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Portfolio optimization in a mean-semivariance framework

Abstract

This paper demonstrates a mean-semivariance approach to measure the downside risk in optimal portfolio selections. The authors measure the return dispersions below the expected value of investment return. Using semivariance for measuring the downside risk is consistent with the intuitive perception of risk of investors. The mean-semivariance framework offers investors a practical guidance in asset allocations and portfolio management that aim to minimize the downside risk in investment. The authors use a sample of seven exchange-traded index funds (ETF) that mimic various categories of securities such as government bonds, municipal bonds, investment grade bonds, high-yield bonds, real estate bonds, mortgage backed securities (MBS), and large capitalization stocks to compare and test the differences between the optimal portfolios and asset allocations constructed out of the mean-semivariance approach and the traditional mean-variance approach. The test results show that the mean-semivariance approach provides certain desirable benefits unavailable to a traditional mean-variance approach. Specifically, optimization under the conditions of the semivariance model produces different portfolio strategies that at least maintain and at best improve the expected return of the portfolio using traditional mean-variance model while minimizing its downside risk exposure. Our findings of the semivariance model have practical implications for both individual investors and institutional investors for asset allocations and optimal portfolio selections, as well as managing their downside risk exposure.

Keywords: downside risk measurement, lower-partial variance, portfolio choice, investment decisions, asset allocations.
JEL Classification: G10, G11, G17, G20, G30, G32.

Introduction

In the wake of 2008-2009 global financial crisis and subprime market meltdown, investors are feeling the pain of heavy losses and becoming more concerned about the downside risk of their investments. If an investor using a buy-hold strategy invested in a broadly diversified index fund or exchange traded-fund (ETF) that mimics a broad market index such as S&P 500 index, Nasdaq, and Dow Jones Industrial Average (DJIA), for a holding period from October 2007 to February 2009, his investment would have lost over 50 percent on his index fund that tracks S&P 500 or Nasdaq, and close to 50 percent on his index fund that tracks Dow Jones Industrial Average. Even for a well-managed Warren Buffet's Berkshire Hathaway investment fund which has consistently outperformed the market lost over 40 percent during this holding period which is in the midst of global financial crises. It seems that diversification alone is no longer sufficient to minimize risk. In a volatile stock market, the challenge for investors is to select an optimal portfolio that minimizes the downside risk while maximizing the upside return.

Financial economists and practitioners in the financial markets have been constantly searching for those investment strategies that can meet this challenge. Modern Portfolio Theory (MPT) founded by the seminal work of Markowitz (1952, 1970, 1987) explicitly addresses the tradeoff between risk and return on an efficient frontier curve. The Markowitz Efficient Frontier simultaneously considers the return, risk of invested assets and the correlation between returns. The curve indicates those portfolios that are at their maximum return for a given level of risk, or at their minimum risk for a given level of return. Markowitz's work has greatly changed the behavior of investors by providing more insights to general investors and fund managers. Today, Markowitz's Mean-Variance framework is widely adopted by practitioners and is perceived as the most standard optimization framework for modern investment management.

While the Markowitz mean-variance framework is sound in theory, there is however, an ongoing criticism as to how its risk is measured. The risk in the Markowitz's mean-variance framework is measured in terms of the variance of expected portfolio returns. The underlying assumption of using variance as the appropriate measure for risk is that investors weigh the probability of negative returns equally against positive returns. As argued by various scholars, variance is a measure that captures both the upside and downside movements of a security's returns (Fishburn, 1977; Tse, Uppal & White, 1993; and Swisher & Kasten, 2005). It is thus an inappropriate risk measurement. Constructing an efficient frontier with an inappropriate risk measure might lead to a nonsensical result in portfolio optimization.

Because of this limitation in using variance as a risk measurement, various downside-risk measurements have been proposed and developed. One of the downside risk measurements is semivariance. By definition, a downside-risk measurement measures only the returns below a certain threshold. This threshold captures the risk perspectives from investors to investors. Unlike standard deviation, downside risk accommodates different views of risk. Suppose an investor is concerned only about losing the initial wealth, that threshold would be zero, and the probability of losing the principal would be viewed as risky. Suppose an investor's minimum required rate of return is 10 percent, any return below ten percent would be considered risky. In the case of institutional investors, this threshold can also be certain peer performance benchmark.

Semivariance, a special case of downside risk measurement, is defined as the weighted sum of square deviations from certain threshold considering only those values below the expected value of returns (Ballestero, 2005). Semivariance also receives early support from Markowitz (1959, 1970), who presented it as an alternative risk measure. Since investors are more concerned about downside risk than about overall volatility, measuring risk by semivariance, instead of variance, produces better portfolios Markowitz (1959). The study of mean-semivariance efficient frontier has been focused on the numerical calculation of the frontier and comparison between mean-semivariance and mean-variance frontiers. To our knowledge, there has been a paucity of research in empirically testing the advantages of the semivariance model over the variance model.

