Larry R. Gorman (USA), Steven G. Sapra (USA), Robert A. Weigand (USA)

The role of cross-sectional dispersion in active portfolio management

Abstract

We derive and interpret the main results of Modern Portfolio Theory and the Theory of Active Portfolio Management from the perspective that, for active investors, the cross-sectional dispersion of returns is more relevant as a measure of risk than time series volatility. We show that all key measures of portfolio risk – total, systematic and idiosyncratic – are positively related to return dispersion, with dispersion primarily affecting idiosyncratic risk. Moreover, active portfolio returns are a function of managers' skill and cross-sectional dispersion, with realized dispersion acting as a leverage factor for realized skill. Regardless of their level of skill, however, active managers will tend to reduce their active weights as the cross-sectional dispersion of returns increases. While higher levels of dispersion represent opportunities to earn higher active returns, managers' information ratios are expected to remain unchanged, as realized tracking error is expected to vary proportionately with dispersion and managers' active returns. Absolute return investors are, therefore, more likely to benefit from tactically adjusting the activeness of their strategies with the level of return dispersion.

The role of cross-sectional dispersion in active portfolio management. Introduction

The ultimate goal of active equity management is to outperform a benchmark index such as the S&P 500 or Russell 1000. Investment managers implement active portfolios by overweighting stocks on which they have positive opinions and underweighting stocks that they view less favorably. Without the ability to determine which securities will outperform and which will underperform, managers' efforts will be futile, and their relative performance disappointing. Beyond the ability to rank winners and losers, however, active portfolio management also requires a reasonable degree of return *dispersion* in order to provide an adequate opportunity set for ranking stocks' relative expected returns. In fact, when active managers predict which stocks will perform better than others, they are essentially forecasting the cross-sectional dispersion (or standard deviation) of returns, which is simply a more formal term for the future distribution of relative winners and losers. It follows intuitively that dispersion – the extent to which stock prices will move in different directions – represents a key consideration in any forecast of relative security returns. For example, as stock returns become more dispersed, the same set of active portfolio weights will generate larger differences in relative performance. Conversely, if stock returns are perfectly correlated – implying zero cross-sectional dispersion – the notion of ranking the cross-section of returns becomes meaningless, since all stocks would yield the same return. Metrics that describe the cross-sectional behavior of asset returns should, therefore, be highly relevant in active portfolio management.

We extend this intuition by developing an analytical framework in which the major results of Modern Portfolio Theory (MPT) and Active Portfolio Management (APM) are derived and interpreted from a perspective in which cross-sectional dispersion, rather than the traditional metric time series volatility, is the relevant measure of risk. We propose that cross-sectional dispersion is a more applicable measure of risk because, in the same sense that relative-return investors are focused on generating portfolio *returns* relative to some mean benchmark return, dispersion measures *volatility* relative to the performance of the same benchmark. More technically, we substitute cross-sectional dispersion for time series volatility as the critical variable in the derivation of investors' optimal portfolio weights and active expected returns. Under the simplified covariance matrix used in this paper, the active management equations retain their simple and intuitive forms, however, which allows us to illustrate how developing a better understanding of cross-sectional dispersion and its drivers can improve active managers' performance. Our major findings and conclusions include:

♦ The cross-sectional dispersion of returns (σ_{CS}) has two main drivers. Dispersion is positively related to the average volatility of individual securities (σ) and negatively related to securities' average correlation (ρ). Understanding the differential effect of σ and ρ upon σ_{CS} has significant implications for active portfolio management.

♦ Active portfolio returns are a function of managers' skill (information coefficients) and cross-sectional dispersion, with realized dispersion acting as a leverage factor for a manager's realized skill. Active returns will be higher (lower) than expected whenever realized dispersion is higher (lower) than expected.

♦ Our analysis shows that all active managers will be averse to cross-sectional risk, regardless of their level of skill. Holding risk aversion constant, managers will tend to reduce their active portfolio weights as the cross-sectional dispersion of returns increases.

♦ Portfolio tracking error is shown to be linear in cross-sectional dispersion, which implies that unexpected changes in dispersion will result in unexpected changes in tracking error. Accurate forecasts of cross-sectional dispersion are, therefore, necessary for a manager to ensure that realized tracking error conforms with expected tracking error.

♦ Our results are consistent with the Fundamental Law of Active Management. We find that when cross-sectional dispersion rises (falls), the increase (decrease) in portfolio expected return will be proportional to the increase (decrease) in tracking error. Higher dispersion periods represent opportunities to earn higher active returns, but managers' information ratios are expected to remain unchanged.

The paper's exposition will proceed as follows. In the next section we introduce our theoretical framework. In the following two sections we illustrate key similarities between the results of time series-based MPT and our cross sectional-based model, beginning with an example of portfolio diversification, and next showing how the key drivers of dispersion (σ and ρ) affect the total, systematic, and indiosyncratic risk of a portfolio. In the final section we show that cross-sectional dispersion is a fundamental variable affecting active managers' choice of portfolio weights and their forecasts of expected returns, and illustrate dispersion's effect on the management of portfolio tracking error and the information ratios managers are able to achieve.

1. Initial perspectives: cross-sectional dispersion in theory and practice

This section provides introductory perspectives on the cross-sectional dispersion of returns. It is important to bear in mind that our analysis proceeds from the view that to active investors, return dispersion is more relevant as a measure of risk than time series volatility, the risk metric usually featured in the theoretical MPT and APM frameworks. We propose

that understanding the drivers of cross-sectional dispersion and the inherent parallels between time series- and cross sectional-based portfolio management models can enhance performance because active investors are essentially forecasting the distribution of the cross-section of returns whenever they attempt to identify future winners and losers.

