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Alternative Monte Carlo simulation models and the growth option with jumps

Abstract

Recent research has revealed the usefulness of Monte Carlo simulation for valuing complex American options which depend on non-conventional stochastic processes. This paper analyzes the possibilities to improve flexibility of traditional real options models by the use of simulation. The authors combine simulation and dynamic programming for valuing American real options contingent on the value of a state variable which evolves according to a mixed Brownian-Poisson process. The paper estimates the optimal exercise strategy using two alternative models, which are based on algorithms developed for financial derivatives. The authors evaluate both valuation proposals using a simple numerical example. The results highlight the need to achieve a *trade-off* between the accuracy of the estimations and the computational effort needed for this type of proposal. They also reveal the existence of non-monotonous and occasionally counterintuitive relations between the value of the growth option and the volatility and frequency of discontinuous jumps, which should be explained by the characteristics of the stochastic process under consideration.

Keywords: real options, Monte Carlo simulation, financial valuation.

JEL Classification: G31.

Introduction

Now the superiority of the real options approach to discounted cash-flow (DCF) models is widely accepted, research on valuation faces a no less important challenge, namely, its diffusion into the practical arena. This objective requires simplifying and making current real options valuation more flexible.

Whereas the DCF approach is directly implemented to virtually all investment opportunities, the option model lacks any similar general formulation. By contrast, the real options approach comprises numerous and complex analytical as well as numerical methods, each of which is suitable for evaluating a particular decision right contingent on specific underlying assets. No “traditional” real options model allows direct treatment of alternative stochastic processes, multiple American-style options or many sources of uncertainty.

Particularly, most real options models assume pure diffusion processes in the evolution of state variables. These processes have been widely used to describe price movements, particularly of commodities and financial assets, yet they are hard to apply to the state variables on which the value of real options depends. Frequently used state variables – such as demand, profit or even costs – could adjust better to mixed processes, which combine continuous Brownian

movement with the probability of discontinuities. These kinds of discontinuities or “jumps” are due, depending on the nature of the variable, to the disruption of an economic crisis, a change in customers’ preferences, a corporate bankruptcy or, quite simply to technological progress.

In addition, the possibility of exercise at more than one future date is probably one of the most common characteristics of real options. The same arguments put forward by advocates of the real options approach when criticising the DCF model for excluding the possibility to defer investment or to decide when to initiate the project (McDonald and Siegel, 1986; Pindyck, 1991; Ingersol and Ross, 1992), are now applicable to “European real options” models.

The possible occurrence of random jumps makes valuing American-type options by traditional techniques – closed-form solutions, analytic approximations and numerical procedures – complicated if not impossible. Merton (1976) derives the analytic solution for the European option when the underlying asset follows a mixed process, comprising a geometric Brownian (continuous) motion subject to discrete Poisson jumps. Merton’s proposal allows valuing options the exercise of which is restricted to its expiration date, although it cannot be used in valuing American-type options.

For their part, analytic approximations and the more widely used numerical procedures – the binomial model (Cox, Ross and Rubinstein, 1979) and finite differences (Brennan and Schwartz, 1977) – can be used to consider the possibility of early exercise, yet their implementation in computational terms is costly when including multiple sources of uncertainty and/or stochastic processes other than the geometric-Brownian family.

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This paper had been benefited by the financial grant from the Ministerio de Ciencia e Innovación (ECO-2011-29144-C03-01) and the financial support of the *Fundación de la Universidad de Cantabria para el Estudio y la Investigación del sector Financiero* (UCEIF) and the Santander Group. The authors appreciate the suggestions received, after the presentation of the paper, from the participants at the 10th Annual International Conference on Real Options. The authors also appreciate the comments by the anonymous referees of the IMFI. Any mistakes or omissions remained are the only responsibility of the authors.

By contrast, models based on Monte Carlo simulation (Boyle, 1977) can be applied to the case of multiple state variables regardless of the stochastic processes to which they are subject, yet they “are not suitable” for American-type options. At least this was held to be true until quite recently. To quote two examples, in the second edition of his well-known options manual, Hull (1993, p. 334) stated that “one limitation of the Monte Carlo approach is that it can be used only for European style derivatives securities”, and Hull and White (1993, p. 1) wrote that “Monte Carlo simulation cannot handle early exercise since there is no way of knowing whether early exercise is optimal when a particular price is reached at a particular time.”

The seeming inability to value American-type options by simulation is due to the nature of the technique. Since exercise of an option at a given date prevents its subsequent exercise, the strategy that determines optimal exercise of American options depends not only on previous paths for the state variables but also on their future value. Future expectations can only be performed through procedures that include *backward induction*, such as binomial trees or finite difference procedures. However, standard Monte Carlo simulation is a *forward induction* procedure, which generates future values for variables based on their previous value and is, therefore, an appropriate technique for assets whose cash-flows at a given moment do not depend on future events, as is the case of European options.

In order to overcome this restriction, recent research has proposed combining simulation with some backward induction procedure that may lead to a valuation model applicable both for European as well as American type options, whatever the number of state variables and the nature of the stochastic processes. The greater flexibility of the Monte Carlo approach is due to the fact that the evaluation problem is overcome by directly approximating the underlying asset process, meaning that the partial differential equation describing the path of the option does not need to be resolved.

The first attempt at applying simulation in valuing American style options is to be found in the work of Tilley (1993)¹, who proposes a model for valuing financial options dependent on the stochastic value of a single state variable coinciding with its underlying asset. Tilley suggests sorting the simulated values of the underlying asset for each exercise date and grouping them into “bundles” for

which a single value of keeping the option alive until the next period is assigned, as the mean of the continuation value of the whole of these paths².