In this paper, we intend to fill this gap by empirically constructing efficient portfolios and efficient frontiers using the mean-semivariance model as well as the mean-variance model so as to compare the differences between the two models' efficient frontiers and asset allocations. While acknowledging the usefulness of other downside risk measurements such as Variance-at-Risk (VaR) and Conditional Variance-at-Risk (VarCVaR), we do not intend to include VaR or CVaR into our portfolio optimization framework for the reasons discussed in the literature review. We focus instead on asset allocations and portfolio optimization in a mean-semivariance framework. Our selection of investment assets are based on the investment portfolios in the insurance industry, the industry which is most concerned with the downside risk in security investment.

The remainder of this paper is organized as follows. Section 1 outlines the literature on downside risk measurements. Section 2 describes the mean-semivariance model; section 3 presents the data and empirical asset allocation using mean-semivariance framework, and discusses empirical results; and the last section concludes this research.

1. Downside risk measurements

1.1. Markowitz's mean-variance framework. Markowitz's mean-variance approach, which leads to optimal investment decision, has two important limitations. Firstly, it assumes that the distribution of investment returns is jointly elliptically distributed, i.e., a symmetric bell-shaped distribution. However, if the underlying return data is not normally distributed, the variance is likely to give misleading results. A number of studies have demonstrated that investment returns are not normally distributed (Fama & Roll, 1968; and Jansen & de Vries, 1991). In the real financial world, security returns tend to be asymmetrically distributed, e.g., approximately lognormal distribution. The skewed distribution of investment returns makes the variance as an inefficient risk measure, because variance treats the favorable upside dispersion of investment return over the mean value of return as a part of risk and penalizes it as much as the unfavorable downside deviation from the mean returns. If the returns are not normally distributed, investors using variance or standard deviation to measure risk are likely to reach wrong asset allocation decisions. Skewness and kurtosis in real return data with non-normal distributions can cause variance or standard deviation to underestimate risk. Secondly, the mean-variance approach ignores the investor's risk aversion. Because the variance can only measure the dispersion of returns distribution around a mean, it cannot be customized for individual investors' aversion.

Moreover, the real-world implementation of Markowitz' mean-variance optimal portfolio construction has many pitfalls. The optimal portfolio constructed in a mean-variance framework may not lead to an optimal portfolio that optimizes expected returns while minimizing risk. Mitchaud (1989) indicates that these real world portfolio optimizers are essentially "error maximizers" because "optimizers" tend to treat the inputs as if they were exact quantities, while in reality they can only be estimated with error. The optimal portfolios constructed based on this framework tend to suggest large bets on stocks with large estimation error in expected returns, often leading to poor-out-of-sample performance. In fact, Markowitz (1970) himself realized the drawback of variance. He showed that both the downside risk measurement and the variance measurement can produce the same correct results when return distributions are normal. However, in situations where return distributions are not normal, the downside risk measurement is more likely to produce a better solution.

1.2. Alternative risk measures. Roy (1952) was the first to discuss the downside risk measure in investment literature. He asserted that it is reasonable for investors to reduce the possibility of disaster as much as possible, and perceived the downside deviation as the "safety-first" rule, which measures the investment risk by the probability of investment value falling below certain target or disaster level. He adds a criterion to Markowitz's efficient frontier which selects the efficient portfolio with the lowest probability to fall short of a given target value. Specifically, given an expected return, *r*, and standard deviation, *s*, investors tend to choose the portfolio that has the lowest possibility of falling below the disaster level, *d*. That is, they will try to maximize the reward-to-variability ratio, $(r-d)/s$. Roy's main contribution – the concept that investors will prefer the principal of safety first when faced with uncertainty – is instructive to the later evolution of the downside risk measurement research. Fishburn (1977) uses a utility function model to incorporate downside risk based on risk aversion level and a target return. Bawa (1978) extended Roy's work from 1st order to a more generalized n^{th} order "safety-first" rule, and showed that the n^{th} order "safety-first "rule is computationally feasible. Tse et al. (1993) further discussed the "safety-first" rule in a dynamic structure.

In the literature of risk management, several risk measures are proposed for measuring downside risks. The most prevalent risk measures debated in the risk management literature are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). CVaR is the conditional expectation of losses that exceed the VaR level and is proposed for direct and asymmetric control over the distribution of residual errors, and for constraining one of its tail means not to exceed some pre-specified value (see Jorion, 1997; Trinidade, Uryasev, Shapiro & Zrazhevsky, 2007; Rachev, Stoyanov & Fabozzi, 2011). In other words, both VaR and CVaR are downside risk measures where the objective is to maximize expected return given a VaR or a CVaR, respectively. Campbell, Huisman, & Koedijk (2001) incorporates VaR as a shortfall constraint into the portfolio selection decision by maximizing expected return subject to the constraint that the expected maximum loss should meet the VaR limits set by the risk manager. They measure risk in terms of the VaR over and above the risk free rate on the initial wealth.

