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## Pricing interest rate derivatives under different interest rate modeling: a critical and empirical comparison

### Abstract

This paper deals with issues related to the choice of the interest rate model to price interest rate derivatives. After the development of the market models, choosing the interest rate model has become almost a trivial task. However, their use is not always possible, so that the problem of choosing the right methodology still remains.

The aim of this paper is to compare some of the most used interest rate derivatives pricing models to understand what the issues are and the drawbacks connected to each one.

It is shown why and in which cases the use of each model does not give appreciable results and when, on the other hand, the opposite occurs. More exactly, it will be shown that the lack of data on the implied volatilities or the inefficiencies in the financial market can prevent the use of the market models, because a satisfying calibration of the interest rate trajectories cannot be guaranteed. Moreover, it is shown how the smile effect in the interest rate options market can affect the price provided by each model and, more exactly, that the difference between the price provided by the models and the observed market prices gets larger, as the strike price increases.

**Keywords:** Cox Ingersoll and Ross, Black Derman and Toy, Libor Market Model, Lognormal Forward-Libor Model, cap, interest rate risk.

**JEL Classification:** G12, C53, C65.

### Introduction

This paper deals with the issues linked to the valuation of interest rate derivatives such as caps and floors and, more precisely, with the question of which model practitioners should choose to obtain the fair value of Over The Counter (OTC) interest rate derivatives.

To begin with, it is possible to point out that an effective pricing process should allow to obtain the unknown price of a financial contract as consistent as possible with the observed prices of other instruments, so that arbitrage opportunities are secluded. For this reason, an effective pricing model should replicate the observed current prices of other financial securities, as far as it is possible.

In the financial markets, after the advent of the “market models”, choosing the right methodology in pricing has become almost a trivial task, because these models offer an easy way to calibrate the future trajectories of interest rates, so that the current market prices can generally be replicated. Remark that not every interest rate model offers this possibility: for example, the Libor Market Model (LMM, Brace et al. 1997), which actually is one of the most popular, allows to obtain prices consistently with the standard market practice of pricing caps, floors and swap-options by using the Black’s formula (Black, 1976).

However, if on the one hand the LMM seems to be a powerful tool in pricing interest rate derivatives, on the other hand, in some circumstances, its usage does not give satisfactory results, so that one could

think to use some other interest rate models to get a “better” price.

The aim of this paper is to point out when LMM does not give appreciable results and to show some empirical criteria on how to choose the right methodology, when practitioners face the task of evaluating interest rate derivatives.

This paper focuses in particular on three interest rate models, which are the most famous and the most used in practice: the first one (model A) is the Cox, Ingersoll and Ross (CIR, 1985) model, and it is one of the first stochastic models of the term structure proposed in literature; the second one (model B) is the Black, Derman and Toy model (BDT, 1990), and it was one of the most popular ones before the advent of Libor Market Model; the third model (model C) is the still mentioned Libor Market Model, and more exactly the Lognormal Forward-Libor Model (LFM), in the version proposed for the first time by Brace, Gatarek and Musiela (Brace et al., 1997).

The paper follows an inductive approach by reporting empirical evidence, whose results can suggest some general rules. Starting by pricing a simple interest rate derivative (e.g., a cap) by means of the three mentioned models, some shortcomings arise, as well as other some interesting aspects involved in pricing derivatives. In fact, by pricing a target contract it will be evident when the use of the LMM could not give appreciable results; moreover, by applying every model to the same target contract, the quantitative differences between the prices generated by each one will be shown. At the end of the comparison, interesting suggestions as to which model practitioners should generally choose will be available.

The paper proceeds as follows. Section 1 offers a review of the existing literature. Section 2 illustrates the target contract, the data used in pricing it as well as a brief description of the interest rates derivatives pricing models mentioned before. Moreover, this section highlights other important aspects concerning the critical application of the LMM, as well as the concept of “appreciable results”. Section 3 reports the comparison of the models and the corresponding results. In particular, this section reports some empirical evidence on the weaknesses of the BDT and of the CIR models, as well as some advice about the selection of the pricing methodology. Finally, the last section provides final remarks and conclusions.

## 1. The background

Although plenty of papers on pricing interest rates derivatives have been written, the same does not hold for topics related to the comparison between interest rate models.

In fact, it is remarkable that this paper originally provides a comparison of models with heterogeneous features, because its aim is closely linked to the necessity of choosing the pricing methodology in pricing interest rate derivatives, from a professional point of view. On the other hand, it is noticeable that the literature about comparisons of interest rate derivatives pricing models appears not to be very wide; moreover, the comparison is often made among models with homogeneous characteristics.