Tilley’s approach has been followed by a growing number of papers that propose different combinations of simulation and backward induction procedures for valuing American-type financial derivatives. Prominent within this approach are Barranquand and Martineau (1995) or Raymar and Zwecher (1997), who propose the use of a partition algorithm on the unidimensional space of the cash-flow yielded by the option, instead of the division of the multidimensional space of the underlying assets defined in Tilley [1993]. Grant, Vora and Weeks (1996) as well as Ibáñez and Zapatero (2001) directly estimate the values of the state variables for which the value of holding the option alive until the following period matches the value of its immediate exercise at each exercise date.

By way of an alternative proposal, Broadie and Glasserman (1997a, 1997b) and Broadie, Glasserman and Jain (1997) propose the use of non-recombinatory simulated trees³ and stochastic meshes to determine two estimates of the option value, one biased “upward” and another biased “downward”, both asymptotically unbiased and convergent towards the certain value. Finally, Longstaff and Schwartz (2001) opt for least square regressions as a method for approaching the expected value of maintaining the option alive at each decision point.

In the light of this type of proposal, recent corporate finance literature has embraced Monte Carlo simulation procedures for valuing real options. Barranquand and Martineau (1995) model and its extension by Raymar and Zwecher (1997) have been used to value American-type options where the state variables follow mean reverting and geometric Brownian processes. This is the case of Cortazar and Schwartz (1998) who resolve the optimal *timing* of the exploitation of oil reserves, and Cortazar (2001) who evaluates the optimal operations of a copper mine, initially modelled by Brennan and Schwartz (1985). Other papers apply the Longstaff and Schwartz algorithm (2001) in valuing real options linked to patents and R&D projects (Schwartz, 2004; Schwartz, and Miltersen, 2002), licences (Albertí et al., 2003, dot-com companies (Schwartz and Moon, 2000; Schwartz and Moon, 2001), pharmaceutical companies (León and Piñeiro, 2004; Rubio and Lamothe, 2006), Internet portals (Sáenz-Diez et al., 2008), and electricity business (Alonso et al., 2009a and 2009b).

¹ Certain authors also refer to Bossaerts (1989), who had previously analyzed early exercise of American options through simulation, and maintain that this is likely to have been ignored as it is still in the form of a *working paper*.

² This procedure suffers from notable drawbacks, such as the need to store all the simulated paths – with the subsequent loss of time – as well as the enormous complexity inherent to the sorting process when considering multiple sources of uncertainty.

³ Unlike binomial and trinomial trees, the values that appear at each node are placed in the order in which they are generated and not following a hierarchic order.

The general purpose of our paper is to examine how simulation techniques can improve flexibility of real options models. We analyze different issues that arise when combining Monte Carlo simulation and dynamic programming to value real options. The optimal exercise strategy is estimated using two different proposals, both based on specific algorithms from financial derivatives valuation. The first proposal is based on Grant, Vora and Weeks (1996), and Ibáñez and Zapatero (2004) algorithms, which approximate the optimal exercise boundary (*critical* values of the state variable) by comparing the payoff from immediate exercise and the expected continuation value. The second approach is based on the Longstaff and Schwartz (2001) algorithm, which focuses on the conditional expected function of the difference between immediate exercise value and continuation value.

Applying these algorithms to the valuation of real options requires their adaptation, both in estimating the underlying asset value from the state variable on which its cash-flow depends, and in approximating the “non pure diffusion” stochastic process followed by this variable. Particular attention is focused on the analysis of pseudo-American options to growth whose underlying variable evolves following a mixed Brownian-Poisson process.

We evaluate both simulation proposals through the valuation of a simple numerical example. Our analysis shows that applying unsuitable techniques to the valuation of this option may significantly affect efficiency of the managerial decision-making process. In particular, our results highlight the advisability of considering the trade-off between accuracy of valuations and effort required in terms of abstraction, modelling and computerization. Moreover, our numerical example underlines the significance of errors when applying traditional models for valuing real options contingent on a state variable whose expected path recommends consideration of discrete discontinuities.

The rest of the paper is organized as follows. The next section describes two alternative proposals of combining Monte Carlo simulation and dynamic programming for estimating the optimal exercise frontier. Section 2 presents numerical results obtained when applying both proposals to valuing a simple numerical example of a growth option. The paper ends with a summary of the main conclusions.

1. Alternative proposals for estimating the exercise frontier in real options

Combining Monte Carlo simulation and dynamic programming to value American options requires estimating the optimal exercise policy. Two alternative estimation procedures have been used in

the area of financial derivatives, which application to real options valuation is not direct. On the contrary, it requires their adaptation, both in terms of determining the value of the underlying asset from the state variable on which its cash-flow depends, as well as of modelling the actual stochastic process followed by the state variable.

The first proposal is based on Grant, Vora and Weeks (1996 and 1997) and Ibáñez and Zapatero (2004) algorithms, and aims to determine what are known as, in the words of Merton (1973), *critical* values of the state variable. These values are estimated by comparing the immediate exercise payoff with the continuation payoff, and serve to estimate the current expanded value (with the option) derived from optimal exercise through a series of simulated paths. The second takes the Longstaff and Schwartz algorithm (2001) as a reference. Thus, it focuses on the estimation of the conditional expected function of the difference between the value from immediate exercise and the value of maintaining the option until the following period at each point early exercise is allowed, therefore obtaining a complete specification of the optimal strategy at each exercise date.