However, both VaR and CVaR require an investor to specify a probability level of cumulative losses (Wiener, 1998). The problem with VaR or CVaR is that the investor has to specify a holding period, typically a very short-term horizon from 1 to 10 days and a level of probability of loss exceeding the VaR limit. Such a short-term horizon is not suitable for asset allocations and portfolio optimization for long-term investors. Moreover, the greater the time horizon is the less precise is the efficient VaR frontier (Campbell, Huisman, & Koedijk, 2001). Campbell et al. (2001) also shows that the optimal portfolios with VaR constraints are sensitive to the confidence level selected. In addition, VaR ignores extreme events below the specified quantile. Indeed, the mean-VaR optimization does not necessarily improve upon the portfolio optimization in a mean-variance framework (Alexander and Baptista, 2002). In fact, when switching from mean-variance framework to mean-VaR framework, we may end up with more volatile portfolios. Many researchers also suggest that VaR is not a coherent risk measure since the subadditivity property is not satisfied. In other words, if we combine several securities into a portfolio, the combined VaR at certain confidence level may not necessarily result in a lower VaR than the sum of the VaRs of individual securities. In fact we may end up with a higher combined VaR than the sum of the VaRs. In addition, VaR is also criticized for its incompleteness in risk measurement since it cannot provide any information about the magnitude of losses once the VaR limit is exceeded. The drawbacks of VaR can be mitigated by CVaR which is a conditional expectation that gives the expected loss beyond the VaR. For example, while VaR at 99% measures the maximum loss in 99% of cases, CVaR at 99% measures the average loss in the 1% of the worst cases. That is, CVaR measures not only the magnitude but also the likelihood of losses. However, for portfolio optimization, CVaR model requires either an assumption about the return distribution or a substantial amount of return observations below the target return which could pose a significant problem of real world data limitations for empirical research. For example, for a sample of 100 real world observations, a CVaR at 99% will be based on 1 observation only, namely, 1% (1 out of 100 observations). Although we can address this limitation using simulation techniques to generate larger samples, this is often a practical disadvantage of this model.

1.3. Semivariance as a downside risk measurement. The two downside risk measurements suggested by Markowitz are of particular interest in finance: 1) the semivariance below the mean value; and 2) the semivariance below the target return. Though he theoretically preferred semivariance, Markowitz still insisted on using variance as the risk measure simply because of the difficulty in computing lower semivariance. The major difficulty lies in gauging the co-movement, or correlation of Lower Partial Moments (LPM), which is the most important provision for investment diversification.

Complication in calculation has not prevented scholars from pursuing the study and research on the downside risk measurement. Based on the concept of downside risk, Fishburn (1977) and Harlow and Rao (1989) introduced a generalized form of lower partial moments (LPM) and developed the "$(\alpha - t)$" model, in which t represents the target return of investment or disaster level as proposed by Roy (1952), and α denotes the investor's risk aversion. The higher the value of α, the greater is the investor's risk aversion. For risk-neutral investors, $\alpha = 1$; for risk seeking investors, $\alpha < 1$; and for risk averse investors, $\alpha > 1$. Fishburn (1977) also introduced the Mean-Lower Partial Moment model (MLPM-model). Note that when $\alpha = 2$ and t equals to the mean value of the investment return, and the "$(\alpha - t)$" model is specified as a LPM model. Harlow (1991) employed LPM as a downside risk measure in portfolio selections. He defined LPM as: $LPM_n = \sum_{R_p=-x}^{T} P_p(\tau - R_p)^n$, where P_p is the probability that return, R_p occurs. He explained that the type of "moment," n, specified in the LPM equation captures an investor's preferences. For $n = 0$, the risk measure becomes a 0^{th}-order moment (LPM_0) which measures the probability of falling below the target rate. However, for $n = 1$, LPM_1 becomes the expected deviation of returns below the target. For $n = 2$, LPM_2 is analogous to variance, in that it is a probability weighting of squared deviations. Thus, LPM_2 can be referred to as a target semivariance. Harlow (1991) further explained that many popular notions of risk are special cases of the generalized LPM_n measure. For example, with $n = 0$ and a target rate = 0%, LPM_0 is simply the probability of a loss. For $n = 2$ and a target rate = mean return, LPM_2 becomes the traditional semivariance. Overall, LPM_1 (target shortfall) and LPM_2 (target semivariance) provide an intuitive set of risk definitions that are more useful than traditional approaches (Harlow, 1991).

Although theoretically and intuitively sound, LPM has its own limit, because it creates much more complexity in computation than does the variance measurement. Because Harlow and Rao (1989) failed to consider the correlation of individual asset returns, i.e., co-movement between individual asset returns that fall below the target return, and because the co-movement is very important for risk diversification, their results can only be limited to those assets whose returns are perfectly or highly correlated.

Foo & Eng (2000)'s downside risk optimization model extended the former work of Harlow & Rao (1989) by incorporating the model with downside covariance of correlated asset returns. But their work is still complicated and computationally burdened. Hogan & Warren (1974) introduced the concept of co-lower-partial-variance, which measures risky asset and market portfolio. Bawa & Lindenberg (1977) further developed this co-lower-partial-variance measure to an n-degree framework called generalized asymmetric co-LPM.