We begin by considering the structure of one of the primary inputs into the portfolio construction process in Modern Portfolio Theory, the variance-covariance (VCV) matrix of stock returns. Generally, the VCV matrix is represented by an $N \times N$ matrix, where N represents the total number of securities in the market. Diagonal elements of the VCV matrix can assume different non-negative values, representing the time series variances for each security, while the off-diagonal elements represent the covariances between the returns of various pairs of assets. When a VCV matrix is unrestricted in its structural form, analytical results are often complex and difficult to interpret, however. Therefore, in the interest of tractability and intuition, our analysis employs a simplified VCV matrix of stock returns. Specifically, we assume that the VCV matrix is described by the two-parameter matrix[1]:

$$\Omega = \sigma^2(1-\rho)I + \sigma^2 \rho 1 1', \qquad (1)$$

where σ is the average time series volatility of individual stocks, ρ is the average stock-by-stock correlation, 1 is an $N \times 1$ vector of ones, and I is the $N \times N$ identity matrix. The VCV matrix Ω is required to be positive semi-definite, which can be achieved by assuming $0 \leq \rho < 1$. Additionally, Ω has the property that all stocks have a variance equal to σ^2 and a covariance with all other assets equal to $\rho\sigma^2$.

Formally, cross-sectional dispersion equals the cross-sectional standard deviation of returns (i.e., the standard deviation of returns measured *across* all stocks on a particular day or month), which takes an intuitive form under the VCV matrix in equation (1). As shown in the Appendix, under Ω, return dispersion (σ_{CS}) equals:

$$\sigma_{CS} = \sigma\sqrt{1-\rho}. \qquad (2)$$

Equation (2) shows that cross-sectional dispersion is a function of two *time series*-based parameters: the

---

[1] The variance-covariance matrix and our other modeling assumptions are the same used in recent research such as Buckle (2004) and Clarke, de Silva, Sapra and Thorley (2008).

average level of individual securities' time series volatility, as measured by σ, and the average correlation between securities, as measured by ρ. Moreover, dispersion is positively related to volatility (σ) and inversely related to the correlation of security returns (ρ), *ceteris paribus*. The differential effect of σ and ρ upon σ_{CS} has significant implications for active portfolio management.

For example, consider periods characterized by dramatic increases in both time series volatility and asset return correlations. Although investors often take for granted that higher volatility increases the payoff to active management as the spread between high- and low-performing stocks widens, and that they just need to identify which stocks will be winners and which will be losers, the above analysis shows that tactically adjusting portfolios towards more aggressive positions based on an increase in time series volatility alone only considers half of the story. Without also assessing how the mean correlation amongst assets may be simultaneously changing, active investors will be unaware that conditions for identifying future winners and losers may have become *more* difficult, despite a higher-volatility environment. If correlations "explode" (approach 1.0), return dispersion can decline as asset returns become increasingly similar, even though volatility may be increasing at the same time. The cross-sectional dispersion of returns can decline under these circumstances, which decreases the expected payoff to active portfolio management. With lower dispersion, active management (tilting portfolio weights toward or away from their inherent index weights) provides diminished value because investors' relative performance will be only negligibly different from their benchmarks. Only absolute return investors who are able to forecast the broad market's general short-term direction will have an opportunity to outperform. In a low-dispersion environment, relative return investors who fail to recognize the dynamics behind the resulting reduction in the value of active management may be disappointed with – and perhaps baffled by – their performance *ex-post*. Equation (2) provides the gateway for understanding how the expected benefits of active management change over time with the drivers of cross-sectional dispersion σ and ρ.

2. Portfolio diversification and systematic and idiosyncratic risk from the cross-sectional perspective

In this section and the one that follows, we further develop our analysis by illustrating parallels between time series- and cross sectional-based depic-

tions of portfolio diversification and systematic and idiosyncratic risk. We provide additional examples of how appreciating both frameworks can enhance a portfolio manager's understanding and performance. This section will focus on how a manager's choice of the number of stocks held in a portfolio (n) affects portfolio risk, while the next section will focus on the effects of the drivers of cross-sectional dispersion – average stock volatility (σ) and return correlations (ρ) – on the relevant measures of risk.

As shown in the Appendix, for the VCV matrix specified in equation (1), the time series (total) risk of any n-asset portfolio can be expressed as:

$$\sigma_P^2 = \rho\sigma^2 + \sigma^2(1-\rho)w_P'w_P, \tag{3}$$

where σ_P^2 is portfolio volatility and w_P is an $n \times 1$ vector of portfolio weights that sum to one[1]. For example, in the case of an equally-weighted portfolio of size n, the *total risk* (variance of returns) of a portfolio equals:

$$\sigma_P^2 = \rho\sigma^2 + \frac{\sigma^2(1-\rho)}{n}. \tag{4}$$

Substituting from Equation (2), which expresses dispersion as $\sigma^2(1-\rho)$, the volatility of an equally-weighted portfolio equals:

$$\sigma_P^2 = \rho\sigma^2 + \frac{\sigma_{CS}^2}{n}. \tag{5}$$

Equation (5) shows our first key result: the *total risk* of any equally-weighted portfolio (usually depicted as a time series-based metric) can also be expressed as a function of the cross-sectional dispersion of returns and its drivers σ and ρ. The limiting cases of Equation (5) are:

1. The equally-weighted market portfolio comprised of all N assets in the market ($n = N$).
2. The single asset ($n = 1$) case.