The real option valuation starts by identifying the ultimate state variable on which the company cash-flow depends, S_t , and estimating its stochastic evolution in time. Without loss of generality, we assume that the state variable follows a mixed process, comprising a continuous geometric Brownian type motion subject to random jumps distributed following a Poisson variable. Therefore, the infinitesimal variation of the state variable dS_t responds to the following equation:

$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dz + (\pi - 1)S_t dq, \quad (1)$$

where α and σ represent the expected drift and volatility of the continuous motion, respectively; λ is the mean frequency of the discrete jumps per time unit; $(\pi - 1)$ is a random variable measuring the size of the proportional jumps in asset value, and k is the mean value of these jumps¹; and dz_t and dq_t represent, respectively, stochastic Wiener and Poisson processes which we assume to be independent and characterised by their usual expressions:

$$dz = \xi \cdot \sqrt{dt}, \quad \xi \rightarrow N(0,1) \quad (2)$$

$$dq = \begin{cases} 0 & \text{prob} = 1 - \lambda \cdot dt \\ 1 & \text{prob} = \lambda \cdot dt \end{cases}, \quad q \rightarrow \text{Poisson}[\lambda] \quad (3)$$

¹ Therefore, the mean growth rate caused by the discrete jumps is λk .

As regards the discrete motion, we assume that jumps are independent and $\log(\pi)$ is normally distributed with mean μ_π and deviation σ_π in such a way that:

$$k = E[\pi - 1] = \exp\left(\mu_\pi + \frac{\sigma_\pi^2}{2}\right) - 1. \quad (4)$$

Discrete discontinuities are linked to “unusual” and important events that give rise to upward or downward variations in the uncertain variable, whereas

$$S_t = S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right)\Delta t + \sigma z_0 \sqrt{\Delta t} + \sum_{i=1}^q \left(\sigma_\pi z_i - \frac{\sigma_\pi^2}{2}\right)\right], \quad (5)$$

where r and δ symbolize the continuous risk-free rate and the convenience yield¹, respectively, z_0 represents a standard normal random variable linked to the diffusion process; z_i are normally distributed independent variables that determine the size of each jump; and q , as has already been pointed out, reflects the number of discrete jumps determined by a Poisson distribution with frequency λ^2 .

1.1. The critical values proposal. One of the differences between financial options and real options is direct knowledge in the former, and lack of knowledge in the latter, of the critical state variable value at expiration. In financial options, this value coincides with the exercise price. By contrast, in real options the critical value at expiration depends on the value of the underlying asset, which in turn depends on future evolution of cash-flows and their state variable.

Hence, determining the series of critical state variable values must be started on the option expiration date and be prolonged, recursively in time

the continuous process – geometric Brownian motion – is linked to the idea of “normal” events. Following standard practice, we assume the direction of the jump to be unknown *a priori* and therefore the effect of the jump in the drift term to be null, in other words, $\mu_\pi = -\sigma_\pi^2/2$ and $k = 0$.

Assuming the existence of complete markets and thus applying the *risk neutral* valuation, the expression representing the future balance value of the “twin” financial asset for the state variable is:

time, at each of the points at which early exercise is allowed. Through simulating a set of M values of the state variable at maturity³, we estimate the critical value, $S_{T^o}^*$, as that for which the value of the underlying project, assuming immediate exercise of the option, matches the non-exercise value. The estimation of this critical value also requires simulating K paths for the state variable to expiration of the underlying project, T , $S_{T^o+\tau}^{i,j}$, $S_{T^o+2\tau}^{i,j}$, ..., $S_T^{i,j}$ for $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, K$, namely, as is shown in Figure 1.

Each of these paths enables us to estimate both cash-flow series, derived from option exercise and expiration without exercise, from option maturity date through underlying project conclusion date (T^O , T). Evidently, discounting each of these cash flows at option expiration, T^O , provides the corresponding contingent value of the investment, $V_{T^O,decision}^i(S_{T^O}^i)$, and the comparison of these M pairs enables us to identify the desired critical value, $S_{T^O}^*$.

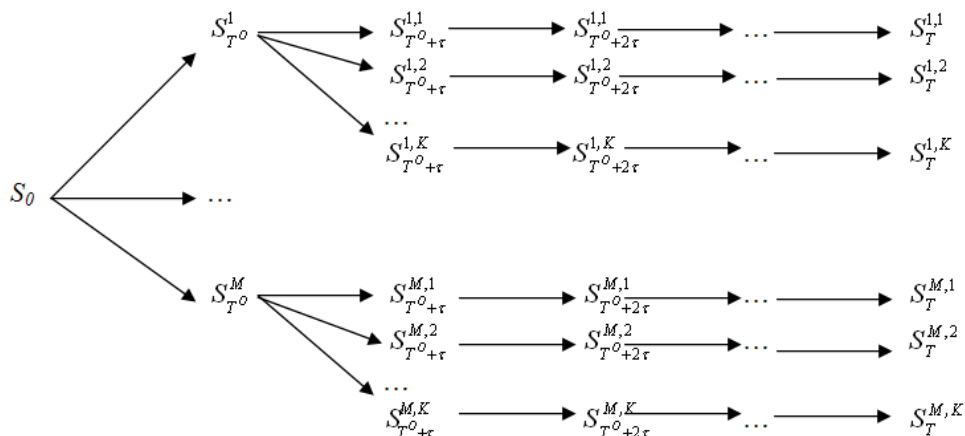


Fig. 1. Simulation paths for the state variable

¹ Following Merton (1976), we assume the risk associated to the discontinuous jump of the state variable to be diversifiable. The risk-neutral simulation would then show a continuous modified drift, $r-\delta$, rather than the initial α drift. This is equivalent to subtracting from the continuous drift the corresponding risk premium (Trigeorgis, 1996, p. 102).