Ballestero (2005) developed a semivariance matrix using a strictly mathematical derivation while relying on the validity of Sharpe's beta regression equation. This approach has greatly eased the computation complexity, allowing one to obtain the computational results of the semivariance model from the traditional mean-variance model.

Since the 1990s, researchers have started to apply the downside risk measures to their empirical research. Sortino & Meer (1991) introduced the downside deviation, i.e., below-target deviation, and the reward-to-downside variability ratio as the tools for the measurement of downside risk. Balzer (1994) discussed the skewness existed in asset returns, and the issues of applying downside variance. Bookstaber & Clarke (1985) and Merriken (1994) tried consecutively to prove how the semivariance could be applied in the downside risk management of different hedging policies using stock options and interest rate swaps.

It is not difficult to see that the academic downside risk research has emphasized in general asset classes, e.g., the optimization of individual stocks, and the performance evaluation of various mutual funds. However, few researchers have applied this theory into other industries such as insurance industry which also needs the portfolio theory to manage asset allocation efficiently. The insurance industry, which has large amount of fund to invest, needs further discussion in the downside risk optimization theorem.

2. Portfolio optimization using mean-semivariance framework

The underlying principle for semivariance model is the same as the variance model, in that investors are willing to bring downside risk as low as possible while keeping the rate of return above a certain level. The definition of semivariance below the mean value can be expressed by the following formula:

$$SV(<) = \sum[(\sum_{j=1}^{n} \tilde{r}_{jt}\omega_j - \sum_{j=1}^{n} r_j\omega_j)^2 p(t)], \quad (1)$$

where for each observation $\tilde{r}_j$ satisfying

$$\sum_{j=1}^{n} \tilde{r}_{jt}\omega_j \le \sum_{j=1}^{n} r_j\omega_j . \quad (2)$$

Similarly, semivariance above the mean value can be expressed as:

$$SV(>) = \sum[(\sum_{j=1}^{n} \tilde{r}_{jt}\omega_j - \sum_{j=1}^{n} r_j\omega_j)^2 p(t)] \quad (3)$$

with

$$\sum_{j=1}^{n} \tilde{r}_{jt}\omega_j > \sum_{j=1}^{n} r_j\omega_j , \quad (4)$$

where $p(t)$ denotes the probability. If we assign the same probability for all observations, then we have $p(t) = 1/T$.

To further derive the simplified formula for semivariance model, we have to make an important assumption, which is the validity of Sharpe (1964) beta regression equation:

$$\tilde{r}_j = \alpha_j + \beta_j \cdot \tilde{r}_M + \tilde{\varepsilon}_j. \quad (5)$$

It states that the random variable of the jth asset's return is related to the market portfolio return, where α_j and β_j are constant, and $\tilde{\varepsilon}_j$ is a random error with zero covariance for $(\tilde{\varepsilon}_j, \tilde{\varepsilon}_h)$, zero covariance for $(\tilde{\varepsilon}_j, \tilde{r}_M)$. The market portfolio is the weighted sum of asset returns.

In addition, we can obtain β_j by:

$$\beta_j = \frac{\operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}, \quad (6)$$

where $\operatorname{cov}(\tilde{r}_j, \tilde{r}_M)$ is the covariance between the return of the jth asset and the return of the market portfolio, σ_M^2 is the variance of the market return.

Based on equation (4), we can get:

$$\tilde{r}_j - r_j = \theta_j + \beta_j \cdot (\tilde{r}_M - r_M), \quad (7)$$

where r_j and r_M are expected return of the jth asset and market portfolio respectively. $\tilde{\theta}_j$ has zero mean value.

Considering all the assets in the portfolio and adding them up based on equation (8),

$$\sum_{j=1}^{n} (\tilde{r}_j - r_j)\omega_j = \tilde{\theta} + (\tilde{r}_M - r_M)\sum_{j=1}^{n} \beta_j\omega_j , \quad (8)$$

where

$$\tilde{\theta} = \sum_{j=1}^{n} \tilde{\theta}_j \cdot \omega_j. \quad (9)$$

Based on (7), we can rewrite the equation of SV (>) as:

$$SV(>) = \sum[(\tilde{\theta} + (\tilde{r}_M - r_M)\sum_{j=1}^{n} \beta_j\omega_j)^2 p(t)]. \quad (10)$$

When the level of diversification goes to infinity, we can prove that:

$$\lim_{L\to\infty} SV(>) = (\sum_{j,h} \beta_j\beta_h\omega_j\omega_h) \cdot SV(\tilde{r}_M > r_M). \quad (11)$$

The definition of SV(>) and SV(<) implies that

$$SV(<) + SV(>) = V = \sum_{t=1}^{t}[\sum_{j=1}^{n} r_{jt}\omega_j - \sum_{j=1}^{n} r_j\omega_j]^2 p(t). \quad (12)$$