In these two cases total risk is expressed, respectively, as:

$$\sigma_P^2\Big|_{n=N} = \rho\sigma^2 + \frac{\sigma_{CS}^2}{N}, \tag{6}$$

and

$$\sigma_P^2\Big|_{n=1} = \rho\sigma^2 + \sigma_{CS}^2. \tag{7}$$

---

[1] The total number of assets in the market is assumed to be N, and hence $1 \le n \le N$ for any equally-weighted portfolio of size n.

In order to partition total risk into its systematic and idiosyncratic components, we express the relation between total, systematic and idiosyncratic risk for any n-asset portfolio as[1]:

$$\sigma_P^2 = \beta^2 \sigma_M^2 + \sigma_e^2 = \sigma_M^2 + \sigma_e^2, \qquad (8)$$

where β represents the portfolio's market beta (which will be equal to 1.0 under the VCV matrix depicted by equation (1)) and σ_e^2 is the portfolio's idiosyncratic risk. Substituting equation (6) (the variance of the market portfolio) into equation (8) allows us to decompose portfolio variance into its systematic and idiosyncratic components, where both are functions of the parameters σ and ρ:

$$\sigma_P^2 = \left(\rho\sigma^2 + \frac{\sigma_{CS}^2}{N} \right) + \sigma_e^2. \qquad (9)$$

Equation (9) shows that the systematic risk of a portfolio (SR_P) of any size n ($1 \leq n \leq N$) can be expressed as:

$$SR_P = \rho\sigma^2 + \frac{\sigma_{CS}^2}{N}. \qquad (10)$$

Equations (9) and (10) also show that the systematic risk of a portfolio is independent of portfolio size, n (recall that the denominator in the second term, N is the total number of assets available for investment *in the market*, which is usually different than the number of assets in the portfolio, n). Equations (9) and (10), therefore, demonstrate, from a cross-sectional perspective, a familiar principle of time series-based MPT: systematic risk is unaffected by diversification[2]. Since the systematic risk of a portfolio is invariant to changes in portfolio size, any reduction in total risk due to *diversification* occurs entirely via the idiosyncratic component of portfolio risk.

Additional insights into the relation between idiosyncratic risk and cross-sectional dispersion can be obtained by directly comparing equations (9) and (10). Specifically, for any n-asset portfolio we have

$$\sigma_P^2 = \rho\sigma^2 + \frac{\sigma_{CS}^2}{n} = \left(\rho\sigma^2 + \frac{\sigma_{CS}^2}{N} \right) + \sigma_e^2, \qquad (11)$$

which implies

$$\sigma_e^2 = \sigma_{CS}^2 \left[\frac{1}{n} - \frac{1}{N} \right]. \qquad (12)$$

Equation (12) expresses our next important result: the idiosyncratic risk of a portfolio can be expressed solely as a scaled version of cross-sectional dispersion. Note how, analogous to the results of time series-based MPT, the magnitude of this cross sectional-based expression for idiosyncratic risk decreases as portfolio size (n) increases. The impact of individual asset idiosyncratic (cross-sectional) risk is, therefore, diversified away as n increases. Moreover, because idiosyncratic risk need to be neither measured nor viewed from the more traditional time series-based framework, the risk diversification tenets of Modern Portfolio Theory can be interpreted from either a pure time series *or* cross-sectional perspective. *From a time series perspective*, diversification reduces the idiosyncratic risk of a portfolio by lowering the net effect of the constituent assets' idiosyncratic risk. *From a cross-sectional perspective*, diversification reduces cross-sectional dispersion's contribution to idiosyncratic risk, which also has a practical interpretation: when active investors increase the number of stocks in a portfolio they are diversifying away *the risk of mis-identifying future winners and losers*.

Further insights into cross-sectional dispersion's contribution to total, systematic and idiosyncratic risk can be obtained by considering the cases when $n = 1$ and $n = N$. For the single asset case ($n = 1$), equation (12) shows that *individual asset* idiosyncratic risk is virtually equivalent to cross-sectional dispersion, since as N becomes large, $1/N$ becomes very small[3]. In the limiting case, as n approaches N, idiosyncratic risk is reduced to zero, which is equivalent to saying that the impact of cross-sectional dispersion on idiosyncratic risk is completely diversified away.

As shown in equation (10), *systematic* risk also depends on cross-sectional dispersion. Therefore, total risk will always contain some exposure to cross-sectional dispersion, even for fully-diversified portfolios. For reasonable values of ρ (0.20 to 0.40), σ (0.30 to 0.70), and $N > 100$, however, the effect of cross-sectional dispersion on systematic risk will be negligible. For example, using the inputs $\rho = 0.3$, $\sigma = 0.5$ and $N = 500$, systematic variance (per

---

[1] Idiosyncratic risk is also commonly referred to as *active risk* and *tracking error*, especially amongst active, relative return-focused investors. At present, we use the more general term idiosyncratic risk. Later in the analysis, when discussing active investing, we employ the term active risk.

[2] However, this does not imply that systematic risk is a constant. As σ, σ_{CS}, or ρ change, systematic risk can change as well.

[3] In fact, for a large number of assets in the market $\left(N \to \infty \right)$, cross-sectional dispersion is equivalent to individual asset ($n = 1$) idiosyncratic risk.

equation (10)) is equal to 0.0754. In this case, the term involving cross-sectional variance equals 0.0004, which comprises less than 1% of systematic risk. Therefore, while cross-sectional risk always has some effect on systematic risk, the majority of dispersion's impact on total risk occurs via the idiosyncratic component, not the systematic component. Diversification reduces cross-sectional dispersion's effect on total risk to negligible levels.