² The number of jumps at a time interval, Δt , is obtained by applying Monte Carlo simulation to the accumulated probability function $P[q \leq X]$.

³ Simulation may be initiated at any moment and for any value of the state variable (Grant, Vora and Weeks, 1996). However, American options optimal exercise at each moment depends on optimal exercise at all future dates, and so the first critical value needs to be that corresponding to expiration.

Having estimated the critical state variable value at maturity, we go back in time to the moment immediately prior to the early exercise of the option, $T^O - \tau$. At this point, we repeat the approximation routine to the state variable value, $S_{T^O - \tau}^*$, in which the payoffs from the option exercise and non-exercise at that date converge, $V_{T^O - \tau, exercise}^i(S_{T^O - \tau}^*) = V_{T^O - \tau, non-exercise}^i(S_{T^O - \tau}^*)$.

Once more, to determine $S_{T^O - \tau}^*$ we generate a set of new M values of the state variable, $S_{T^O - \tau}^{i=1, \dots, M}$ – at a range close to the previously determined critical value – from which we simulate other K paths up to the conclusion of the project – values of $S_{T^O}^{i,j}, S_{T^O + \tau}^{i,j}, \dots, S_{T^O + 2\tau}^{i,j}, \dots, S_{T^O}^{i,j}$ for $i=1, 2, \dots, M$ and $j=1, 2, \dots, K$.

These paths in turn serve to determine the cash-flow generated by the project from this date in both exercise and non-exercise.

In this case, the procedure requires not only considering whether to exercise the option at $T^O - \tau$, but also the possibility of exercising the option at a subsequent moment, which mainly influences evaluation of the expected value of the project in the following period. If the option is exercised at $T^O - \tau$, the expected value of the project at the following period, $E[V_{T^O, exercise}^i]$, should be determined bearing in mind the early exercise that has already taken place, which in turn prevents any new decisions to exercise being taken at subsequent dates.

However, if the option is not exercised at $T^O - \tau$, obtaining the expected value of the project involves considering the possibility of adopting a new contingent decision at the following period, T^O , comparing the simulated value of the variable $S_{T^O}^{i,j}$ with the critical value obtained at the previous step, $S_{T^O}^*$. As a result, for certain $S_{T^O}^{i,j}$ values the option will be exercised at T^O , whereas for the remainder the option will expire without being exercised. Thus, we calculate the expected value of the project at T^O , $E[V_{T^O}^i]$ – without exercise subscript – averaging the estimated j values, whether with or without exercise of the option. Once the expected value of the project at T^O has been evaluated, assuming non-early exercise of the option at $T^O - \tau$, $E[V_{T^O}^i]$, to determine the value of the project, $V_{T^O - \tau, non-exercise}^i$, merely requires adding the flow generated at that date. What remains is to compare the value from early exercise, $V_{T^O - \tau, exercise}^i$ with the

value from continuation, $V_{T^O - \tau, non-exercise}^i$ with the aim of identifying the critical value at $T^O - \tau$, $S_{T^O - \tau}^*$, for each $S_{T^O - \tau}^i$.

We repeat the procedure to determine $S_{T^O - \tau}^*$ at each of the dates in which early exercise is possible until we find the remainder of the values that make up the optimal exercise frontier. Evidently, as we approach to the initial moment and although the logic of the estimation is always the same, the complexity and number of operations involved in determining the critical values grows exponentially.

Having determined the critical state variable values at the discrete dates for which option exercise is possible, $S_0^*, S_\tau^*, S_{2\tau}^*, \dots, S_{T^O - \tau}^*, S_{T^O}^*$, we estimate the value of the American option by traditional simulation as if it were a European option. In this case, simulation involves estimating a sufficient number of paths for the state variable from the current moment to the project expiration date, S_t^h for $h = 1, 2, \dots, H$ and $t = \tau, 2\tau, \dots, T$, and the moment of optimal exercise along each path is determined in accordance with the optimal early exercise boundary. Finally, the current value of the option is estimated discounting the resulting payoff from each path, and then taking the average over all paths.

1.2. The OLS regression proposal. To approach the optimal exercise frontier by least-squares regression we estimate the conditional expected function of the difference between the values from immediate exercise and continuation. We only regress the simulated paths for the state variable that are in the money at any particular moment. By estimating this conditional expected function at each exercise date, we obtain a complete specification of the optimal exercise strategy.

Estimating the expected function of this difference marks a contrast not only with the proposal developed by Longstaff and Schwartz (2001) for valuing American financial options, but also with other real options applications (Schwartz, 2004; León and Piñero, 2003; Schwartz and Moon, 2000; Schwartz and Moon, 2001; Lamothe and Aragón, 2002), which basically focus on valuing abandon options, where the payoff from immediate exercise at each moment does not depend on stochastic evolution of the state variable. In these cases, what is estimated is the function of the expected continuation value, in such a way that the optimal exercise strategy is provided by merely comparing at each exercise date the expected value and the value derived from immediate exercise.

The procedure for estimating the successive functions of the conditional expected difference follows a recursive process which takes expiration date of the option as the starting point. Then, at each exercise point, it requires the simulated paths for the state variable that are in the money to be determined since, *a priori*, these are the only ones for which it is worth considering the decision to exercise or not¹.

