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Managing gap risks in iCPPI for life insurance companies: a risk return cost analysis

Abstract

Individualized constant proportion portfolio Insurance (iCPPI) products are attractive alternatives to traditional unit linked products offering a guaranteed minimum return, such as variable annuities. They offer high potential returns whilst limiting the downside risk by implementing a dynamic allocation strategy between risky and risk-free assets tailored to the risk attitude of the beneficiary. But performance evaluation of iCPPI products should not rely on the unrealistic assumptions of continuous market price variations and continuous rebalancing of asset allocations. We adopt a more general and realistic price jump model and examine several dynamic strategies as well as gap put options to mitigate the risk that the value of the product falls below the guaranteed minimum.

Key words: CPPI, dynamic multiplier, jump processes and gap risk, vanilla and gap options

Introduction

Increased market volatility and falling interest rates triggered by the 2008-9 financial crisis reduced the performance of traditional long-term investment products, increased their risks and, where applicable, their capital requirements. In this context the new iCPPI products provide an attractive alternative to many traditional long-term investment products offering a guaranteed minimum return, such as variable annuities, for several reasons: lower exposure to uncertain volatilities and extreme market price movements, lower costs, and lower regulatory capital requirements, to name a few.

Already, with rising life expectancies, current provisions for retirement may not be sufficient for many people to secure acceptable life standards after retirement. To achieve sufficiently high investment returns together with low risks over the long term, funds should remain invested in stocks and other risky assets as well as in the safer bonds over an extended period well into retirement. The design of long-term investment products should also reflect the requirements and risk attitudes of individual investors.

Constant proportion portfolio insurance (CPPI) is the name given to an investment strategy that provides a minimum guaranteed return, the "floor" (usually defined as the discounted value of a final capital guarantee) and aims to maintain at all times an exposure to a risky asset equal to a constant multiple of the "cushion" defined as the excess value of the fund above the floor. The final capital guarantee and the multiplier are chosen to satisfy the risk attitude of the investor. Various authors, among which Perold (1986), Merton (1971) and Black and Jones (1987), proved that, assuming a geometric Brownian process for the risky asset price dynamics, a constant rate of return for the risk-free asset, and continuous rebalancing at no cost between the two assets, i.e, under Black-Scholes conditions, the CPPI payoff is optimal for an investor with a coefficient of risk tolerance varying linearly with wealth. Specifically, the CPPI payoff is equal to the chosen floor plus a cushion value proportional to a power of the risky asset price equal to the chosen multiplier; the floor and the multiplier are chosen according to the two parameters of a HARA utility function so as to maximize the expected utility of the investor. Additional advantages offered by CPPI strategies over more traditional investments with minimum guaranteed returns are: price transparency, open time-horizon, no early redemption penalty, wide range of alternative investments for the risky asset, and flexibility to add other guarantees such as ratchets (see II.1.4).

iCPPI is a CPPI strategy adapted to evolving individual needs and market conditions. The floor and the multiplier are modified accordingly. Thus iCPPI may combine most of the advantages of CPPI with the need for flexibility and enhanced risk management.

However the provider of an iCPPI product (typically, an insurance company) faces many challenges in the implementation of the dynamic strategy that replicates the guaranteed payoff. The rebalancing of the risky asset/risk-free asset allocation can only be made at discrete times, there are transaction costs, and risky asset prices may jump. Thus there is likely to be a difference between the realized compared to the theoretical value of an iCPPI strategy under hypothetical conditions of continuous price movements, unfettered zero-cost trading, and continuous rebalancing. In particular, here is a finite probability for the value of the fund to fall below the guaranteed floor. We call such shortfall the gap risk. Managing or insuring the gap risk may be delegated to a third party (e.g. a bank).

The analysis of the gap risk has often been limited to simple conditions to preserve analytical tractability:

- Unrealistic modelling of the risky asset price market including continuous price dynamics, zero-cost trading and unlimited liquidity.

- Simple parameterization of the CPPI strategy such as constant capital guarantee and multiplier.
- Simplistic rebalancing strategies such as constant frequency.

As a result, the iCPPI offers a mechanism that takes advantage of the specific advantages of both stocks and bonds, while complying with growing needs of flexibility as experienced by policyholders.