Hence, we can get the expression of SV(<) by subtracting SV(>) from V:

$$\lim_{L\to\infty} SV(<) = V - \lim_{L\to\infty} SV(>), \quad (13)$$

$$\lim_{L\to\infty} SV(<) = \sum_{j,h}[V_{jh} - \beta_j\beta_h \cdot SV(\tilde{r}_M > r_M)]\omega_j\omega_h. \quad (14)$$

Note that the definition of the level of diversification is: a portfolio is considered to reach a level L of diversification if: $\max_{j\in 1,\dots,n} \omega_j = 1/L$ and $n/L = q$ with $Q > q \ge 1$ and $\sum_{j=1}^{n} \omega_j = 1$, $\omega_j \ge 0$ for all j. Q is a constant that defines the bound of q. Then the higher the level of diversification is reached, i.e., the higher the value of L, the lower is the greatest weight.

From equation (13), we can see the whole calculation is much simplified, and thus, all the data or parameter can thus be obtained.

Hogan & Warren (1972) presents the essential mathematical properties of mean-semivariance models, where they prove the convexity and differentiability of this model. Their contributions make the theoretical and computational viability of mean-semivariance model guaranteed.

3. Data and empirical tests

In this section, we employ a sample of seven exchange-traded index funds (ETF) that mimic various categories of securities, such as government bonds, municipal bonds, investment grade bonds, high-yield bonds, real estate bonds, mortgage backed securities, and common stocks to compare and test there is any difference between the two models in asset allocation and optimal portfolios constructions. From the CRSP database, we obtain the monthly return data for each category of ETFs for the period from August 2002 to December 2007. The rationale for selecting this period is that this period does not have extreme downside movements nor upside movements as we have witnessed in the period from 2008 to 2010.

The rationale for using ETFs rather than individual stocks and bonds is that ETF is an index fund that attempts to track a basket of stocks, bonds or certain indexes but can be traded like a stock on the market, where it experiences price fluctuations throughout the trading day. This provides a broad benchmark to measure the return of a particular asset type and enables us to approach the real market return for a certain type of investments as a whole.

3.1. Application of Markowitz's mean-variance efficient frontier. The descriptive statistics of our sample are reported in Table 1 and the optimal portfolio asset allocation results constructed out of the Markowitz mean-variance efficient frontier model are reported in Table 2.

Table 1. Summary statistics of the monthly returns on investment assets

Asset types	N	Mean return	Median return	Std. dev.
Government bond	53	0.333%	0.360%	1.792%
Muni bond	53	0.554%	1.066%	3.544%
High yield bond	53	0.662%	1.103%	5.845%
Real estate	53	1.979%	3.111%	4.437%
MBS	53	0.584%	1.027%	3.383%
Investment grade bond	53	0.510%	0.713%	1.816%
Large cap stocks	53	1.037%	1.252%	3.171%
Value-weighted market returns	53	1.049%	1.282%	3.251%

Table 2. Results of the Markowitz Mean-Variance Efficient Frontier

c	Portfolio Std.dev	Portfolio expected returns	Government bond	Muni bond	High yeld bond	Real estate	MBS	Investment grade bond	Large cap stocks
0.744	1.896	0.930	0.000	0.000	0.000	25.000	0.000	65.000	10.000
0.753	1.899	0.930	0.000	0.000	0.200	25.000	0.000	64.800	10.000
0.761	1.910	0.931	0.000	0.000	0.850	25.000	0.000	64.150	10.000
0.770	1.923	0.932	0.000	0.000	1.590	25.000	0.000	63.410	10.000
0.779	1.938	0.934	0.000	0.000	2.410	25.000	0.000	62.590	10.000
0.788	1.957	0.935	0.000	0.000	3.360	25.000	0.000	61.640	10.000
0.797	1.979	0.937	0.000	0.000	4.450	25.000	0.000	60.550	10.000
0.805	2.024	0.940	0.000	0.000	5.670	25.000	1.650	57.680	10.000
0.814	2.085	0.944	0.000	0.000	7.120	25.000	3.970	53.910	10.000
0.823	2.165	0.948	0.000	0.000	8.900	25.000	6.800	49.290	10.000
0.832	2.273	0.954	0.000	0.000	11.140	25.000	10.360	43.500	10.000
0.841	2.380	0.960	0.000	0.060	13.780	25.000	12.500	38.660	10.000
0.849	2.519	0.967	0.000	3.240	17.270	25.000	12.500	31.990	10.000
0.858	2.707	0.975	0.000	6.940	21.750	25.000	12.500	23.810	10.000
0.867	2.986	0.987	0.000	12.020	27.920	25.000	12.500	12.550	10.000
0.876	3.336	1.000	0.000	17.110	35.390	25.000	12.500	0.000	10.000
0.885	3.435	1.004	0.000	13.800	38.700	25.000	12.500	0.000	10.000
0.893	3.567	1.008	0.000	9.710	42.790	25.000	12.500	0.000	10.000
0.902	3.748	1.014	0.000	4.510	47.990	25.000	12.500	0.000	10.000
0.911	3.916	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.920	3.916	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.929	3.916	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.937	4.233	1.026	0.000	0.000	61.370	25.000	3.630	0.000	10.000
0.946	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.955	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.964	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.973	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.981	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.990	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.999	4.371	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000

3.2. Application of mean-semivariance efficient frontier. In this section, we illustrate our computation of the semivariance matrix. The first step is to derive the beta coefficients of various assets. We calculate the variance and covariance matrix of different asset types. Results are shown in Table 3.