With these paths we propose a regression in which the dependent variable is calculated as the difference between the underlying asset values from immediate exercise and from maintaining the option. The independent variables are based on the simulated values of the state variable (whether raised to the square, to the cube, ..., or as a result of another type of function). By means of this regression, we estimate the coefficients that make up the optimal exercise boundary. Thus, for example, at any given moment t , and considering a parabolic regression, the equation to be estimated is

$$Y_t^{l_i}(S_t^{l_i}) = \beta_0 + \beta_1 S_t^{l_i} + \beta_2 (S_t^{l_i})^2,$$

where the superscript l_i represents the paths for the state variable which are in the money at the moment t of the total set of H approximated simulations in each period; and the dependent variable is calculated as:

$$Y_t^{l_i}(S_t^{l_i}) = V_{t,exercise}^{l_i}(S_t^{l_i}) - V_{t,non-exercise}^{l_i}(S_t^{l_i}).$$

The value of the underlying asset when the option is exercised at t (with $t \leq T^O$) is the discounted cash-

$$V_{t,non-exercise}^{l_i}(S_t^{l_i}) = \sum_{\tau=t+1}^{t^*} F_{\tau,non-exercise}^{l_i} \exp[-r(\tau-t)] + \sum_{\tau=t^*+1}^T F_{\tau,exercise}^{l_i} \exp[-r(\tau-t^*)].$$

This procedure is repeated at each moment at which exercise is possible. It should also be remembered that the effort required in terms of computerization depends lineally on the number of opportunities considered for early exercise, thus overcoming some of the operational drawbacks inherent in the previous proposal.

Once the regression coefficients that make up the optimal exercise boundary are estimated, we approximate the value of the American option by traditional simulation as if it were a European option. There is also the possibility of using the same sample of state variable simulated values to estimate both the optimal exercise strategy as well as the option value, thus reducing the computational effort required. *A priori*, any efficient algorithm should provide similar evaluations when the exercise is applied to the same set of simulations or

flow of the “altered” project between t and the project expiration date, also considering the exercise price, X , received (put option) or paid (call option):

$$V_{t,exercise}^{l_i}(S_t^{l_i}) = \pm PE + \sum_{\tau=t+1}^T F_{\tau,exercise}^{l_i} \exp[-r(\tau-t)].$$

To determine the value of the underlying project if the option is not exercised it is necessary to distinguish between the expiration date of the option, and the remaining moments at which early exercise is allowed. Thus, if $t = T^O$, the underlying asset value if the option is not exercised and expires without exercise, is the discounted cash-flow as generated by the “unaltered” project between the option maturity and the investment maturity,

$$V_{T^O,non-exercise}^{l_i}(S_t^{l_i}) = \sum_{\tau=T^O+1}^T F_{\tau,non-exercise}^{l_i} \exp[-r(\tau-T^O)].$$

Moreover, if the option is not exercised at a moment t prior to expiration (in other words, $t < T^O$) and is maintained until the following period, the value of the underlying asset is calculated by considering the possible optimal exercise at a subsequent date. In other words, the value of the underlying asset would be calculated from the discounted cash flows resulting from the “unaltered” investment between t and optimal exercise of the option at a later date, t^* , plus the discounted cash-flow as generated by the “altered” project between the moment of optimal exercise, t^* , and its maturity,

to any other new set². Nevertheless, to avoid evaluation biases, Broadie and Glasserman (1997a) recommend simulating a second set of paths.

2. A numerical example of the American option to grow contingent on discontinuous processes

The advisability of previous simulation proposals in valuing real options can be evaluated most easily with a simple numerical example. We consider a finite-life investment in the installation of production capacity to meet demand, S , of a specific product which is assumed to evolve stochastically according to equation [5]. The initial demand value is established as equal to 10 million physical units, market share of the project is 50%, and cash-flow is determined by a known and constant margin equal to one monetary unit. The project life is 5 years and the total initial investment is equal to 25 million monetary units. In addition, we assume complete capital markets and a risk-free rate of 6%.

¹ Those paths for which the intrinsic value were positive would be in the money, assuming expiration of the option at that moment.

² Longstaff and Schwartz (2001) show that, in financial option valuation, differences between both ways of applying the algorithm are minimal.

To compare valuation results for different process specifications, ranging from a pure diffusion motion to a mixed process including random size jumps, we consider a wide of parameters values. With regard to geometric Brownian motion, we assay with alternative volatilities of 10%, 20% and 30%, coupled with an annual drift of 15%. For the discontinuous part of the mixed process, we consider a range of volatilities between 25% and 500% with an average number of annual jumps ranging between 0.20 and 1¹.

It can be seen in Table 1 that the NPV of this project is negative for all process specifications, and decreases as discrete volatility increases. This result is due to the substantial increase in total volatility of the process when highly volatile discontinuities are included – even when only one such jump occurs

during the whole period of analysis. Thus, for a discrete volatility of 500%, the total volatility of the process reaches values of 602.82% or 602.16% depending on whether the typical deviation of the continuous motion is 30 or 10%, respectively².

In addition, we consider that the initial investment provides the right to expand the original size of the project through a new expenditure. This right may be exercised at three specific dates at the end of the second, third and fourth years. Therefore, the possibility of extending the initial project size is similar to quasi-American call option with expiration at the end of the fourth year of operations. The exercise of this option involves an increase in project sales equal to 50% of the existing level, and an exercise price equal to 20% of the initial outlay.