To begin with, it may be pointed out that a relevant work that tries to compare interest rate models with heterogeneous characteristics can be found in Khan et al. (2008), where the comparison involves the Hull-White and the Black-Karasinski model. However, the most popular market models are not considered in that work, also because its aim is linked to risk management rather than pricing issues.

It can be highlighted that important consideration on the drawbacks of the models, which are of fundamental importance to establish whether and to what extent an interest rate model can be successfully used in pricing derivatives, can be found in the works of the authors that for the first time developed the models themselves, and in particular some attention can be paid to the works of Cox, Ingersoll and Ross (1985), Black, Derman and Toy (1990), and Brace, Gatarek and Musiela (1997), and their successive developments.

Some other works that deal with the comparison between models have been developed, both, from a theoretical and empirical points of view. A comparison of valuation models can be found in Jacobs

(2007), where one of the key issues faced by the author is to establish criteria for model quality; issue which is somewhat linked to this work. However, it can be pointed out that Jacobs focuses his attention on continuous-time stochastic interest rate and stochastic volatility models, such as CIR model and Heath, Jarrow and Morton (HJM, 1992) model, but he does not take into account the LMM or any other discrete-time stochastic interest rate model.

Another important work related to this, which is closely linked in particular to the market models, can be found in Plesser, de Jong and Driessen (2001), where nevertheless, the attention is focused on the Libor Market model and on the Swap Market model only, and no comparison is made between continuous-time and discrete-time interest rate models; comparison that, on the other hand, is central in this work.

A broader, interesting analysis on interest rate derivatives pricing models is carried out by Barone (2004), where almost each kind of model is studied, included continuous and discrete time models, equilibrium and arbitrage models, one factor and multifactor models as well. However, the comparison between all these models is based on a theoretical point of view only, where aspects linked to the concrete application to pricing, hedging and risk management issues are not central in those work.

This work will in fact try to use an approach similar to that followed by Jacobs and Plesser, which is based on empirical analysis, without renounce to report some important considerations on the financial theories on which models are based; considerations which can moreover be used to carry out some important conclusions about the use of interest rate models themselves.

Finally, it is noticeable that consideration about the usage of the BDT model can be found in Bali et al. (1999), where a comparison between two different approaches in determining the volatility parameters is offered. Also, if the approach to estimating the volatility is completely different, this issue is faced in this paper too.

## 2. Pricing interest rate derivatives

In this section target contract, necessary data and applied models are presented. To begin with, the target contract and the models will be presented (par. 2.1 and 2.2 respectively); data will be shown (par 2.3) also after, because necessary data depend both, on the kind of contract and on the model considered. The models will be sketched since we want to understand how to use it in pricing, and to highlight qualities and drawbacks of each one for the comparison that will be done in the next section.

**2.1. Target contract.** To put in place the comparison between models, a simple contract is chosen also because in this way it will be possible to understand how large is the difference between the price provided by each model and the price provided by the Black's formula. In this way it will also be possible to understand what are the reasons for such differences in prices, and some advice on how to minimize this difference could arise. This is a key consideration when a practitioner faces the task of choosing the pricing model, and also because the use of the Black and Scholes' (1973) approach is recommended by the central banks. So it can be important to understand if, and for what reasons, the model under observation produces a price considerably different from market standards.

For these reasons, the contract that will be priced in the next section is a one year plain vanilla cap (that can be easily priced by using the Black formula), written on the three-month Euribor and made of four paid-in-arrears caplet. This means that the pay off of each caplet  $C(T_j + \delta)$ , with maturity date  $T_j$ , with  $j=1,2,3,4$ , and tenor  $\delta = .25$ , at the settlement date  $T_j + \delta$ , will be:

$$C(T_j + \delta) = \max\{L(T_j, T_j + \delta) - K; 0\}N\delta, \quad (1)$$

where  $T_1 = .25$ ,  $T_{j+1} = T_j + 0.25$ ,  $L(T_j, T_j + \delta)$  is the three-month Euribor at the reset date  $T_j$ ,  $K$  is the strike rate, and  $N$  is the notional amount, and it equals €100,000. If this is the case, the value of the caplet at each maturity date  $C(T_j)$  will be the present value of  $C(T_j + \delta)$ :

$$C(T_j) = \frac{\max\{L(T_j, T_j + \delta) - K; 0\}N\delta}{1 + L(T_j, T_j + \delta)\delta}. \quad (2)$$

At the valuation date  $t = 0$  the value of the cap will be given by the sum of each  $t$ -time caplet.

**2.2. Interest rate derivatives pricing models.** As told before, the models considered in this work are the CIR, the BDT and the LMM.

