] =$$
$$= \int 1_{\left\{\tilde{X}(\tau)>\frac{(\beta-\delta)\tau}{\theta}\right\}}\left(e^{-\beta\tau} - e^{-\delta\tau-\theta y}\right)dF_{\tilde{X}(\tau)}^{(y)},$$

thus following Lemma 2 we get

$$e^{-\beta\tau}\left[1-\Phi\left(\frac{\frac{(\beta-\delta)\tau}{\theta}-(xe^{-\alpha\tau}+\gamma(1-e^{-\alpha\tau}))}{\sqrt{h(\tau)}}\right)\right] -$$
$$-e^{-\delta\tau}\Gamma_{\tilde{X}(\tau)}^{(\theta)}\left[1-\Phi\left(\frac{\frac{(\beta-\delta)\tau}{\theta}-[xe^{-\alpha\tau}+\gamma(1-e^{-\alpha\tau})-h(\tau)\theta]}{\sqrt{h(\tau)}}\right)\right].$$

Now given that $\tau = T$ for $T < t_0$, and that the value of this option is 0 for $T > t_0$, taking expectation with respect to $\tau =$ T, we get the required result of (10) which is provided in (11).

We are now interested in evaluating P_{ri}^{OU} which is the value of this option under the OU process.

Lemma 4. *Let* $X(t)$ *follow the OU process as described in* (2). *Then for* $\theta > 0$, *we can write* (*10*) *by:*

$$P_{ri}^{\text{OU}}(0) = B\int_{t=0}^{t^0} e^{-\beta t}\left(1-\Phi\left(\frac{\frac{(\beta-\delta)t}{\theta}-(xe^{-\alpha t})}{\sqrt{h(t)}}\right)\right)dF_T^{(t)}$$
$$-B\int_{t=0}^{t^0} e^{-\delta t}\Gamma_{X(t)}^{(\theta)}\left[1-\Phi\left(\frac{\frac{(\beta-\delta)t}{\theta}-[xe^{-\alpha t}-h(t)\theta]}{\sqrt{h(t)}}\right)\right]dF_T^{(t)}, \quad (12)$$

where $dF_T^{(t)}$ *is the cumulative distribution function of T,* $\Phi(\cdot)$ *is the cumulative distribution function of the normal distribution, and* $\Gamma_{X(t^0)}^{(\theta)}$ *is the LT of the OU process given in* (3).

Proof. Following the proof of Lemma 3 and letting $\gamma = 0$, we obtain the required result.

Numerics and conclusions

We now consider two cases of the random variable of the total lifetime of an individual, *T*. The first case is an exponential lifetime, where $\Pr(T > t^0) = e^{-\tau * t^0}$ for positive constants τ, t^0. The second case is Gompertz's low: $\mu_{age} = wc^{age}$ for positive constants *w* and *c*. In this case, we can write the survival lifetime as: $\Pr(T > t^0) = e^{-\frac{w}{\ln c}(c^{age}(c^{t^0}-1))}$, where the parameter "age" refers to the present age of the insured. In each case, we will find the prices of the put options $P_{pe}^{OU}(0)$, $P_{pe}^{Vasicek}(0)$ and $P_{ri}^{OU}(0)$, $P_{ri}^{Vasicek}(0)$, i.e. formulas (4), (9) and formulas (11), (12) respectively, under several assumptions of the parameters. We compare these results to the prices, where the probability of the insured to survive through the exercise date is 1, i.e. $\Pr(T > t^0) = 1$.

Now suppose the constant parameters of the processes are: $\delta = 0.05$, $\sigma = 0.01$, $\alpha = 0.02, \theta = 0.1, \gamma = 0.7, x_0 = 0.05$ and that the benefit insurance is *B* = \$1. The results will be given for a scenario of the fixed interest rate *β*, for some positive t^0, τ, and for the *age* parameter in the following tables (t^0 is given in years). Also, suppose that the constant parameters of Gompertz's low are: w = 10^{-4}, c = 1.1. Table 1 presents the results for the European put option defined on pure endowment insurance, i.e. formulas (4), (9) and Table 2 presents the results for the American put option defined on risk insurance, i.e. formulas (11), (12).

Table 1 outlines the prices of the European put option defined on pure endowment insurance and the sensitivity to the constant parameters. As mentioned in Section 2, the strike price for this option on the exercise date is $Be^{\beta t^0}$, meaning that the present value of the gain from exercising is $BE\left[e^{-\beta t^0} - e^{-\delta t^0 - \theta X(t^0)}\right]^+$. We can look, for example, at the first case, where the constants are $\beta = 0.03$, $\tau = 0.01$, $t^0 = 5$. Note that assuming $\beta = 0.03$ is equal to assuming a fixed interest rate of 3.0455% per year. The prices of this option contract in case of the OU process, where the survival lifetime is Pr(T > 5) = 1 is \$0.0823 and in case of the Vasicek process is \$0.0828. The price of this option in the exponential case is \$0.0782 in the OU process and \$0.0787 in the Vasicek process. This means a decline of about 5% of the price of the option from the certain lifetime case, and a decline of about 2.2% comparing Gompertz's case with the insured age 30 and 3% to the insured age 40. Further, note that the option holder pays at time zero \$0.0782 in the OU process and will receive an amount of money with a present value of \$0.0823. This means that the interest rate on the investments is 5.24% in case of survival through the exercise date.