Table 1. Value of the net present value

Continuous volatility	Discrete volatility								
	0%	25%	50%	75%	100%	200%	300%	400%	500%
$\sigma = 10\%$	-2 117 062	-2 341 963	-2 318 721	-2 249 759	-2 728 270	-3 965 568	-4 561 367	-10 388 764	-10 335 590
$\sigma = 20\%$	-2 120 406	-2 337 170	-2 299 687	-2 439 157	-2 576 769	-3 352 074	-5 788 946	-10 624 907	-11 356 142
$\sigma = 30\%$	-1 999 903	-2 359 443	-2 296 945	-2 006 384	-2 292 343	-3 371 138	-6 198 091	-11 789 391	-11 883 307

Note: Initial project demand is 10 million of physical units, market share is 50%, unitary margin is one monetary unit, and life span is 5 years. We assume complete capital markets and a risk-free rate of 6%.

The number of paths simulated to obtain the value of this option, H , is 400,000, resulting from 200,000 direct approximations plus another 200,000 estimations using the variance reduction technique of the “antithetical variables”³. Additionally, the M and K parameters are equal to 400 in the “critical value” proposal.

The valuation results yielded by both simulation proposals and different process parameters – from pure diffusion motion to mixed process – are shown in Table 2⁴. Furthermore, two different values are estimated by the regression procedure. The first set of options values, *Regression I*, derives from employing the same simulation paths to obtain the optimal exercise strategy and the option value. Whereas the second set of values, *Regression II*, is calculated by simulating two different groups of paths both for determining the optimal exercise and for

estimating future cash flows. In all these cases, we have assumed that average number of annual jumps is 0.20 ($\lambda = 0.2$), in other words, that only one jump occurs during the five-year life span of the project.

The similarity of valuations reached by both proposals is particularly clear in cases with lower jump volatility⁵. Thus, for jump volatility values below 100%, the relative distance between both proposals does not generally exceed 1%. However, as discrete volatility increases, option value estimates depends on the selected simulation proposal. Moreover, regression estimates differ depending on the number of simulation sets for jump volatilities of 400% or 500%, and the option value provided by the critical values procedure is normally located in an intermediate position. We should also bear in mind the substantial increase in total volatility of the process when highly volatile discontinuities are included.

Table 2. Value of the option to grow

Continuous volatility		Discrete volatility								
		0%	25%	50%	75%	100%	200%	300%	400%	500%
$\sigma = 10\%$	Critical values	4 541 824	4 480 622	4 594 030	4 698 266	4 753 891	4 872 654	4 308 987	2 149 137	2 258 166
	Regression 1	4 562 905	4 463 662	4 565 867	4 748 962	4 762 368	4 691 797	4 516 021	2 200 317	2 294 227
	Regression 2	4 563 104	4 444 285	4 545 118	4 790 180	4 751 705	5 158 746	4 098 685	2 588 903	2 166 808

¹ As mentioned in the model description, we assume that jump risk belongs to the category of specific risk or non-systematic.

² The estimation of the total volatility of the uncertain variable for the mixed process considered has been performed based on the expression obtained in Navas (2004), which corrects the one originally obtained in Merton (1976).

³ The technique of antithetical variables consists of generating two symmetrical observations at zero for each of the random simulations of the normal accumulated distribution, with which both values of the derivative are obtained.

⁴ Note that a level of discrete jump volatility of 0% implies a pure diffusion process.

⁵ The results of the Student T test enable us to refuse the presence of significant differences among the different valuations when the jump volatility is equal to or below 100% at 95% confidence level.

Table 2 (cont.). Value of the option to grow

Continuous volatility		Discrete volatility								
		0%	25%	50%	75%	100%	200%	300%	400%	500%
$\sigma = 20\%$	Critical values	4 589 035	4 524 324	4 607 096	4 661 720	4 899 131	5 022 732	4 201 530	2 693 936	2 127 076
	Regression 1	4 575 170	4 539 631	4 550 657	4 714 420	4 907 684	4 850 880	4 369 977	2 481 018	2 146 372
	Regression 2	4 558 581	4 520 780	4 559 007	4 697 119	4 852 479	5 183 542	5 122 898	2 704 963	1 977 476
$\sigma = 30\%$	Critical values	4 720 324	4 591 795	4 792 212	4 900 641	4 991 093	5 205 075	3 933 065	2 402 946	2 157 500
	Regression 1	4 739 826	4 595 652	4 741 438	4 977 156	4 963 822	5 179 860	3 999 496	2 551 498	2 186 396
	Regression 2	4 702 300	4 588 513	4 703 947	4 943 696	5 086 998	4 508 421	4 376 591	2 148 260	2 401 196

Notes: Initial project demand is 10 million of physical units, market share is 50%, unitary margin is one monetary unit, and life span is 5 years. We assume complete capital markets and a risk-free rate of 6%. The option to grow is a quasi-American type call option, which can be exercised at the end of the second, third and fourth years. Its exercise implies an outlay of the 20% of initial investment, and it increases the project sales by 50% of the existing level. Option values are estimated by both *Critical values* proposal and *Regression* proposal. *Regression 1* uses the same simulated paths to estimate the optimal exercise strategy and the option value, whereas *Regression 2* employs different sets of simulations. We consider a mixed Brownian-Poisson process. Geometric-Brownian drift is 15%, with alternative volatilities of 10%, 20% and 30%. For the jump motion, we consider a range of volatilities between 25% and 500% with an average number of annual jumps of 0.20. The number of simulated paths, H , is 400,000 (200,000 from direct approximations plus 200,000 antithetical estimations. M and K in the “critical value” proposal are equal to 400.

Greater dispersion reflected in the regression based results contrasts with greater flexibility inherent to this procedure, where computational effort increases lineally with the number of exercise opportunities. These results highlight the need to achieve a *trade-off* between the accuracy of estimates and the requirements – in terms of modelling and use of computer resources – needed for implementing each proposal.