The CIR model (model A) is a continuous-time equilibrium model where the instantaneous short rate dynamic under the risk-neutral probability measure is described by the following stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma_{cir}\sqrt{r_t}dW_t, \quad (3)$$

where  $r_t$  is the  $t$ -time value of the instantaneous short rate,  $k$ ,  $\theta$ , and  $\sigma_{cir}$  are positive constants representing respectively the mean reversion rate, the long period mean and the volatility of  $r_t$  in the

CIR model;  $dW_t$  is a Wiener increment. The Feller condition  $2k\theta > \sigma_{cir}^2$  ensures that the origin is inaccessible to the process (3), so that the short rate will never be negative.

One of the main problems of the CIR model is how to estimate the constants  $k$ ,  $\theta$ , and  $\sigma_{cir}$ . In fact, it is generally known that an estimate of these parameters ensuring a perfect fitting of the observed term structure is extremely difficult and not always satisfying. This drawback can be, however, removed by using some particular extension of the model, such as the CIR++ (Brigo and Mercurio, 2002), where a correction term is added to the short rate so that the bond prices provided by it are identical to those observed in the market. Although it is possible to improve the model, this extension will not be considered in this work.

However, to estimate the parameters of the (3) a procedure based on current market data is put in place. The approach is similar to those used by Brown and Dybvig (1986): the vector of the parameters  $\beta = [\phi_1, \phi_2, \phi_3, r_t]$  is estimated by minimizing the squared differences between the observed bond prices  $v(t, T_j)$  and the theoretical bond prices  $v(t, T_j, \beta)$  provided by the model,

$$v(t, T_j, \beta) = A(t, T_j)e^{-B(t, T_j)r_t}, \quad (4)$$

where  $A(t, T_j)$  and  $B(t, T_j)$  are respectively a state contingent cash flow and a temporal parameter generated on the base of the (3), and are defined as follows:

$$A(t, T_j) = \left[ \frac{\phi_1 \exp\{\phi_1(T_j - t)\}}{\phi_2(\exp\{\phi_1(T_j - t)\} - 1) + \phi_1} \right]^{\phi_3},$$

$$B(t, T_j) = \frac{\exp\{\phi_1(T_j - t)\} - 1}{\phi_2(\exp\{\phi_1(T_j - t)\} - 1) + \phi_1}.$$

In this way, we can obtain the following parameters

$$\phi_1, \phi_2, \phi_3 \text{ where: } \phi_1 = k - \lambda, \phi_2 = k, \phi_3 = -\frac{\theta}{\lambda};$$

with  $\lambda$  constant representing the market price of risk, and where the volatility parameter of the process is given by:

$$\sigma_{cir} = \sqrt{2(\phi_1\phi_2 - \phi_2^2)}.$$

To obtain the vector  $\beta$  it will be assumed that:

$$Y = v(t, T, \beta) + \varepsilon, \quad (5)$$

where  $Y$  represents the vector of the observed market prices,  $v(t, T, \beta)$  is the vector of the theo-

retical prices and  $\varepsilon$  is the vector of the errors. In this way, the vector  $\beta$  can be obtained by solving the following problem by means of Marquard's algorithm:

$$\min_{\beta} [Y - v(t, T_j, \beta)] [Y - v(t, T_j, \beta)]$$

$$ZBC(t, T_j, T_i, X) = v(t, T_i, \beta) \chi^2 \left( 2\bar{r} [\rho + \psi + B(T_j, T_i)] \frac{4k\theta}{\sigma_{cir}^2}, \frac{2\rho^2 r_t \exp\{h(T_j - t)\}}{\rho + \psi + B(T_j, T_i)} \right) + Xv(t, T_j, \beta) \chi^2 \left( 2\bar{r} [\rho + \psi] \frac{4k\theta}{\sigma_{cir}^2}, \frac{2\rho^2 r_t \exp\{h(T_j - t)\}}{\rho + \psi} \right), T_j < T_i, \tag{6}$$

where  $\chi^2(x; a, b)$  is the noncentral chi-squared distribution function with  $a$  degrees of freedom and non-centrality parameter  $b$ , and where:

$$\rho = \frac{2h}{\sigma_{cir}^2 \exp\{(T_j - t)h\} - 1}, \quad \psi = \frac{k + h}{\sigma_{cir}^2},$$

$$\bar{r} = \frac{\ln\left(\frac{A(T_j, T_i)}{X}\right)}{B(T_j, T_i)}, \quad h = \sqrt{k^2 + 2\sigma_{cir}^2};$$

Secondly, the price of the corresponding put bond option can be obtained by using the put-call parity (Black and Scholes, 1973):

$$ZBP(t, T_j, T_i, X) = ZBC(t, T_j, T_i, X) - v(t, T_i, \beta) + Xv(t, T_j, \beta) \tag{7}$$

Thirdly, the price of the caplet is obtained by the following relations:

$$C(t) = N(1 + X\delta)ZBP(t, T_j, T_i, X) \tag{8}$$

The BDT model (model B) is an arbitrage free discrete-time short rate model, which allows to obtain a binomial tree for the dynamic of the short rate. Once the tree is obtained, the fundamental theorem of the finance (Duffie, 2001) can be applied to calculate the price of a wide range of interest rates derivatives. Despite the CIR model, it can not allow to obtain closed form formulas and the price of interest rates derivatives shall be evaluated numerically.