Table 3. Covariance matrix of assets

	Government bond	Muni bond	High yield bond	Real estate	MBS	Investment grade bond	Large cap stock
Government bond	0.031%	0.040%	0.048%	0.037%	0.039%	0.026%	-0.010%
Muni bond	0.040%	0.123%	0.074%	0.070%	0.063%	0.039%	-0.010%
High yield bond	0.048%	0.074%	0.335%	0.107%	0.077%	0.047%	0.002%
Real estate	0.037%	0.070%	0.107%	0.193%	0.091%	0.026%	0.029%
MBS	0.039%	0.063%	0.077%	0.091%	0.112%	0.033%	0.022%
Investment grade bond	0.026%	0.039%	0.047%	0.026%	0.033%	0.032%	-0.007%
Large cap stock	-0.010%	-0.010%	0.002%	0.029%	0.022%	-0.007%	0.099%
Std. dev	1.775%	3.511%	5.789%	4.395%	3.351%	1.799%	3.141%
Skewness	-0.782	-1.374	-0.001	-2.147	-1.133	-0.774	-0.521
E(M)	1.049%						
Var(M)	0.104%						
Cov(j,M)	-0.012%	-0.010%	-0.003%	0.027%	0.020%	-0.008%	0.101%
Beta	-0.111	-0.098	-0.027	0.262	0.197	-0.072	0.972
V(>)	0.046%						

The second step is to compute the market mean and variance values as follows:

$$E_M = \frac{1}{T}\sum_{t=1}^{T} vwretd_t \, , \tag{15}$$

$$\sigma_M^2 = \frac{1}{T-1}\sum_{t=1}^{T}(vwretd_t - E_M)^2 \, , \tag{16}$$

where *vwretd* is the value-weighted-return of the market. The covariance of different assets with the market return is calculated as follows:

$$Cov(X_j, X_M) = \frac{1}{T}\sum_{t=1}^{T}(\tilde{R}_{jt} - R_j)(vwretd_t - E_M). \tag{17}$$

Then beta is calculated according to equation (18):

$$\beta_j = \frac{Cov(j,M)}{\sigma_M^2}. \tag{18}$$

The third step is to calculate the market portfolio's semivariance above the mean return:

$$V_M(>) = \frac{\sum_{t=1}^{T}\max(\tilde{R}_{Mt} - E_M, 0)^2}{T}. \tag{19}$$

The results of equations (15)-(19) are shown in Table 3.

The fourth step is to calculate the required semivariance matrix. Alternatively, the semivariance matrix can also be calculated as a result of the following matrix:

$$V_{n\times n}(<) = Cov_{n\times n}() - V_M(>) \cdot \beta_{n\times 1} \cdot \beta_{n\times 1}^{\ T} \, , \tag{20}$$

where:

$$Cov_{n\times n}() = \begin{pmatrix} \sigma_1^2 & \cdots & \mathrm{cov}_{1,n} \\ \vdots & \ddots & \vdots \\ \mathrm{cov}_{n,1} & \cdots & \sigma_{\mathrm{n}}^2 \end{pmatrix}, \tag{21}$$

$$\beta_{n\times 1}^{\ T} = (\beta_1, \beta_2, ..., \beta_n). \tag{22}$$

The result of the semivariance matrix is presented in Table 4.

Table 4. Downside semivariance matrix V (<)

	Government bond	Muni bond	High yield bond	Real estate	MBS	Investment grade bond	Large cap stock
Government bond	0.031%	0.039%	0.048%	0.038%	0.040%	0.026%	-0.005%
Muni bond	0.039%	0.123%	0.073%	0.071%	0.064%	0.039%	-0.006%
High yield bond	0.048%	0.073%	0.335%	0.107%	0.078%	0.047%	0.003%
Real estate	0.038%	0.071%	0.107%	0.190%	0.089%	0.027%	0.018%
MBS	0.040%	0.064%	0.078%	0.089%	0.111%	0.033%	0.014%
Investment grade bond	0.026%	0.039%	0.047%	0.027%	0.033%	0.032%	-0.003%
Large cap stock	-0.005%	-0.006%	0.003%	0.018%	0.014%	-0.003%	0.055%
Std. dev. of semivariance	1.759%	3.504%	5.789%	4.359%	3.324%	1.792%	2.349%
Std. dev of variance	1.775%	3.511%	5.789%	4.395%	3.351%	1.799%	3.141%

The optimal asset allocation results constructed from the semivariance model are reported in Table 5. Comparing the results of the traditional variance model (see Table 2) and the semivariance model (see Table 5), we show that the investor can generate almost the same portfolio returns as the traditional variance investor while lowering the downside risk below the level of that of the variance investor. For example, when the risk free rate *c* is 0.7614%, a semivariance investor will have a portfolio return of 0.931% as compared to a variance investor's return of 0.931%; while at the same time the semivariance investor insurers will have a lower downside risk of 1.891%, as compared to 1.910% for the variance investor.