3. How the drivers of cross-sectional dispersion affect total and idiosyncratic risk

The analysis presented in this section will show that changes in the key drivers of cross-sectional dispersion σ^2 and ρ will affect total, idiosyncratic and cross-sectional risk, but not necessarily in similar ways. Equation (5) shows that we can express the total risk of any equally-weighted portfolio of size n as:

$$\sigma_P^2 = \rho\sigma^2 + \frac{\sigma^2(1-\rho)}{n}. \tag{13}$$

From equation (2), we know:

$$\sigma_{CS}^2 = \sigma^2(1-\rho). \tag{14}$$

After substituting equation (14) into equation (12), the idiosyncratic risk of an equally-weighted portfolio of size n can be expressed as

$$\sigma_e^2 = \sigma^2(1-\rho)\left[\frac{1}{n} - \frac{1}{N}\right]. \tag{15}$$

Equations (13), (14), and (15) show that all three primary risk measures (total, cross sectional, and idiosyncratic) can be expressed in terms of the model parameters σ^2 and ρ.

We examine how changes in either σ^2 or ρ affect the three risk measures by taking the partial derivative of each expression. With three risk measures and two input variables, a total of six partial derivatives are of interest. We, therefore, compute partial derivatives for total risk:

$$\frac{\partial\sigma_P^2}{\partial\sigma^2} = \rho + \frac{1-\rho}{n} > 0, \tag{16}$$

$$\frac{\partial\sigma_P^2}{\partial\rho} = \sigma^2\left[1 - \frac{1}{n}\right] \geq 0, \tag{17}$$

idiosyncratic risk:

$$\frac{\partial\sigma_e^2}{\partial\sigma^2} = (1-\rho)\left[\frac{1}{n} - \frac{1}{N}\right] \geq 0, \tag{18}$$

$$\frac{\partial\sigma_e^2}{\partial\rho} = -\sigma^2\left[\frac{1}{n} - \frac{1}{N}\right] \leq 0, \tag{19}$$

and cross-sectional risk:

$$\frac{\partial\sigma_{CS}^2}{\partial\sigma^2} = (1-\rho) > 0, \tag{20}$$

$$\frac{\partial\sigma_{CS}^2}{\partial\rho} = -\sigma^2 < 0. \tag{21}$$

In Table 1 we report the magnitude and direction of the impact for reasonable levels of σ^2, ρ, n and N: $\sigma^2 = (0.5)^2 = 0.25$ (50% annualized volatility for a typical stock), $\rho = 0.30$, $n = 100$, and $N = 1000$. For these input values, the partial derivative values are:

Table 1. The drivers of dispersion and portfolio risk

Risk measure: VCV Input:	$\partial\sigma_P^2$	$\partial\sigma_e^2$	$\partial\sigma_{CS}^2$
$\partial\sigma^2$	0.3070	0.0063	0.7000
$\partial\rho$	0.2475	−0.0023	−0.2500

Notes: The impact of changes in σ^2 and ρ on total, idiosyncratic and cross-sectional risk, as measured by partial derivatives. The derivatives are evaluated for values of $\sigma^2 = 0.25$, $\rho = 0.30$, $n = 100$ and $N = 1000$.

As shown in Table 1 and Equations (16) through (21), the parameters that drive cross-sectional dispersion – σ^2 and ρ – make distinctly different contributions to each measure of risk. While all three measures of risk are positively related to the average level of time series variance, σ^2, this is not the case for ρ. Total risk is positively related to ρ, but idiosyncratic risk and cross-sectional dispersion are negatively related to ρ. Total risk rises with ρ because as asset correlations rise, the positive impact upon the systematic component of total risk $\rho\sigma^2 + \frac{\sigma^2(1-\rho)}{N}$ is larger in magnitude than the negative impact upon the idiosyncratic component of total risk $\sigma^2(1-\rho)\left[\frac{1}{n} - \frac{1}{N}\right]$, which leads to a net overall increase in total risk. Conversely, idiosyncratic risk declines as correlations rise because an increase in ρ compresses cross-sectional dispersion (per equation (2)) and, in our model, the idiosyncratic risk of a portfolio is simply a scaled measure of dispersion (per equations (12) and (15)). In-

tuitively, as stocks become increasingly correlated (or, equivalently, as cross-sectional dispersion declines), the risk of misidentifying high- and low-performing stocks also declines. The net result is that a rise in the correlation among securities reduces a portfolio's idiosyncratic risk, *ceteris paribus*.

The impact of dispersion on idiosyncratic and total risk can be further understood by noting that the directional impacts (signs) of the partial derivatives are the same for both idiosyncratic risk and cross-sectional risk, but the values in Table 1 differ by approximately two orders of magnitude, i.e., by approximately 100. This occurs due to interaction with the number of securities in the portfolio (n), the factor analyzed in the previous section. Equations (12) and (15) show that the link between cross-sectional risk and idiosyncratic risk depends only on N and n, and not on σ^2 or ρ. The approximate two orders of magnitude difference is explained by the factor $\left(\dfrac{1}{n} - \dfrac{1}{N}\right)$, which equals 0.009 with $N =$ 1000 and $n = 100$, with an inverse of 111.1[1]. We see that as the number of securities in a portfolio rises, cross-sectional dispersion's effect on a portfolio's idiosyncratic risk component decreases, since as noted earlier, dispersion represents the diversifiable component of returns.