On the other hand, this model provides for an excellent calibration to the observed bond prices which, in every time, can be perfectly replicated from the model. Unfortunately, the same does not hold for the price of caps and floors, which cannot be perfectly duplicated by the model, as it will be shown after.

To obtain an interest rate tree, it is necessary to solve a system of  $n$  non-linear equation in  $n$  unknown, where  $n$  depends on the length of the tree. To obtain a tree, arbitrage free prices of zero coupon

Once the parameters are estimated, the price of a paid-in-arrears caplet can be obtained firstly by calculating the price of a call bond option written on a coupon bond with strike price  $X = \frac{1}{1 + K\delta}$ , by using the following formula:

bonds, as well as a term structure of the volatility, are necessary. For example, to obtain a steps tree for the one year Libor rate  $L_j$ , with  $j=0,1,2$  it is necessary to solve the following system:

$$\left\{ \begin{aligned} v(t, T_1) &= \frac{1}{1 + L_0} \\ v(t, T_2) &= E_t^p \left[ \frac{1}{(1 + L_0)(1 + L_1)} \right] \\ v(t, T_3) &= E_t^p \left[ \frac{1}{(1 + L_0)(1 + L_1)(1 + L_2)} \right] \\ \sigma_{bdt}(t, T_1) &= \frac{1}{2} \ln \left( \frac{L_1^u}{L_1^d} \right) \\ \sigma_{bdt}(t, T_2) &= \frac{1}{2} \ln \left( \frac{L_2^{uu}}{L_2^{ud}} \right) \\ \frac{L_2^{uu}}{L_2^{ud}} &= \frac{L_2^{du}}{L_2^{dd}} \\ L_2^{du} &= L_2^{ud} \end{aligned} \right. ,$$

where  $v(t, T_{j+1})$  is the arbitrage free price of a zero coupon bond with maturity  $T_{j+1}$ ,  $L_j^m$  is the one-year Libor rate at the reset date  $T_j$  in the state of world  $m$ ,  $\sigma_{bdt}(t, T_{j-1})$  is the observed volatility, used by the CIR model, of the short rate for the maturity  $t_i$  and where  $E_t^p$  indicates the conditional expected value, at the information available at time  $t$ , under the risk-neutral probability  $p$ , so that we can obtain, for example:

$$v(t, T_2) = \frac{1}{(1 + L_0)} \left( p \frac{1}{(1 + L_1^u)} + (1 - p) \frac{1}{(1 + L_1^d)} \right)$$

In this way, the following tree can be extracted from the market information:

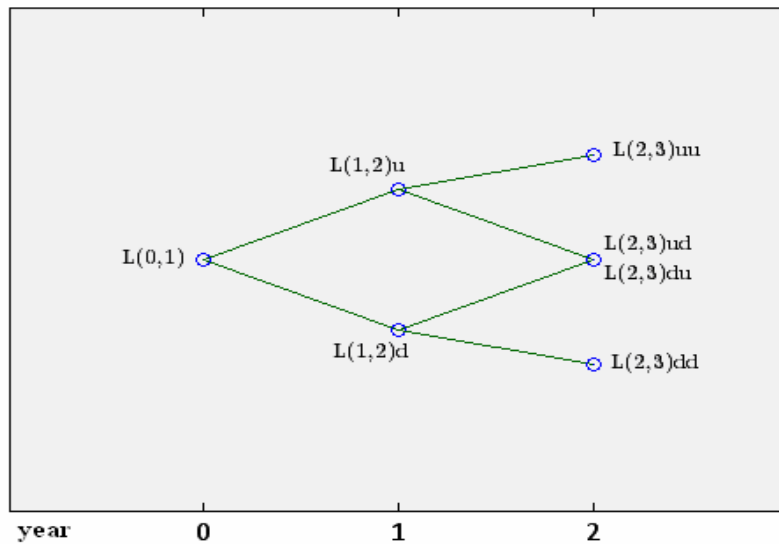


Fig. 1. The BDT tree

Once the tree is obtained, it can be used to get, for example, the  $t$  price of a paid-in-arrears caplet with maturity  $T_2=2$  years, written on the one-year Libor rate (assuming  $p = \frac{1}{2}$  constant through the time):

$$C(t) = \left[ \frac{C_{T_2}^{uu}}{(1+L_0)(1+L_1^u)(1+L_2^{uu})} + \frac{C_{T_2}^{ud}}{(1+L_0)(1+L_1^u)(1+L_2^{ud})} + \frac{C_{T_2}^{du}}{(1+L_0)(1+L_1^d)(1+L_2^{du})} + \frac{C_{T_2}^{dd}}{(1+L_0)(1+L_1^d)(1+L_2^{dd})} \right] \frac{1}{4},$$

where  $C_{T_2}^m = \max\{L_2^m - K; 0\}$ .