However, the implementation of iCPPIs at insurance companies levels suffers from a number of operational constraints on the asset management: the rebalancing occurs through regular albeit discontinuous (at most daily) checks between the insurance company and a bank; depending on the design of the iCPPI and the discontinuous rebalancing frequency, the magnitude of the earnings at extreme risk may require the externalization of the gap risk management to the bank. As a result the main issue experienced by the insurance company remains to minimize the downside risk and keep control of the gap risk, which involves three main challenges: This article extends previous analyses of the gap risk by introducing:

- Price jump dynamics.
- A dynamically adjusted multiplier.
- Advanced rebalancing strategies, vanilla and gap put options to mitigate the gap risk.

Review of CPPI mechanism basics. Consider at time t a risky asset (e.g., a share) with price S_t and a risk-free asset (e.g., a Treasury bond) with price B_t returning a constant rate r. The CPPI fund is invested into these two assets so that part of its value, called the "floor" F_t, is guaranteed whilst the excess value above the floor, called the "cushion" $C_t = V_t - F_t$, remains exposed to the risky asset price fluctuations. At any time, the exposure to the risky asset, e_t, is kept at a constant multiple, m, of the cushion, that is: $e_t = mC_t$.

The rest of the value of the fund is invested (or, if negative, borrowed) at the risk-free rate (Note that the exposure e_t may be acquired at no cost if using an off-balance sheet instrument such as a future, which may be advantageous because of liquidity and low transaction costs). The floor is often chosen to increase over time at the risk-free rate (it could not be made to increase faster indefinitely), that is:

$$F_t = F_0 e^{rt}. \quad (1)$$

In theory, when the risky asset price follows a geometric Brownian motion, and with continuous, zero-cost rebalancing (Black-Scholes conditions), the value of the cushion is path independent and proportional to S_t^m In other words, it is the value of a power option. It is convex when $m > 1$ (like a long call option), linear when $m = 1$, and concave when $m < 1$, like a short put option. But unlike standard call and put options there is no need to fix an expiry date, a CPPI strategy is open-ended. Under the above assumptions, the value of the cushion would never fall to zero; in practice, if it does fall to zero or below zero (e.g., because of a price jump or of discrete rebalancing), the entire fund is monetized, i.e., is entirely invested in the risk-free asset, and the product provider must make up the shortfall to deliver the floor value. In practice there may also be other constraints such as no borrowing or additional features such as ratcheting up the floor. In those cases, the path independency and open-endedness of the product are lost and the payoff profile becomes more complex.

2. Methodolody and results

2.1. CPPI in theory and practice. *1.1.1. Continuous-time framework.* The risky asset S is defined by the diffusion equation $dS_t[\mu dt + \sigma dW]$ where W is a standard Brownian motion. The previous hypothesis for the risk-free asset is kept.

In such context, and assuming continuous time CPPI, the cushion C is log-normally distributed with drift $m(\mu - r) + r$ and volatility $m\sigma$:

$$C_t = C_0 \exp\left(\left(m(\mu - r) + r - \frac{m^2\sigma^2}{2}\right)t + m\sigma W_t\right). \quad (2)$$

and the portfolio value V has the path independent expression:

$$V_t = F_t + (V_0 - F_0)\exp\left(\left(m(\mu - r) + r - \frac{m^2\sigma^2}{2}\right)t + m\sigma W_t\right). \quad (3)$$

However, such assumptions are unrealistic and not consistent with market practice. To remedy these unrealistic hypothesis, two alternatives are studied: modeling in a discrete-time framework and in a Lévy framework.

2.1.2. Discrete-time CPPI. In practice the CPPI is rebalanced in discrete time, where the shortfall probability is no longer equal to 0, which implies to monetize more often.