Table 5. Results of the mean-semivariance efficient frontier

c	Portfolio semi-dev.	Portfolio expected returns	Govern-ment bond	Muni bond	High yeld bond	Real estate	MBS	Investment grade bond	Large cap stocks
0.744	1.879	0.930	0.000	0.000	0.000	25.000	0.000	65.000	10.000
0.753	1.879	0.930	0.000	0.000	0.000	25.000	0.000	65.000	10.000
0.761	1.891	0.931	0.000	0.000	0.724	25.000	0.000	64.276	10.000
0.770	1.904	0.932	0.000	0.000	1.443	25.000	0.000	63.557	10.000
0.779	1.919	0.933	0.000	0.000	2.256	25.000	0.000	62.744	10.000
0.788	1.937	0.935	0.000	0.000	3.183	25.000	0.000	61.817	10.000
0.797	1.972	0.937	0.000	0.000	4.193	25.000	1.457	59.350	10.000
0.805	2.019	0.941	0.000	0.000	5.377	25.000	3.444	56.179	10.000
0.814	2.080	0.945	0.000	0.000	6.799	25.000	5.831	52.370	10.000
0.823	2.159	0.949	0.000	0.000	8.539	25.000	8.751	47.710	10.000
0.832	2.267	0.955	0.000	0.000	10.744	25.000	12.473	41.784	10.000
0.841	2.340	0.959	0.000	0.000	13.262	25.000	12.500	39.238	10.000
0.849	2.463	0.965	0.000	2.238	16.537	25.000	12.500	33.726	10.000
0.858	2.643	0.973	0.000	5.811	20.870	25.000	12.500	25.819	10.000
0.867	2.911	0.985	0.000	10.731	26.838	25.000	12.500	14.931	10.000
0.876	3.311	1.000	0.000	17.326	35.174	25.000	12.500	0.000	10.000
0.885	3.409	1.004	0.000	14.069	38.431	25.000	12.500	0.000	10.000
0.893	3.538	1.008	0.000	10.035	42.465	25.000	12.500	0.000	10.000
0.902	3.716	1.014	0.000	4.908	47.592	25.000	12.500	0.000	10.000
0.911	3.898	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.920	3.898	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.929	3.898	1.019	0.000	0.000	52.500	25.000	12.500	0.000	10.000
0.937	4.156	1.024	0.000	0.000	59.666	25.000	5.334	0.000	10.000
0.946	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.955	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.964	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.973	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.981	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.990	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000
0.999	4.360	1.029	0.000	0.000	65.000	25.000	0.000	0.000	10.000

Figure 1 shows the graph of the two efficient frontiers, in which the semivariance efficient frontier is moving outward to the left of the variance efficient frontier. That is, the efficient frontier based on mean-semivariance framework has higher return-risk tradeoff than the efficient frontier based on mean-variance framework. Although the improvement from the mean-variance efficient frontier to the mean-semivariance efficient frontier is not substantial, this does not necessarily indicate that the real differences between the variance-oriented investors and the semivariance-oriented investors are negligible. If the optimizing process is fed with more assets instead of just seven ETF index funds, the ultimate difference between the two frontiers might be much more obvious. Moreover, the inter-correlations between asset classes are much higher than those of individual securities, and the skewness may be reduced through diversification.