It is remarkably that the application of the BDT model requires to specify the values of  $\sigma_{bd}(t, T_{j+1})$ . In practice, the implied volatility is largely used to calibrate this model, because this measure of volatility is affected only by the information at the valuation date, and the past information cannot influence its value. However, from a theoretical point of view, the implied volatility obtained by using the Black formula, should represent the volatility of the forward rate dynamic, not of the spot rate, which is the lonely risk factor considered in the model. On the other hand, since no closed-form formula is attainable from the model, it is not possible to get an equivalent implied volatility using BDT.

The Libor Market model (model C) is an arbitrage free, multifactor continuous-time forward rate model which can allow, in every instant of time to reproduce both, the observed arbitrage free prices of bonds and of standard derivatives such as caps and floors. This is the case because, as demonstrated by Brace, Gatarek and Musiela (1997), if the forward Libor rate at the time  $t$ ,  $F_t \equiv F(t, T_j, T_j + \delta)$ , defined as:

$$F(t, T_j, T_j + \delta) = \left[ \frac{v(t, T_j)}{v(t, T_j + \delta)} - 1 \right] \frac{1}{\delta} \quad (9)$$

follows, under the working  $T_j + \delta$  forward measure probability, the process:

$$dF_t = F(t, T_j, T_j + \delta) \sigma_{LMM}(t, T_j) dW_t^{T_j + \delta}, \quad (10)$$

where  $\sigma_{LMM}(t, T_j)$  is the volatility of  $F_t$  used in the LMM, the  $t$ -time price of a paid-in-arrears caplet with maturity  $T_j$ , and settlement date  $T_j + \delta$  is given by the Black formula. Assuming the volatility to be constant  $\sigma_{LMM}(t, T_j) = \sigma_{LMM}$ , this means that the price of such caplet is:

$$C(t) = v(t, T_j + \delta) [F(t, T_j, T_j + \delta) N(d_1) - KN(d_2)] \delta N, \quad (11)$$

where  $N(x)$  is the normal standard distribution function, with parameters:

$$d_1 = \frac{\ln[F_t / K] + \sigma_{LMM}^2 (T_j - t) \frac{1}{2}}{\sigma_{LMM} \sqrt{(T_j - t)}};$$

$$d_2 = d_1 - \sigma_{LMM} \sqrt{(T_j - t)}.$$

It is remarkably that, equation (10) can be used to evaluate numerically the price of derivatives, written on the  $\delta$ -Libor rate, for what a closed formula does not exist. The resulting price will be consistent not only with the observed term structure of interest rates, but also with the observed arbitrage free prices of caps and floors. This also means that this model needs the implied Black and Scholes volatility for the calibration, and using another measure of volatility does not ensure a perfect replication of the observed arbitrage free prices. As a consequence, if

the implied volatility is not available, the use of other measures of volatility produces results that, in general, cannot be consistent with observed prices so that, in this case, all the remarks about the difficulties in the calibration for the CIR model, would hold for the Libor Market model too.

In general, it is possible to assert that the Libor Market model can be considered as a powerful tool to create an association between observed prices and prices of not listed contract, where the link between them is represented by the implied volatility. If the observed prices are not efficient, included when the market is not liquid enough, the resulting price will be consistent with a price that is not considerable as “fair value”, and it should not be considered fair value as well.

**2.3. Data.** Different models require different input data to price a target contract. The model which requires less information is the CIR, because all the parameters are estimated using only the information from the term structure of interest rates observed at the valuation date. Because the risk driver of the contract is, in our application, the three-month Euribor, to estimate the parameters, all the maturities in the Euribor yield curve are used. On the other hand, for the Libor Market model and the BDT model also data about the volatilities of interest rates are necessary.

$$\sigma_{bdt}^h(day) \equiv \sqrt{\text{var}[L(0;3)]} = \sqrt{\frac{\sum_{j=1}^n [L_j(0;3) - \bar{L}(0;3)]^2}{n-1}}, \quad (12)$$

where  $L_j(0;3)$  is the 3-month Euribor at the date  $T_j$ , with  $j = 1, 2, \dots, n = 252$ ,  $\bar{L}(0;3)$  is the sample mean of  $L_j(0;3)$ . Since the first price is calculated on 14/11/2008, the time series of the 3-month Euribor have to begin on 14/11/2007. On the other hand, for all the models, only the observed rates are necessary; so the time series of the whole Euribor yield curve is necessary only over the period in which the comparison is made (from 14/11/2008 to 15/05/2009). The time series of the Euribor is available on [www.euribor.org](http://www.euribor.org).