Table 1. Final value metrics: buy & hold strategy vs CPPI with $m = 3$ vs CPPI with $m = 6$

	Buy & hold strategy	CPPI with $m = 3$			CPPI with $m = 6$		
		Daily	Weekly	Monthly	Daily	Weekly	Monthly
Mean	126.97	123.31	122.39	119.75	124.10	124.87	125.01
Std-dev	7.18	31.58	32.66	36.86	42.62	43.88	48.10
95% quantile	116.90	100.48	99.98	97.01	99.99	99.13	89.69

Table 1 (cont.). Final value metrics: buy & hold strategy vs CPPI with $m = 3$ vs CPPI with $m = 6$

	Buy & hold strategy	CPPI with $m = 3$			CPPI with $m = 6$		
99.5% quantile	113.42	100.02	99.88	91.47	99.98	95.20	74.28
5% quantile	140.21	194.37	195.23	197.94	216.51	218.50	225.46
0.5% quantile	150.63	266.47	284.07	282.58	291.49	293.75	311.46
Rebalancing cost	0.01	0.91	0.44	0.26	0.78	0.46	0.31
pBh	0	0.0018	0.0947	0.5289	0.2016	0.5730	0.6555

A sequence of equidistant refinements of the interval $[0, T]$ is defined:

$$\Theta = \left\{ t_0 = 0 < ... < t_{N-1} < t_N = T \right\}. \quad (4)$$

where $t_{k+1}^N - t_k^N = \frac{T}{N}$ for $k = 0,\dots N-1$. The number of shares is constant on the intervals $]\ t_i,\ t_{i+1}]$. Let $t_s = \min\left\{ t_k \in \Theta \mid V_{t_k} - F_{t_k} \le 0 \right\}$. The first time the portfolio value touches the floor. The discrete-time cushion follows.

$$C_{t_{k+1}} = e^{r(t_{k+1} - \min\{t_s, t_{k+1}\})} (V_{t_0}^{\Theta} - F_{t_0}) \prod_{i=1}^{\min\{s,k+1\}} \left(m \frac{S_{t_i}}{S_{t_{i-1}}} - (m-1) e^{r\frac{T}{N}} \right). \quad (5)$$

or recursievely:

$$C_{t_{k+1}} = \begin{cases} C_{t_k} \left(m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1) e^{r\frac{T}{N}} \right) & \text{if } C_{t_k} > 0 \\ C_{t_k} e^{r\frac{T}{N}} & \text{if } C_{t_k} \le 0 \end{cases}. \quad (6)$$

V_{tk} is given through the relation $V_{tk} = C_{tk} + F_{tk}$. In order to comply with the CPPI algorithm and respect practical constraints, the number of shares of the risky and safe assets (α and β) are as follows:

$$\alpha_{t_k} = min \left(\max\left(\frac{mC_{t_k}}{S_{t_k}}, 0 \right), \frac{V_{t_k}}{S_{t_k}} \right).$$

$$\beta_{t_k} = \frac{V_{t_k} - \alpha_{t_k} S_{t_k}}{B_{t_k}}. \quad (7)$$

When adding transaction costs, these are taken as a proportion of the change in the risky exposure (i.e $Proportional\ cost = (\alpha_{t_k} - \alpha_{t_{k-1}}) \times S_{t_k}$). So at time t_k, the number of shares of the risky asset will be reduced:

$$\tilde{\alpha}_{t_k} = \alpha_{t_k} - |\alpha_{t_k} - \alpha_{t_{k-1}}| \times nb\ of\ bps. \quad (8)$$

The CPPI capital guarantee is ensured as long as the bond floor is not breached through, enabling to fully invest the portfolio into the non risky assets. The probability of breaching the floor is defined as the probability that the portfolio value falls below the floor (i.e. $P^{BF} := \mathbb{P}(V_T \le G) = \mathbb{P}(\exists t \in [0,T] : V_t \le F_t)$

The local shortfall probability is the conditional probability defined as: $P_{t_i,t_{i+1}}^{LBF} = \mathbb{P}\left(V_{t_{i+1}} \le F_{t_{i+1}} \mid V_{t_i} > F_{t_i} \right)$. The two are related as follows $P^{BF} = 1 - \prod_{i=1}^{i=N} (1 - P_{t_i,t_{i+1}}^{LBF})$.

This probability which was equal to zero in the contnuous Black-Scholes model, is now greater than zero. Assuming the portfolio hasn't breached the floor up to time t_k, the probability of breaching the floor at time t_{k+1} is that of a downside jump in the risky asset of more than about 1/m. Its mathematical expression is:

$$P_{t_i,t_{i+1}}^{LSF} := \mathbb{P}\left(\frac{S_{t_{i+1}}}{S_{ti}} \le \frac{m-1}{m} e^{r\frac{T}{N}} \right). \quad (9)$$

where the evolution of the risk-free part with rate r is taken into account.