Table 1. Prices of the European put options on pure endowment insurance: formulas (4), (9)

(β, τ, t^0, age)	$e^{-\tau * t^0}$		$\Pr(T > t^0)$ 1		$e^{-\frac{10^{-4}}{\ln 1.1}(1.1^{age}(1.1^{t^0}-1))}$	
	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$
(0.03, 0.01, 5 30) 40	0.0782	0.0787	0.0823	0.0828	0.0813 0.0799	0.0819 0.0804
(0.03, 0.015, 5 30) 40	0.0763	0.0768	0.0823	0.0828	0.0813 0.0799	0.0819 0.0804
(0.03, 0.01, 15 30) 40	0.1424	0.1431	0.1654	0.1663	0.1561 0.1423	0.1569 0.1430
(0.03, 0.015, 15 30) 40	0.1321	0.1328	0.1654	0.1663	0.1561 0.1423	0.1569 0.1430
(0.03, 0.01, 30 30) 40	0.1359	0.1365	0.1835	0.1842	0.1358 0.0840	0.1363 0.0843
(0.03, 0.015, 30 30) 40	0.1170	0.1175	0.1835	0.1842	0.1358 0.0840	0.1363 0.0843
(0.05, 0.01, 5 30) 40	0.0008	0.0011	0.0009	0.0012	0.0008 0.0008	0.0012 0.0012
(0.05, 0.015, 5 30) 40	0.0008	0.0011	0.0009	0.0012	0.0008 0.0008	0.0012 0.0011
(0.05, 0.01, 15 30) 40	0.0006	0.0011	0.0007	0.0013	0.0007 0.0006	0.0012 0.0011
(0.05, 0.015, 15 30) 40	0.0006	0.0010	0.0007	0.0013	0.0007 0.0006	0.0012 0.0011
(0.05, 0.01, 30 30) 40	0.0003	0.0006	0.0004	0.0009	0.0003 0.0002	0.0006 0.0004
(0.05, 0.015, 30 30) 40	0.0003	0.0006	0.0004	0.0009	0.0003 0.0002	0.0006 0.0004

Table 2 indicates the prices of the American put option defined on risk insurance. If we take, for example, these constant parameters $\beta = 0.03$, $\tau = 0.015$, $t^0 = 30$, we can see the prices of this option contract in the case of the OU process of a certain lifetime is \$4.4297 and in the case of the Vasicek process is \$4.2706. The price of this option in the exponential case is \$0.0356 in the OU process and \$0.0358 in the case of the Vasicek pro cess, constituting a tremendous difference between the prices. This difference derives from the lifetime probability of over 30 years. Additionally differences exist between the option prices under the Gomperz low of mortality and the exponential case, predominantly due to the lack of memory with regard to the age of the insured parties attributed to the exponential lifetime.

Table 2. Prices of the American put options on risk insurance: formulas (11), (12)

(β, τ, t^0, age)	$e^{-\tau * t^0}$		$\Pr(T > t^0)$ 1		$e^{-\frac{10^{-4}}{\ln 1.1}(1.1^{age}(1.1^{t^0}-1))}$	
	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$	$P_{pe}^{OU}(0)$	$P_{pe}^{Vasicek}(0)$
(0.03, 0.01, 5 30 40)	0.0021	0.0022	0.2211	0.2225	0.0004 0.0010	0.0004 0.0010
(0.03, 0.015, 5 30 40)	0.0032	0.0032	0.2211	0.2225	0.0004 0.0010	0.0004 0.0010
(0.03, 0.01, 15 30 40)	0.0139	0.0140	1.5309	1.5397	0.0026 0.0064	0.0026 0.0065
(0.03, 0.015, 15 30 40)	0.0200	0.0201	1.5309	1.5397	0.0026 0.0064	0.0026 0.0065
(0.03, 0.01, 30 30 40)	0.0356	0.0358	4.2497	4.2706	0.0067 0.0148	0.0067 0.0149
(0.03, 0.015, 30 30 40)	0.0491	0.0493	4.2497	4.2706	0.0067 0.0148	0.0067 0.0149
(0.05, 0.01, 5 30 40)	0.0000	0.0000	0.0037	0.0047	0.0000 0.0000	0.0000 0.0000
(0.05, 0.015, 5 30 40)	0.0001	0.0001	0.0037	0.0047	0.0000 0.0000	0.0000 0.0000
(0.05, 0.01, 15 30 40)	0.0001	0.0002	0.0118	0.0175	0.0000 0.0001	0.0000 0.0001
(0.05, 0.015, 15 30 40)	0.0002	0.0002	0.0118	0.0175	0.0000 0.0001	0.0000 0.0001
(0.05, 0.01, 30 30 40)	0.0002	0.0003	0.0202	0.0338	0.0000 0.0001	0.0001 0.0001
(0.05, 0.015, 30 30 40)	0.0002	0.0004	0.0202	0.0338	0.0000 0.0001	0.0001 0.0001

To conclude, we note that the suggestion of these options could lead insurance companies, if we think of them as option writers, to be more involved in the capital market, an objective that is very important to all parties involved, particularly in a country as small as Israel.

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