The above result is intuitively comparable with time series-based MPT as well. In both frameworks, holding a more diversified portfolio (increasing n) lowers idiosyncratic risk, and for portfolios consisting of all of the stocks in the benchmark (N), idiosyncratic risk equals zero (*i.e.*, the case of an index fund). Larger portfolios will always diversify away a greater amount of cross-sectional dispersion, *ceteris paribus*. And, the more cross-sectional dispersion is diversified away, the less changes in cross-sectional dispersion can affect idiosyncratic (and total) risk.

4. Cross-sectional dispersion and active portfolio management

In the previous sections we showed that: 1) total, systematic and idiosyncratic risk are positively related to cross-sectional dispersion; 2) cross-sectional dispersion primarily affects the idiosyncratic component of portfolio risk; and 3) in the limit, as $n \rightarrow N$, the effect of cross-sectional dispersion on the idiosyncratic component of total risk is completely di-

versified away[2]. We next focus on the role of cross-sectional dispersion in active portfolio management, where investors attempt to outperform a benchmark such as the Russell 1000. Specifically, we derive investors' vectors of optimal active portfolio weights and benchmark-relative expected returns in the context of a cross sectional-based active management framework.

The essence of active investing involves constructing portfolios with position weights that differ from the benchmark weights, which induces *tracking error*[3]. Of course, investors employ active position weightings with the expectation that they will subsequently be compensated with positive benchmark-relative returns. More formally, active investors are assumed to solve the following optimization problem to determine their vector of active holdings:

$$\max_{w_A} U = w_A'E(r) - \frac{a_0}{2} w_A'\Omega w_A, \qquad (22)$$

where $E(r)$ is a mean-zero $n \times 1$ vector of benchmark-relative expected returns, a_o is the manager's coefficient of absolute risk aversion, Ω is the variance-covariance matrix given by equation (1), and w_A is an $n \times 1$ vector of active weights that represent the difference between the weight in the manager's portfolio and the weight in the benchmark index. By definition, tracking error, σ_A^2, is equal to $w_A'\Omega w_A$. In the Appendix we show that tracking error can also be expressed in terms of cross-sectional risk and active weights:

$$\sigma_A^2 = \sigma_{CS}^2 w_A'w_A. \qquad (23)$$

Making this substitution into equation (22) and maximizing with respect to w_A yields the vector of optimal weights, as given by[4]:

$$w_A^* = \frac{E(r)}{a_o\sigma_{CS}^2}. \qquad (24)$$

---

[1] This can also be seen by comparing Equations 18 and 20 and equations (19) and (21), each of which differ by the factor $\left(\dfrac{1}{n} - \dfrac{1}{N}\right)$.

[2] See equations (10) and (12).

[3] In the context of active portfolio management, idiosyncratic risk relative to a benchmark index is typically referred to as tracking error. The two are equivalent when the benchmark index is the entire market, and the beta of the portfolio equals one. From here on we will use the terminology *tracking error* when discussing active management versus an equally-weighted benchmark index of size n.

[4] The solution to equation (22) is also shown in Clarke et al. (2008) as $w = \dfrac{E(r)}{a_o\sigma^2(1-\rho)}$. With the substitution $\sigma_{CS}^2 = \sigma^2(1-\rho)$, the two expressions are equivalent.

Notice that, thus far, we have made no assumptions regarding the nature of the relative expected return vector, $E(r)$. Following Grinold (1994), Clarke, de Silva and Thorley (2006) show that an appropriate formulation for $E(r)$ is given by:

$$E(r) = IC \cdot V^{1/2} \cdot z, \tag{25}$$

where V is an $n \times n$ general VCV matrix of asset returns, IC (the information coefficient) is the assumed correlation between expected returns and actual returns (a measure of manager skill), and z is a mean-zero $n \times 1$ vector of information, assumed to distributed as $N \sim (0,1)$, that reflects a manager's opinion on the relative attractiveness of securities. Under equation (1), $V = \Omega$, and thus:

$$E(r) = IC \cdot \Omega^{1/2} \cdot z. \tag{26}$$

In the Appendix, it is shown that equation (26) is equivalent to:

$$E(r) = IC \cdot \sigma_{CS} \cdot z. \tag{27}$$

The intuition behind equation (27) is as follows. For a given level of skill (IC), the magnitude of a manager's expected returns will rise as cross-sectional dispersion increases. In the limiting case, when cross-sectional dispersion is equal to zero ($\rho = 1$ or $\sigma = 0$), the expected return vector will consist only of zeros, since there is no cross-sectional difference amongst security returns. In this case there is no opportunity to add value, and thus, no role for active management, because it is not possible to earn returns that differ from the benchmark. The potential for active management to add value is, therefore, positively related to the level of cross-sectional dispersion.

Using the expression for expected returns given by equation (27), we can re-write the vector of optimal weights (equation (24)) as:

$$w_A^* = \frac{IC}{a_o \sigma_{CS}} z. \tag{28}$$

Equation (28) shows that managers with higher levels of skill (IC) will optimally hold portfolios with larger active weights than managers with lower levels of skill. Further note that all managers, regardless of their levels of skill, are averse to cross-sectional risk, however. Given their personal level of risk aversion α_o, managers will tend to reduce their active weights as the level of cross-sectional dispersion increases.