The backtesting is based on the period Q1-2006 to Q4-2010 on S&P500 index. Simulating paths (N = 10,000) in the Black & Scholes model is made using the 3-month realized volatility based on the standard deviation (see Figure 1), a constant asset return m = 8%. The rate of the risk-free asset is r = 4%. Three rebalancing frequencies are being compared regarding the distribution of the final portfolio value (daily, weekly and monthly), with the following assumptions:

- Initial investment/Guarantee: \$100, and \$100
- Duration: 5 years
- Transaction costs: 10 bps.

The CPPI strategy under daily rebalancing performs better against a bear market than the weekly and monthly ones due to its reactiveness to decrease the risky exposure whenever needed. With such frequency, the guarantee is almost ensured; the less frequent we rebalance the more we are exposed to breaching the floor (as illustrated by fatter left tails (see Figure 2 bottom, right). The backtesting (Figure 2 top) and Table 2 illustrate the following remarks:

- In periods of mild market conditions, transaction costs negatively affect the performance of a daily rebalancing, although not to a significant extent.
- During a market crash, the three strategies monetize, with the daily rebalancing having less losses than the two others.
- The empirical probability of breaching the floor decreases when the rebalancing frequency increases.

- The cost of rebalancing increases with the frequency and with the multiplier. However, in our results, the cost of daily rebalancing for $m = 6$ is lower than the one with $m = 3$. This is explained by the fact that such a high multiplier allows for a total risky exposure and thus no rebalancing reducing the cost.

When comparing different strategies (buy & hold, CPPI with $m = 3$ and $m = 6$), we have the following results:

- The Buy & Hold strategy has higher expectation and lower standard deviation (Table 2). This is mainly due to the low exposure to the risky asset. Its performance is highly correlated to the non-risky return (chosen to be 4%).
- The 5% and 0.5% quantiles show that the CPPI with $m = 6$ has a larger right tail and thus, performs better than the two others in bullish market. This remark is also illustrated in Figure 8.

Daily rebalancing almost prevents the bond floor from being breached, which ensures the capital guarantee at maturity. However, constant volatility and log-normal distribution modeling are not consistent with empirically observed jumps during extreme market moves likely to breach the bond floor. In order to relax these unrealistic assumptions, jumps are thus added through Lévy processes as developed in the next section.

2.1.3. Adding jumps. We assume that the process of the risky asset follows a Lévy process:

$$\frac{dS_t}{S_t} = dZ_t, \quad (10)$$

where Z is a Lévy process. The risk-free asset F_t is still deterministic.

Let $\tau = \inf\{t : V_t \le B_t\}$ the time where the portfolio value is fully invested in the risk-free asset. Until τ the actualized cushion $(C_t^* = \frac{C_t}{F_t})$ is as follows: $C_t^* = C_0^* \varepsilon(mL)_t$, where ε denoting the stochastic exponential:

$$\mathcal{E}(Z)_t = Z_0 e^{Z_t - \frac{1}{2}<Z>_t} \prod_{s \le t, \Delta Z_s \ne 0} (1 + \Delta Z_s) e^{-\Delta Z_s}, \quad (11)$$

which gives us the portfolio value:

$$V_t = \begin{cases} V_t \left\{ 1 + \left(\frac{V_0}{F_0} - 1 \right) \mathcal{E}(mL)_t \right\} & \forall t \le \tau \\ V_\tau e^{r(t-\tau)} & if\ t > \tau \end{cases}. \quad (12)$$

The probability of breaching the floor can be expressed as:

$$P^{BF} = \mathbb{P}(\exists t \in [0,T], V_t \le B_t) = 1 - \mathbb{P}\left(\forall t, \Delta L_t < \frac{1}{m} \right). \quad (13)$$

$$P^{BF} = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx) \right), \quad (14)$$

which is illustrated by the fact that the number of ownside jumps of size more than $\frac{1}{m}$ follows a Poisson distribution with intensity $T\nu(-\infty, 1/m)$.