Once the daily volatility has been estimated to obtain the volatility at 3, 6, 9 and 12 months, the square root rule is used:

$$\sigma_{bdt}^h(T) = \sigma_{bdt}^h(day) \sqrt{T}, \quad (13)$$

where  $\sigma_{bdt}^h(T)$  is the historical volatility for the maturity  $T$  expressed in days.

On the other hand, when the implied volatility is used to calibrate the BDT model, the reverse problem occurs: the implied volatility can be considered as one year volatility and thus the issue of determin-

The comparison is made over a period of about six months and, more exactly, it is made by calculating the price, using all the three models, from 14/11/2008 to 15/5/2009, for a total amount of 121 observations.

It is interesting to highlight that:

- ◆ the use of the implied volatility from the Black formula is *generally* recommended only if the market is efficient because, otherwise, the price of the target contract would not be efficient as well, and this holds independently from how the calibration is done;
- ◆ the efficient implied volatility should be used always to calibrate the Libor Market model, but the same does not hold for the BDT model where, in particular cases, the historical volatility allows to fit better the observed caps and floors prices.

To provide some evidence about the second statement, in the next section the BDT price of the target contract will be calculated twice for each date, using two different measures of volatility: the historical volatility and the implied Black formula volatility.

The historical daily volatility  $\sigma_{bdt}^h(day)$  will be estimated using the following estimator (that is known to be an unbiased estimator):

ing the 3, 6 and 9 months volatility arises. To solve this problem, the volatility for the other maturities will be interpolated using a cubic spline interpolation method.

To calibrate the BDT model and Libor Market model, the implied volatility of caps written on the three-month Euribor is used. The volatility data are provided by Bloomberg. In particular, the volatility used in pricing the target contract is a “flat volatility”, i.e. the volatilities which solves the Black formula with respect to the whole cap. For the comparison, the implied volatility for three strike prices are also available: ATM, 2% and 6%, so it can be evaluated how the money-ness of the option can affect the resulting prices.

Finally, it is remarkable that, since the pay off of the last caplet in the target contract will be paid only after 15 months from the valuation date, the time series of the Euribor appear not sufficient. To complete the data, the Eurirs curve (the swap curve on the Euribor) will be used to interpolate the 15 months rate. In this case a linear interpolation method is used.

### 3. Results

First of all, the comparison is made through the period mentioned in the previous section, by fixing the strike price at the value of 2%. Afterward, the time will be fixed and the price for each contract will be calculated for three strike prices: ATM, 2% and 6%.

Figure 2 shows the price of the target contract, with a strike price of 2%, calculated by using all the three methodologies, from 14/11/2007 to 15/05/2007. It is noticeable that, by the end of the series, the price by the CIR model appears to be very near to the price by the Libor Market model (Black formula), and very far in the first part of the period considered. This can suggest that the CIR model can provide a results very near to the market standard when the market

appears to be stable. In fact, from 14/11/2008 to 16/03/2009 the three-month Euribor decreased from 4.22% to 1.69%, while from 16/03/2009 to 15/05/2009 the same rate has decreased till 1.25%. This can suggest that during periods in which the interest rate is volatile, the CIR model does not produce appreciable results, because it can generate prices too far from the observed prices.

By looking at the BDT price, it is firstly noticed that the price obtained by using the implied volatility is always higher than the Black's price, while the price obtained by using the historical volatility is always below it. It could be straightforward to remark that this is the case because the implied volatility is always higher than the historical volatility.