The result that investors are averse to cross-sectional risk (depicted by the denominator of equation (28)) may appear counterintuitive at first. After all, dispersion drives the value of active investing, which might lead one to surmise that greater dispersion would lead to larger active weights, as managers attempt to "cash in" on the opportunity set provided by a higher dispersion environment. Because equation (28) is the result of maximizing expected returns *net of tracking error variance*, however, when dispersion increases, there is a trade-off between increasing utility via higher portfolio expected returns ($w_A' \cdot IC \cdot \sigma_{CS} \cdot z$) and decreasing utility via higher tracking error *variance* ($0.5 \cdot a_o \cdot \sigma_{CS}^2 \cdot w_A' w_A$). Dispersion, therefore, affects utility from different directions, with the net result being that the vector of optimal weights declines in magnitude as dispersion increases.

In practice, it is more common for active investors to optimize portfolios subject to a specific tracking error constraint than as a function of investors' (largely unobservable) risk aversion level. In the Appendix we show that equation (28) can be re-written so that a manager's vector of active weights is a function of the desired level of tracking error[1]:

$$w_A^* = \frac{\sigma_A}{\sigma_{CS} \sqrt{n}} z. \tag{29}$$

Equation (29) allows us to directly relate our cross-sectional framework to a key result of the theory of Active Portfolio Management. Substituting equations (27) and (29) into $w_A' E(r)$ results in the well-known *Fundamental Law of Active Management* (Grinold, 1989):

$$\frac{E(r)}{\sigma_A} = IC \sqrt{n}. \tag{30}$$

The left-hand-side of equation (30) is commonly referred to as the *information ratio*, and is equal to IC times the square root of n (the number of active positions in a portfolio, or "breadth")[2]. Intuitively, the Fundamental Law results because the information ratio is invariant to cross-sectional dis-

---

[1] The link between equations (29) and (28) implies that the optimal level of tracking error depends upon skill (IC), breadth (n) and risk aversion (α_0), such that $\sigma_A = a_o^{-1} IC \sqrt{n}$.

[2] Although the variable n is commonly referred to in the literature as "the number of *independent* bets," our results show that stock returns do not need to be independent in order for the fundamental law to hold. In our model stocks are correlated by ρ, yet the fundamental law still results.

persion – when dispersion rises (falls), the increase (decrease) in portfolio expected return is proportional to the increase (decrease) in tracking error, thus leaving a manager's information ratio unchanged. This result is driven by equations (27) and (29), which show that both expected returns and portfolio allocations scale directly with cross-sectional dispersion.

Equation (29) shows that a portfolio's active position weights depend on a manager's *estimates* of the critical model inputs σ_{CS} (return dispersion) and IC (skill), which in practice will be subject to error. Next we illustrate how realized values of both active returns and tracking error (the numerator and denominator, respectively, of the information ratio) also depend on *realized* values of dispersion and skill. In the Appendix we derive an expression relating a manager's realized tracking error to expected tracking error:

$$\sigma_A = E(\sigma_A)\left[\frac{\sigma_{CS}}{E(\sigma_{CS})}\right] \qquad (31)$$

Equation (31) shows that realized tracking error (σ_A) is linearly related to realized dispersion, and that tracking error will be higher or lower than expectations whenever realized dispersion is higher or lower than expectations. In practical terms, equation (31) shows that when markets are characterized by unexpected positive (negative) shocks to cross-sectional dispersion, such as those that occur prior to an economic recession, a rise (decline) in realized tracking error will result (see Gorman, Sapra and Weigand (2010) for empirical evidence). Accurate forecasts of cross-sectional dispersion are, therefore, necessary for a manager to ensure that realized tracking error is consistent with expectations.

Equation (31) provides insight into the relation between the denominator of the information ratio, σ_A, and realized cross-sectional dispersion. Not surprisingly, the numerator of the information ratio, the portfolio's expected active return, is also dependent upon realized and expected values of σ_{CS}, as well as the key input variables IC and z. Recall that the vector z describes a manager's relative opinion on the cross-section of returns, and is assumed to be distributed as a standard normal variable. The relation between opinions z, and realized benchmark relative returns r, can be measured as:

$$r = \hat{a} + \hat{\gamma}z + \hat{e}, \qquad (32)$$

where r is the $n \times 1$ realized return vector, z is the $n \times 1$ analyst opinion vector, and $\hat{a}$ is a mean-zero

vector of residuals. In this context, $\hat{\lambda}$ is the product of the manager's realized IC and realized cross-sectional dispersion[1].

In order to better understand how equation (32) relates to portfolio performance, note that a portfolio's realized active return r_A is equal to the product of active portfolio weights w_A, and realized returns, $r_A = w_A^{*\prime}r$, where w_A^* and r are given by equations (29) and (32), respectively. In the Appendix we use this relation to show that a manager's realized active return is given by:

$$r_A = E(\sigma_A)IC\sqrt{n}\,\frac{\sigma_{CS}}{E(\sigma_{CS})}. \qquad (33)$$

Similar to equation (31), equation (33) reveals that active portfolio returns are a function of realized IC and realized cross-sectional dispersion. Further note that the last term in equation (33) – the ratio of realized to expected dispersion – acts as "leverage" for a manager's realized skill. When skill is positive, higher (lower) dispersion results in higher (lower) realized returns. Conversely, if a high dispersion environment manifests when a manager experiences a negative *IC*, portfolio performance will be significantly negative. Therefore, realized cross-sectional dispersion magnifies a manager's realized skill, causing active returns to be higher (lower) than expectations when realized dispersion is higher (lower) than expectations. Actual returns and expected returns will be equal only when actual and expected dispersion are equal $\left(\sigma_{CS} = E(\sigma_{CS})\right)$ and actual skill equals expected skill $\left(IC = E(IC)\right)$.