For computation tractability, we choose the double exponential Kou model (see Kou [2002]). Under the risk neutral probability, the risky asset is modeled as follows:

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW + d\left(\sum_{i=1}^{N_t} e^{Y_i} - 1 \right), \quad (15)$$

where W is a standard brownian motion, N is a poisson process with rate A, the constants μ and $u > 0$ are drift and volatility of the diffusion part and the jump sizes $\{Y_1, Y_2,...\}$ are i.d.d random variables with a common asymmetric double exponential distribution of density:

$$f_Y(y) = (1-p)\eta^+ e^{-\eta^+ y} 1_{y \ge 0} + p\eta^- e^{\eta^- y} 1_{y<0}. \quad (16)$$

η^+ is intensity of positive jumps while η^- and p are the intensity of negative jumps and the probability of their occurrence.

Under this jump model, and assuming a continuous rebalancing frequency, the probability of breaching the floor takes the following form:

$$P^{BF} = 1 - \exp\left(-Tp\lambda(1 - \frac{1}{m})^{\frac{1}{\eta^-}} \right). \quad (17)$$

In this section, the CPPI strategy keeps the same characteristics except for the risky asset which is modeled through a Kou process calibrated on implied volatility smile (between 2006 and 2011 on a 1-month implied volatility on a weekly). We carried out the calibration by minimizing the quadratic error $\sum_{i=1}^{9} \left(C_{t_i}(T,K_i)^{\text{Market}} - C_{t_i}^{\text{Kou}}(T,K_i,\sigma,p,\eta^+,\eta^-,\lambda) \right)^2$, (18)

where T is 1-month maturity, K_i strikes from 80 to 110 and $(p,\eta^+,\eta^-,\lambda,\sigma)$ are the jump parameters (more details about). We give different statistics for these parameters in the table below:

	Average	5% Quantile	Std-Dev
p	0.64	0.84	0.24
η^+	0.16	0.28	0.06
η^-	0.15	0.28	0.07
λ	0.62	2.44	0.12
σ	18.29%	29.64%	0.08

In order to avoid instability in parameters, we chose several starting points and set boundary conditions. An example of the result on the calibration is shown in Figure 4. A few remarks on the calibration can be made:

- Since the upward-sloping part of the smile is very small, the positive jumps are hardly calibrated in a reliable manner. However, the pricing of the gap option (section II.2.2) only needs the negative jumps intensity (i.e the downward-sloping part of the smile).
- The calibration is better on close-to-maturity options (as mentioned in Tankov, 2010). It allows a better capture of instantaneous jump.
- The calibrated parameters will be used for hedging gap risk in the last section

Figure 5 compares different discrete rebalancing frequencies with a jump modeling:

- Even for daily rebalancing, breaching the floor is unavoidable with the same probability as the two other frequencies.
- The three rebalancing frequencies give similar results when taking transaction costs into account.

	Kou model		
	Daily	Weekly	Monthly
Mean	146.28	147.10	147.57
Std-Dev	52.84	52.93	53.11
95% quantile	92.19	92.21	92.03
99.5% quantile	59.38	59.08	59.23
5% quantile	238.13	238.67	239.41
0.5% quantile	349.41	350.92	350.37
Rebalancing cost	0.92	0.45	0.26

The previous illustrations show that both the frequency of the rebalancing and the modeling affect the final value. The two metrics previously defined for different modeling assumptions

- The local probability of breaching the floor:

$$P^{LBF}_{t_i,t_{i+1}} := \mathbb{P}\left(V_{t_{i+1}} \leq F_{t_{i+1}} \mid V_{t_i} > F_{t_i}\right). \quad (19)$$

- The overall probability of breaching the floor:

$$P^{BF} := \mathbb{P}\left(\exists t \in [0,T] : V_t \leq F_t\right) = \mathbb{P}\left(V_T \leq F_T\right). \quad (20)$$

- For B&S model in discrete-time rebalancing:

$$P^{LBF}_{t_i,t_{i+1}} = \mathcal{N}\left(-\frac{log(\frac{m}{m-1}) + (\mu - r)\frac{T}{N} - \frac{1}{2}\sigma^2\frac{T}{N}}{\sigma\sqrt{\frac{T}{N}}}\right) \quad (21)$$

and $P^{BF} = 1 - \prod_{i=0}^{N-1}\left(1 - P^{LBF}_{t_i,t_{i+1}}\right).$ (22)

- For Kou jump process in continuous time:

$$P^{BF} = 1 - \exp\left(-Tp\lambda(1 - \frac{1}{m})^{\frac{1}{\eta^-}}\right). \quad (23)$$

Results depend on the model parameters and discretization time step:

- Gap risk goes to 0 as the rebalancing tends to be more frequent.
- When considering a discontinuous path (jump models), even in continuous rebalancing the gap risk value > 0.