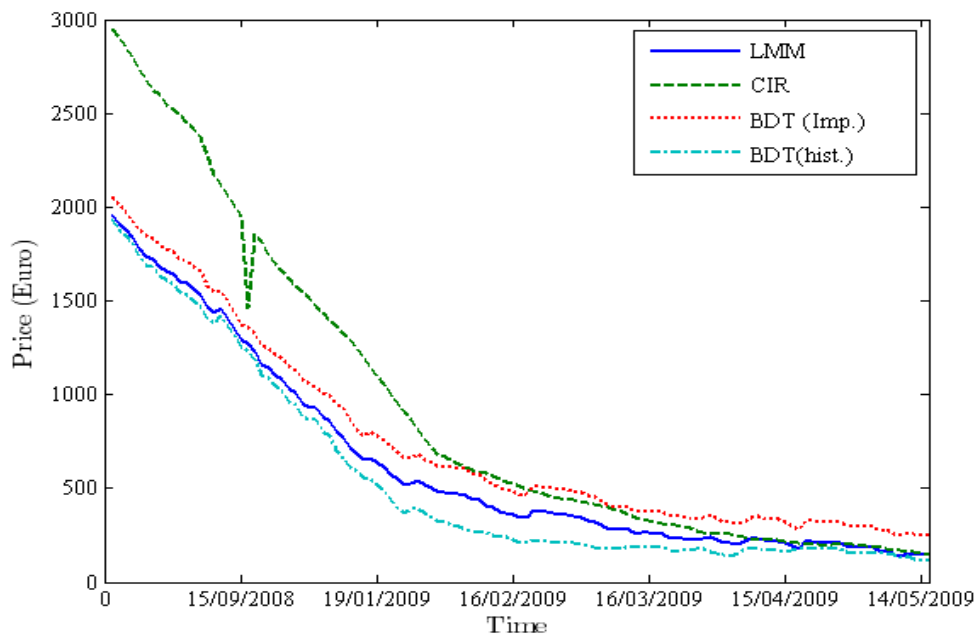


Fig. 2. The price of the target contract through the time

However, it is also possible to notice that the market price of the target contract is almost always higher than the historical volatility price and always less than the implied volatility price, so that a mean between them may be very near to the market model price.

Another important issue is that, in most cases, the BDT model provides prices that are nearer to the market standard if the historical volatility is used rather than the implied one. In fact, on 121 observations, only thirty times the use of the implied volatility produces a price nearer the Black price than the use of the historical volatility. This is consistent with the theory, which suggests that the implied Black volatility can be referred to the

dynamic of the forward rate, and not also of the short rate. In fact, this assumption implies that the interest rates of the yield curve are perfectly correlated or, better, that the term structure is flat and that it moves only according to additive shift.

A little evidence to support the previous statement can be found by analyzing Figure 3. It shows the term structure of interest rates in two different dates, on 12/02/2009 and on 15/05/2009. The regression line is also shown for both the yield curve and the R-squared index. It can be easily inferred that the slope of the regression line on 12/02/2009 is 0.0350 with  $R^2=0.75$ , and is flatter than the slope in the other regression line that is 0.0599 with  $R^2=0.82$ .

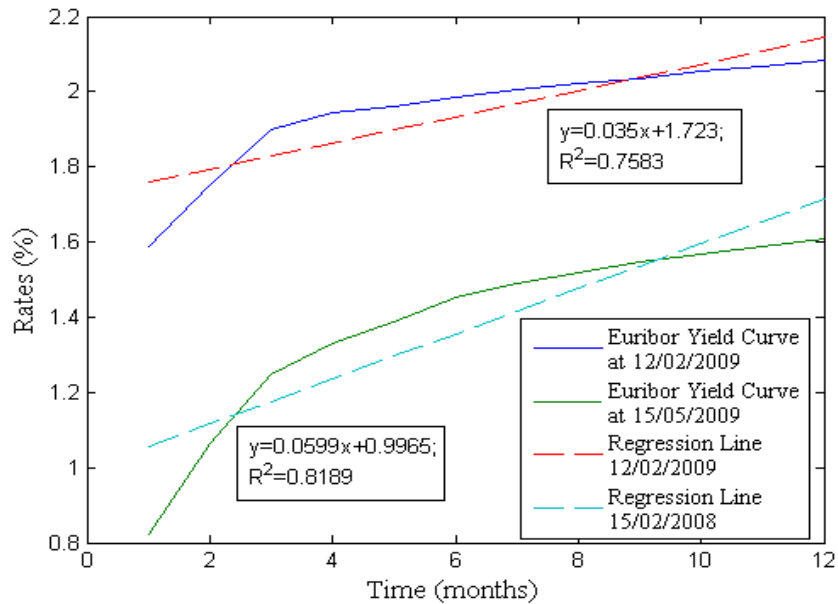


Fig. 3. Euribor Yield Curve on 12/02/2009 and on 15/05/2009, regression lines and R-squared values

However, independently of the considerations on the slope of the yield curve, it can be noticed that, from 22/01/2009 to 04/03/2009, the use of the implied volatility produces more efficient result than the use of the historical volatility. This is probably due to the fact that the historical volatility, as obvious, takes into account the information that is not actual. As the theory suggests (Fama, 1970), if the financial markets are efficient, the current value of prices and rates keep all the information available, while the old prices are, however, affected by past information. For this reason, when new information is available on the market, it is immediately reflected by the new value of prices and, thus, of the volatility. The same does not hold for the historical volatility.

If attention is paid to the comparison between prices when the moneyness of the option changes, some other issues can be noticed. In Tables 1 and 2 the results of such comparison are shown. Table 1 shows how the prices for each model on 15/05/2009 changes as the strike price changes, passing from 1.7% to 2% and to 6%, while Table 2 shows the difference, in percentage, for each strike price, between the price from Libor Market model and the price from other models.