The previous two sections analyzed the role of cross-sectional dispersion in managing portfolio risk. In this section we illustrated dispersion's role as a fundamental variable driving the performance of active portfolios. Unexpected shocks to cross-sectional dispersion have direct implications for a manager's tracking error and active returns. For example, when dispersion becomes unexpectedly elevated, one would expect to observe an increase in the tracking error of active managers. Cross-sectional dispersion is also directly related to the magnitude of active portfolio returns. Holding IC fixed, active returns are linear in cross-sectional

---

[1] Note that the regression coefficient can be expressed as $\hat{\gamma} = \dfrac{IC}{\sigma_z}\sigma_r = IC \times \sigma_{CS}$, since $\sigma_z = 1$ and $\sigma_r = \sigma_{CS}$. The second equality is due to the fact that equation (32) is a cross-sectional regression, and thus, the standard deviation of r is equal to the cross-sectional dispersion of returns.

dispersion: when dispersion is high (low), active returns will similarly be high (low). Finally, we find that because both tracking error and active returns are linear in cross-sectional dispersion, shocks to cross-sectional dispersion do not affect managers' information ratios. This occurs because dispersion shocks result in a change in tracking error that is proportional to the change in active return.

Conclusions

We derive and interpret the key results of Modern Portfolio Theory and the Theory of Active Portfolio Management starting from the perspective that, for active investors, the cross-sectional dispersion of returns is more relevant as a measure of risk than time series volatility. Our analysis demonstrates how developing a better understanding of the role of cross-sectional dispersion in active management can enhance managers' performance.

We find that cross-sectional dispersion is driven by two *time-series* based parameters: the average level of time-series volatility of individual securities (σ) and the average correlation between securities (ρ). Because return dispersion is positively related to σ but inversely related to ρ, however, it is important for active investors to remain aware of how both of these underlying variables may be changing as they forecast their high- and low-conviction stock selections and the weightings they will assign to these securities.

Interpreting portfolio diversification from a cross-sectional perspective, we find that all key measures of risk (total, systematic and idiosyncratic risk) are positively related to dispersion, and that cross-sectional dispersion primarily affects the idiosyncratic component of portfolio risk. We show that active portfolio returns are a function of managers' skill and cross-sectional dispersion, and that realized cross-sectional dispersion serves as a leverage factor for a manager's realized skill. Active returns will, therefore, be higher or lower than expected whenever realized dispersion is higher or lower than expected. Regardless of their level of skill, however, *all* active managers will be averse to cross-sectional risk. Holding risk aversion constant, managers will tend to reduce their active weights as return dispersion increases.

Portfolio tracking error is also shown to be directly related to cross-sectional dispersion, which implies that unexpected changes in dispersion will result in unexpected changes in tracking error. Accurate forecasts of cross-sectional dispersion are, therefore, necessary for a manager to ensure that realized tracking error conforms with expectations.

Because active returns and portfolio tracking error are both linearly related to cross-sectional dispersion, the results of our model are consistent with the well-known Fundamental Law of Active Management. Changes in cross-sectional dispersion represent opportunities to earn higher active returns, but managers' information ratios are expected to remain unchanged, as realized tracking error is expected to be proportional to managers' active returns. Therefore, absolute return investors are more likely to benefit from tactically adjusting the activeness of their strategies with the level of return dispersion.

References

1. Buckle, David. "How to Calculate Breadth: an Evolution of the Fundamental Law of Active Portfolio Management", *Journal of Asset Management*, 4 (6), April 2004, pp. 393-405.
2. Clarke, Roger, Harindra de Silva, and Steven Thorley. "The Fundamental Law of Active Portfolio Management", *Journal of Investment Management*, 4 (3), Q3 2006, pp. 54-72.
3. Clarke, Roger, Harindra de Silva, Steven Sapra, and Steven Thorley. "Long-Short Extensions: How Much is Enough?" *The Financial Analysts Journal*, January/Februrary 2008, pp. 16-30.
4. Gorman, Larry R., Steven G. Sapra and Robert A. Weigand. "The Cross-Sectional Dispersion of Stock Returns, Alpha and the Information Ratio". Forthcoming, *The Journal of Investing*, 2010.
5. Grinold, Richard. "The Fundamental Law of Active Management", *The Journal of Portfolio Management*, Spring 1989, pp. 30-37.
6. Grinold, Richard. "Alpha is Volatility Times IC Times Score", *The Journal of Portfolio Management*, Summer 1994, pp. 9-16.

Appendix

1. Proof of equation (2)

Assume that stock returns are distributed as $r \sim N(0, \Omega)$, where Ω is given by equation (1). Thus, the return vector r, can be expressed as $r = \Omega^{1/2} z$, where z is an $N \times 1$ vector of standard normal z-scores. Let a represent the diagonal term of $\Omega^{1/2}$ and let b represent the off-diagonal elements of $\Omega^{1/2}$. Since $\Omega^{1/2} \Omega^{1/2} = \Omega$, then a and b solve the following two equations:

$$a^2 + (N-1)b^2 = \sigma^2, \tag{34}$$

$$(N-2)b^2 + 2ab = \rho\sigma^2. \tag{35}$$

Subtracting equations (35) from (34) yields $(a-b)^2 = \sigma^2(1-\rho)$. Using the formula for the population variance $V(r) = N^{-1}\sum_i (r_i)^2$ and the zero mean property of standard normal z-scores, we have $V(r) = N^{-1}\sum_i (az_i + b(z_1 + z_2 + ... + z_{i-1} + z_{i+1} + ... + z_n))^2 = N^{-1}\sum_i (az_i - bz_i)^2$. Using the property $E(z^2) = 1$, this can be written as $V(r) = (a-b)^2 = \sigma^2(1-\rho)$. Taking the square root yields the cross-sectional standard deviation of r, $\sigma_{CS} = \sigma\sqrt{1-\rho}$.