As regards the relation between the value of the option and the jump volatility, the valuation results reflect that its sign depends on the level of the latter. Thus, in those scenarios with lower jump dispersion the value of the option increases¹, as a result of a higher total volatility. This result is due to the fact that we assume the average size of the jumps to be null and not affecting the average drift of the state variable. However, as jump volatility increases, this relation is inverted and is more prominent in higher levels of discrete volatility. This result remains as long as the average size of the jumps, k , is null and does not affect the drift term for the process followed by demand. This assumption implies that the mean of the logarithm of the jump depends inversely on the value designated to the volatility of the jump ($\mu_{\pi} = -\sigma_{\pi}^2/2$) and, therefore, an increase in the latter for values over 100% reduces this average and with it the value of the underlying.

We also find certain unusual features in the relation between continuous volatility and the value of the option. Specifically, we observed a negative influence of volatility on option values for some high jump volatility scenarios, which is apparently contrary to the *ceteris paribus* relation established

by the OPT. Nevertheless, this outcome is coherent with stochastic evolution of the state variable and the discretisation method employed. In addition to the value reduction suffered when jump dispersion is over the 100% level, the hypothesis that the state variable evolves in the continuous field following a lognormal diffusion process leads to its relative variation being distributed normally with a tendency reduced 0.5 times the variance of the process. As a result, the volatility parameter not only affects the volatility of future values, but also the average simulated values. It follows that the increase in volatility not only widens the range of possible future values of the underlying asset, but also reduces its average simulated value and, hence, the possibilities of optimal exercise of the option to expand.

In addition, we have analyzed how the value of the option is affected by the existence of an upper absorbing boundary of 50,000,000 physical units for the state variable, which may be a reasonable feature in this type of options. This boundary can be derived from economic factors outside the scope of the enterprise, but also from technical reasons such as the maximum production capacity of plants and equipment. The results are shown in Table 3.

A quick look confirms the expected reduction in the value of the growth option when an upper limit restricts upward variations of the state variable. In addition, as was to be expected, the differences are more significant the greater the possibility of the limit being reached, i.e. for the higher levels of continuous volatility considered. Greater volatility means larger dispersion of the lower values, whereas the higher values remain bounded by this maximum capacity and, as a result, any increase in volatility will reduce the expanded value.

¹ Only when jump volatility is 25% is any slight reduction observed in the value of the option that is recouped as volatility increases.

Table 3. Value of the option to grow (state variable bounded by an upper limit)

Continuous volatility		Discrete volatility								
		0%	25%	50%	75%	100%	200%	300%	400%	500%
$\sigma = 10\%$	Critical values	4 553 757	4 528 903	4 564 327	4 564 403	4 559 350	3 361 724	2 492 501	2 156 461	1 918 284
	Regression 1	4 565 153	4 500 405	4 563 103	4 588 026	4 542 885	3 372 517	2 565 782	2 159 300	1 984 446
	Regression 2	4 571 992	4 513 612	4 511 240	4 522 015	4 407 836	3 355 618	2 512 140	2 159 739	1 939 450
$\sigma = 20\%$	Critical values	4 538 402	4 493 742	4 500 246	4 440 375	4 484 839	3 338 236	2 423 226	2 068 322	1 917 960
	Regression 1	4 583 037	4 522 017	4 461 144	4 410 490	4 458 645	3 410 736	2 492 306	2 096 498	1 902 715
	Regression 2	4 547 279	4 498 620	4 378 561	4 407 881	4 394 089	3 330 945	2 498 419	2 103 522	1 975 644
$\sigma = 30\%$	Critical values	4 687 126	4 568 101	4 592 566	4 420 479	4 247 401	3 233 265	2 391 753	1 999 073	1 955 058
	Regression 1	4 656 870	4 586 813	4 531 310	4 479 680	4 258 302	3 329 831	2 436 804	2 099 870	1 990 820
	Regression 2	4 653 151	4 597 764	4 648 257	4 492 770	4 285 706	3 366 924	2 535 404	2 140 281	2 022 933

Notes: Initial project demand is 10 million of physical units with an upper absorbing boundary of 50 million, market share is 50%, unitary margin is one monetary unit, and life span is 5 years. We assume complete capital markets and a risk-free rate of 6%. The option to grow is a quasi-American type call option, which can be exercised at the end of the second, third and fourth years. Its exercise implies an outlay of the 20% of initial investment, and it increases the project sales by 50% of the existing level. Option values are estimated by both *Critical values* proposal and *Regression* proposal. *Regression 1* uses the same simulated paths to estimate the optimal exercise strategy and the option value, whereas the *Regression 2* employs different sets of simulations. We consider a mixed Brownian-Poisson process. Geometric-Brownian drift is 15%, with alternative volatilities of 10%, 20% and 30%. For the jump motion, we consider a range of volatilities between 25% and 500% with an average number of annual jumps of 0.20. The number of simulated paths, H , is 400,000 (200,000 from direct approximations plus 200,000 antithetical estimations). M and K in the “critical value” proposal are equal to 400.

The upper barrier likewise reduces the dispersion in the valuations obtained with each proposal, enabling the accuracy of the estimations. This lack of any significant difference as well as the lower requirements involved in the implementation of the regression based procedure lead us to opt for the latter, whereas the critical values method provides a useful *benchmark* for the valuations obtained.

Finally, we analyzed the influence of multiple smaller size discontinuities on the value of the growth option. We specifically consider the case of up to five jumps, by way of an average, during the life span of the underlying investment ($\lambda = 1$), with 50% dispersion.