Table 1. The price of the target contract on 15/05/2009 with different strike prices

Strike price	LFM	CIR	BDT (imp)	BDT (hist)
ATM(1,7%)	243,47	253,89	340,92	208,66
2%	143,20	145,36	249,06	112,44
6%	1,18	0,02	34,44	0,00

Note: (imp) is for implied volatility; (hist) is for historical volatility.

Table 2. Differences, in percentage, between the LMM and other models

Strike price	CIR/LMM	BDT (imp)/LMM	BDT (hist)/LMM
ATM(1,7%)	-4,28%	-40,03%	14,30%
2,00%	-1,51%	-73,93%	21,48%
6,00%	98,69%	-2814,36%	100,00%

It is possible to show that the distance in percentage of the Black price from the prices by the other models increases as the moneyness decreases. In fact, the higher the strike price, the higher the difference in prices. This effect is true almost always, except when the CIR price passes from the strike 1.7% to 2%. Only in this case the decreasing of the moneyness produces an increase in the price. However, it can be generally noticed that the price generated by the models gets further from the market price as the moneyness decreases. This effect is due to the fact that the CIR and the BDT models do not take into account the smile effect, in spite of the Libor Market model which is perfectly calibrated by using the value of the implied volatility related to the moneyness of the contract. This suggests that the less is the moneyness of the contract, the less efficient will be the price provided by CIR and BDT models. An easy way to improve the efficiency of the BDT price is to use always the ATM volatility which, being less than the other volatility (because of the smile effect), can allow to obtain a price nearer to the market standard price.

**Remarks and conclusion**

The first remark concerns the procedure used in estimating the parameters in the CIR model because, by using the Marquard’s algorithm, the optimization problem can have an infinite number of solutions, depending on the low and up bound used for the



iterations, and by the start value of the parameters from which the iterations begin. An objective criterion would be necessary on how to choose those values, so that the theoretical term structure provided by the CIR model could fit the observed term structure as well as possible.

Secondly, it can be argued that, in the BDT model, implied volatility and historical volatility along the different maturities are determined by using different methodologies: the cubic spline for the implied volatility and the square root for the historical one. However, this choice allows to obtain a value for the implied 3, 6 and 9 month volatility as lower as the value obtainable by using the square root rule and, so doing, to obtain a price nearer the market standard than otherwise.

Thirdly, it is remarkable that the comparison among strike prices is done just for one maturity; to keep stronger results it should be necessary to calculate the prices through the time. However, this remark can be neglected because the aim of the comparison is to highlight that the CIR and the BDT models were not thought to take into account problems linked to the volatility smile.

A final remark concerns the connection between the movements in the slope of the term structure and the possibility to use the implied volatility in the BDT model. In fact, the evidence provided in the previous section is quite not strong, because the R-squared index, especially in the first case, is not high enough, so that to assert a difference in the slope, a higher order interpolation method should be necessary.

It is possible to conclude by illustrating that the evidence provided in this paper suggests that the Libor Market model can provide a price, for the target derivative, rigorously consistent with the prices observed in the market of caps and floors; however, it can be used if, and only if, data on the implied volatilities are available and are based on efficient prices. If the market is not arbitrage free, or if it is not liquid enough, the price will be consistent

with a cap/floor price that cannot be considered a fair value.

Furthermore, if implied volatility is not available or useful, it could be necessary to use other pricing models because, in this case, all the three models considered in this work cannot ensure a satisfactory calibration with respect to the observed caps and floor prices. In fact, there is no reason to suggest to use other measures of volatility to calibrate the Libor Market model, so that its use does not produce appreciable advantages in respect to other models, especially when a closed form formula is available.

By observing the prices obtained, it is noticeable that if the implied volatility is not available, and the interest rates appear to be not volatile, the CIR model, as well as BDT model, can offer a price not far from the market standard. Moreover, despite the lack of data on the implied volatilities, it is still possible to adopt the BDT model and, on the other hand, it is remarkable that the use of such a kind of volatility measure is not always recommended, for the BDT, especially if the term structure appears not to be flat, so that the dynamic of the short rate can be logically considered different from the dynamic of the forward rate. This seems to be the case since the risk factor considered by the BDT model (the short rate) is somewhat different from the one used by the Black's formula (the forward rate), and the coincidence of their dynamics is not always verified. In this case, the use of the historical volatility measure, notwithstanding its weaknesses, can still allow a good fit of the BDT price to the market price, if the market is not considerably volatile.

It would finally be very interesting to know if the CIR++ can provide a price nearer to the market standard in respect to the CIR model, so to understand if the use of a more complicated model can be justified from a higher precision. Generally, it would be interesting to extend this kind of analysis to other, more sophisticated pricing models and particularly to the models with a stochastic volatility.

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