2. Proof of equations (4) and (5)

We first prove equation (4) for a general vector of portfolio weights and then substitute the equally-weighted portfolio into the solution. The variance of a portfolio is given by $w_P'\Omega w_P$, where w_p is an $nx1$ vector of total portfolio weights and Ω is given by equation (1). Since the vector of portfolio weights sum to one, $w_P'\Omega w_P = \sigma^2\left[(w_1^2(1-\rho) + w_1\rho) + (w_2^2(1-\rho) + w_2\rho) + ... + (w_N^2(1-\rho) + w_N\rho)\right]$ and thus,

$$w_P'\Omega w_P = \sigma^2(1-\rho)\sum_{i=1}^n w_i^2 + \rho\sigma^2\sum_{i=1}^n w_i .$$

Expressing in vector notation, we have $w_P'\Omega w_P = \rho\sigma^2 + \sigma^2(1-\rho)w_P'w_P$, which proves equation (3). To arrive at equation (5), we simply need to substitute $w_P = (n^{-1}1)$, where 1 is an $nx1$ vector of ones, which yields the result

$$\rho\sigma^2 + \sigma^2(1-\rho)w_P'w_P = \rho\sigma^2 + n^{-1}(\sigma^2(1-\rho)).$$

3. Proof of equation (23)

The definition of active variance is $\sigma_A^2 = w_A'\Omega w_A$. Under the simplified covariance matrix of equation (1), active variance can be expressed as $\sigma_A^2 = \sigma^2\sum_{i=1}^n \left[w_{A,i}^2 - \rho w_{A,i}^2\right] = \sigma^2(1-\rho)\sum_{i=1}^n w_{A,i}^2$, where we have used the property that active weights are zero-mean. Taking the square root and using vector notation, active portfolio risk is $\sigma_A = (\sigma\sqrt{1-\rho})\sqrt{w_A'w_A}$ and using equation (2) this yields $\sigma_A = \sigma_{CS}\sqrt{w_A'w_A} \Rightarrow \sigma_A^2 = \sigma_{CS}^2 w_A'w_A$.

4. Proof of equation (27)

By the same logic as in the proof of equation (2), the term $\Omega^{1/2}z$ can be expressed as $(a-b)z = (\sigma\sqrt{1-\rho})z$. Thus, $E(r) = IC\cdot\Omega^{1/2}\cdot z = IC\cdot\sigma\sqrt{1-\rho}\cdot z = IC\cdot\sigma_{CS}\cdot z$.

5. Proof of equation (29)

From equations (24) and (27), $w_A^* = [\alpha_0\sigma_{CS}^2]^{-1}E(r) = [\alpha_0\sigma_{CS}]^{-1}IC\cdot z$. From the definition of tracking error variance, equation (23), we have $\sigma_A = ICa_o^{-1}\sqrt{z'z} = ICa_o^{-1}\sqrt{n}$. Solving this for a_o and substituting back into equation (24), we have $w_A^* = \left[\dfrac{\sigma_A}{IC\sqrt{n}}\right]\left[\dfrac{IC\sigma_{CS}}{\sigma_{CS}^2}\right]z = \dfrac{\sigma_A}{\sigma_{CS}\sqrt{n}}z$.

6. Proof of equation (31)

Optimal weights are generated using (29), where we augment the random variables σ_{cs} and σ_A with the expectation operator $E(\ )$ to denote the fact that optimal weights are based on ex-ante *expected* values of the relevant input vari-

ables. Thus, equation (29) is written as $w_A^* = \dfrac{E(\sigma_A)}{E(\sigma_{CS})\sqrt{n}} z$. A portfolio's realized tracking error is calculated using

equation (23), where both σ_{cs} and σ_A are considered realized values and, therefore, are not prefaced by the expectations operator. Substituting equation (29) into equation (23), and using the fact that $E\left[z'z\right] = n$, we have

$$\sigma_A^2 = \sigma_{CS}^2 \left[\frac{E(\sigma_A)^2}{E(\sigma_{CS})^2 n}\right] z'z \Rightarrow \sigma_A = E(\sigma_A)\left[\frac{\sigma_{CS}}{E(\sigma_{CS})}\right].$$

7. Proof of equation (33)

Realized portfolio active return is given by $r_A = w_A^{*\prime} r = \dfrac{E(\sigma_A)}{E(\sigma_{CS})\sqrt{n}} z'r$, where r is given by $r = \hat{a} + \hat{\gamma}z + \hat{e}$.

Let $\dfrac{E(\sigma_A)}{E(\sigma_{CS})\sqrt{N}} = a$. Thus, active return is given by $r_A = a\left[\hat{a}\sum_{i=1}^{n} z_i + \hat{\gamma}\sum_{i=1}^{n} z_i^2 + \sum_{i=1}^{n} e_i z_i\right]$. The first term in brack-

ets is zero by the zero-mean property of z-scores, the second term is equal to $\hat{\gamma}N$ since $E(z^2) = 1$, and the last term is

zero by regression orthogonality. Substituting back for a and noting that $\hat{\gamma} = IC\sigma_z^{-1}\sigma_r = IC\sigma_{CS}$, we

have $r_A = E(\sigma_A) IC\sqrt{n} \dfrac{\sigma_{CS}}{E(\sigma_{CS})}$.