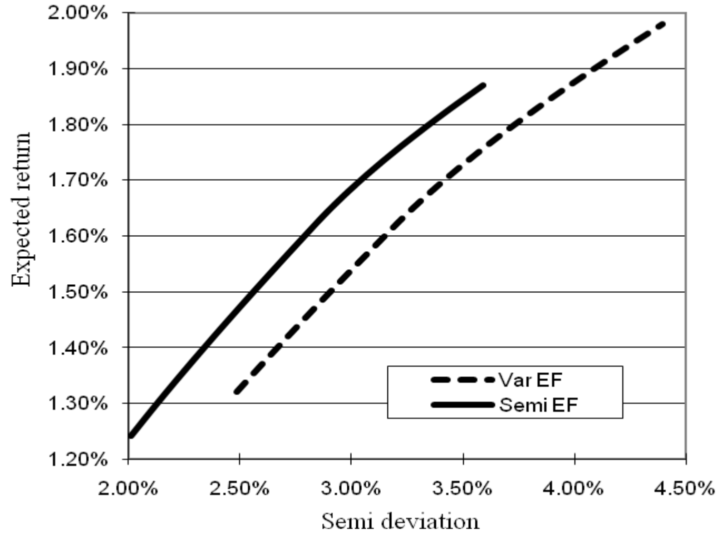

Fig. 1. Mean-variance and mean-semivariance efficient frontiers

In terms of asset weight allocation, we can observe some similarities as well as differences between the Markowitz mean-variance model and the mean-semivariance model. As risk-free rate goes higher, the real estate and common stock investments always stay at their highest investment constraints, 25% and 10% respectively, while at the same time, each model allocates a zero percentage to government bonds (see Table 2 and Table 5). For the municipal bond allocation, neither model allocates any capital until risk-free rate goes higher. However, the semivariance model allocates slightly less than the variance model when the risk-free rate is below 0.867%. When the risk-free rate is above 0.867%, the semivariance model allocates more than the variance model, but both models suggest dropping this asset from the portfolio after the risk-free rate goes above 0.911%. For the high yield bond allocation, as the risk-free rate goes up, the optimal asset allocation in this investment category goes up, but the semivariance model always indicates to maintain a lower investment ratio in this investment category than does the variance model. For the Mortgage-Backed Securities (MBS), the investment allocations suggested by both models are all increasing initially, but dropping dramatically to a zero percentage after the risk-free rate rises above a level of 0.946%. Interestingly, for this investment category, the semivariance model suggests a much higher allocation to the MBS than does the variance model. For the investment grade corporate bond investment, things get a little twisted. The allocation suggested by the semivariance model starts out lower than that of the variance model, but the allocation of the semivariance model soon exceeds that of the variance model as the risk-free rate increases. In the end, both models allocate a zero percentage to this asset category.

The differences in the asset allocation strategy suggested by the two models may be partly explained by each asset's standard deviation and semi-deviation of different asset returns. Municipal bonds and high yield bonds both have relatively higher standard deviation and standard semivariance than those of MBS and investment grade corporate bonds. The difference between the standard semivariance and the standard deviation of the municipal bonds and high yield bonds are smaller than that between the MBS and investment grade corporate bonds. So the semivariance model suggests a smaller asset allocation in these two assets than what the variance model suggests.

The relative difference in the asset allocation in the portfolio has practical economical implications. The mean-semivariance model allows investors to select a portfolio that can achieve an expected portfolio return that matches and even exceeds what the mean-variance model can offer while keeping the risk level at the minimum variance and the semivariance level. These results are consistent with Ballestero's findings.

Conclusion

In general, Markowitz's mean-variance model has been one of the most commonly used methods in real-world portfolio management and asset allocation. However, it has received criticisms for its strict mathematical assumption that the returns of the assets in the portfolio are normally distributed. When portfolio is composed of assets with skewed returns, the results of the mean-variance model will be ineffective. Moreover, variance which is used as a risk measurement in the mean-variance framework treats both the returns above and below the mean return equally. Thus, variance as a risk measurement tends to give a misleading estimation for the downside risk which investors weigh more heavily than the upside volatility of security returns.

Research on the downside risk, including the semivariance below the mean value or certain target value, has undergone for many years. The difficulties in the calculation of the downside risk measure have made the downside risk optimization not as popular as the mean-variance optimization algorithm. However, the introduction of a strictly derived method has simplified the process: the symmetric co-semivariance matrix is derived from the empirical validity of Sharpe's beta regression equation. Few prior studies have devoted to empirically testing these two models and compare their efficient frontiers. In this paper, we empirically test these two models using real world data and compare the two efficient frontiers in investment portfolio and asset allocation.

The efficient frontiers comparison seems to indicate that the mean-semivariance framework could provide clearer indications in terms of asset allocations that lead to an optimal portfolio that not only matches if not exceeds those expected returns from the traditional mean-variance framework, but also lowers downside risk. In other words, the results indicate that the mean-semivariance model could provide investors with asset allocation strategy that minimizes asset allocation not only in those assets with higher variance but also in those assets with higher semivariance in particular and with higher correlation with each other than with other assets. By constructing a portfolio as suggested by the mean-semivariance model, investors can simultaneously minimize the downside risk and achieve similar or better expected returns. In contrast, the mean-variance model tends to lead to a portfolio that has limited upside gain and higher downside risk.

Our findings of the semivariance model have practical implications for both individual investors and institutional investors for asset allocation and portfolio optimization while managing their downside risk exposure. It is especially important for insurance and banking sectors, the industries that have a higher risk aversion for downside risk. Insurance companies and commercial banks in the United States, for example, are required by regulators to maintain a certain level of capital, which is determined by the level of the risk in their invested assets. While insurance companies and banks seek to reduce the required capital to a minimum level, they are very concerned about minimizing the downside risk while maintaining a certain level of return on investment. The mean-semivariance model could be very instrumental to these companies for their risk management. Moreover, the mean-semivariance model allows portfolio managers to have a clear definition of risk that combines the objectives and constraints of the entire investment portfolio.

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