Valuation results are shown in Table 4. Once again, the estimates to emerge from the proposals analyzed offer little dispersion and only when there is no upper barrier in the state variable and the jump frequency takes higher values, is the relative variation among the various valuations above 3%. It should also be noted that, as was the case with jump dispersion, the increase in their frequency implies an increase in the total volatility of the process, although for the values of continuous variation considered, $\sigma = 30\%$, as well as discrete, $\sigma_\pi = 50\%$, this only ranges between 38% and 60%, depending on whether λ takes values of 0.2 and 1, respectively.

Table 4. Value of the option to grow depending on the number of discrete jumps

$\sigma = 30\%, \sigma_\pi = 50\%$		Number of discrete jumps per time unit					
		0	0.2	0.4	0.6	0.8	1
Without upper limit	Critical values	4 720 324	4 792 212	4 886 447	4 959 297	5 052 782	5 551 509
	Regression 1	4 739 826	4 741 438	4 970 723	5 041 511	5 018 144	5 660 201
	Regression 2	4 702 300	4 703 947	4 917 413	4 850 528	5 189 245	5 504 847
With upper limit	Critical values	4 687 126	4 592 566	4 579 373	4 526 308	4 458 098	4 501 015
	Regression 1	4 656 870	4 531 310	4 649 051	4 555 648	4 520 735	4 586 699
	Regression 2	4 653 151	4 648 257	4 761 222	4 497 494	4 488 916	4 623 271

Notes: Initial project demand is 10 million of physical units, market share is 50%, unitary margin is one monetary unit, and life span is 5 years. We assume complete capital markets and a risk-free rate of 6%. The option to grow is a quasi-American type call option, which can be exercised at the end of the second, third and fourth years. Its exercise implies an outlay of the 20% of initial investment, and it increases the project sales by 50% of the existing level. Option values are estimated by both *Critical values* proposal and *Regression* proposal. *Regression 1* uses the same simulated paths to estimate the optimal exercise strategy and the option value, whereas the *Regression 2* employs different sets of simulations. We consider a mixed Brownian-Poisson process. Geometric-Brownian drift is 15%, with alternative volatilities of 10%, 20% and 30%. For the jump motion, we consider a range of volatilities between 25% and 500%, with an average number of annual jumps ranging of 0.20 to 1. The number of simulated paths, H , is 400,000 (200,000 from direct approximations plus 200,000 antithetical estimations). M and K in the “critical value” proposal are equal to 400.

Once more, the desire to strike a trade-off between the accuracy of the estimations and the computational requirements would seem to justify the use of the regression based procedure. The results also reveal that

the value of the option increases with the number of discrete variations, thus increasing the underestimation error when its influence is not reflected. However, when including an upper boundary in the evolution

of the demand, the option tends to lessen the effect mentioned. In this case, omitting jumps leads to an overestimation of the chance to extend the project, possibly giving rise to the option being exercised before the optimal date and even to accepting non-profitable projects.

Conclusion

Throughout this paper, we have addressed the issue of valuing American-type real options contingent on a continuous stochastic process subject to random jumps. This combination of American-type and random jumps prevents any closed-form solution for the fundamental pricing equation. Usual numerical techniques, such as binomial trees or finite differences, also fail to provide satisfactory results.

Strange though it may seem, an option pricing technique which had for many years been restricted to the analysis of European-type derivatives has recently been re-appraised and proposed to value real complex options such as those described. The reason behind this apparent delay is the nature of the traditional simulation models which prevents identification of the optimal exercise policy. Monte Carlo simulation is a *forward* induction procedure, which generates future values of the variable from its previous value and therefore is not suitable for valuing assets generating cash flows contingent on future events, such as is the case of American-type options. Recent research has proposed overcoming this restriction through the joint use of simulation and some backward induction technique that enables its application to valuing American-type options.

In order to evaluate two alternative simulation proposals, and after adapting them for the real option problem, we value a numerical example which consists of a finite-life project that incorporates a growth option contingent on a state variable following a geometric Brownian motion subject to average null size random jumps.

Our numerical results reflect the need to consider a trade-off between the accuracy of the estimations

and the effort required in terms of abstraction, modelling and computerization, particularly in the case of high volatility of the discontinuities. In these cases, the valuations that emerge from the regression based procedure show greater dispersion, although implementation in this case is less costly as it depends linearly on the number of opportunities for exercise. By contrast, the proposal based on critical values is generally found in an intermediate position in comparison with the previous ones, yet requires the simulation of new paths for the state variable at each point at which exercise is allowed.

In addition, we have considered an upper boundary in the state variable evolution and multiple smaller size discontinuities. Our results have revealed no significant differences in valuations offered by both proposals, and consequently, the regressions based procedure has proved more convenient as it involves a lower cost in terms of resources.

As regards the evolution of the state variable, we observed that the possible occurrence of a single random jump during the project life span leads to an increase in the growth option value, for the lower levels of discrete volatility considered. In such cases, omitting discrete discontinuities can lead to an underestimation of the option value, with the subsequent deferral in exercise, and even the execution of profitable projects being discarded.

Nevertheless, for higher jump volatilities the effect of a single random jump on the option value is inverted and may even fall below the value obtained from conventional evaluation models that do not contemplate discontinuities. Moreover, the combined effect of the jump and an upper boundary in the evolution of the state variable also adds differences in the valuation results. In contrast to what occurs in the absence of this kind of restriction, there is a reduction in the option value as the number of random jumps increases. As a result, the over-valuing to which the omission of these discontinuities leads may cause exercise prior to the optimal date and even the acceptance of non-profitable projects.

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