Model	Frequency	P^{BF}
B&S	Monthly Weekly Daily	9.07×10^{-5} 1.2×10^{-10} ~
Kou	Continuous	0.00410

Consider the stopping time t as the first time the portfolio value breaches the floor which does not depend on the bond floor level. The distribution of t is the same in case of adding the ratchet, i.e. the probability of breaching the floor is not usually affected by the ratchet feature in theory. However, in our simulations, this probability in higher for the monthly rebalancing. This might be.

2.1.4. Impact of the ratchet feature. The ratchet feature is used by insurance companies to attract investors as it periodically locks in profit (see Brigo and Mercurio [2006] and Andersen and Piterbarg, 2010 for more details): at anniversary dates the guarantee is set to the highest value so far. The guarantee G becomes a time dependent function.

$$G_t = \begin{cases} V_0 & if\ t = 0 \\ \max(G_{t^*_{k-1}}, V_{t^*_k}) & if\ t = t^*_k \\ G_{t^*_k} & if\ t \in [t^*_k, t^*_{k+1}] \end{cases}. \quad (24)$$

The bond floor is then defined as $F_t = G_t e^{-r(T-t)}$.

This feature has advantages and drawbacks. Locking-in the cash will ensure a higher guarantee but also reduces the cushion, the risky exposure and thus the upside potential risk.

The main results from Figure 6 are:

- The mean and standard deviation of the final value increase with the rebalancing frequency (see Table 2). This is justified by the path dependency of the guarantee which has a larger distribution with higher rebalancing frequency.
- The quantiles on the two tails of the final value distribution increase with the rebalancing frequency, while the distribution is shifted to the right with narrower body.

2.2. Mitigating downside risk: Preventing from breaching the floor. *2.2.1. Adjusting the multiplier to market conditions.* By focusing on managing returns in downside markets, CPPI effectively manages portfolio volatility. Over the 5-year data (which included one bullish market, one bear market and a recovery), the CPPI strategy resulted in a slightly lower return – but also a significantly lower volatility. Additionally, the worst one year return for the CPPI strategy was significantly less than that of the index portfolio.

Table 2. Final value metrics: comparison between a CPPI without and with the ratchet feature

	Without ratchet			With ratchet		
	Daily	Weekly	Monthly	Daily	Weekly	Monthly
Mean	123.82	124.26	124.17	145.46	143.01	134.03
Std-Dev	41.96	43.29	47.25	100.08	81.75	45.60
% quantile	99.99	99.68	90.88	100.61	100.53	99.99
.5% quantile	99.99	97.57	77.84	99.99	99.94	98.27
% quantile	214.18	216.42	222.23	268.74	261.73	219.52
.5% quantile	289.15	292.33	314.58	700.97	603.28	359.59
P^{BF}	0.11	0.47	0.64	0.11	0.48	0.84

The manager usually sets the multiplier at the beginning of the period. The risky exposure depends then on the evolving cushion. As the probability of breaching the floor may surge in market crash, or the manager might miss the subsequent market recovery, the multiplier needs to be adjusted accordingly with the market conditions.

A first approach to define a dynamic multiplier is the choice of the optimal *m*, deduced from the closed form solutions for optimal payoffs, and optimal certainty equivalent returns (CERs) using HARA utilities and log-normal distribution (see Pezier, 2011). The authors give the following formula $m^* = \eta(\mu - r)/\sigma^2$ (η here is the investor's sensitivity of risk tolerance to wealth). A particular case is the growth optimal leverage with $\eta =$ 1 which is resulted in optimizing the growth rate of the leveraged strategy (cushion).

An alternative to the optimal multiplier is a value-at-risk based multiplier where investors choose the confidence level according to their risk tolerance as well but focused on tail risks. Based on the weight w_t^R of the value-at-risk based portfolio insurance (VBPI) introduced by Jiang et al. (2009), and the expression of the risky exposure in both strategies $E_t = m_t C_t = w_t^R V_t$ the expression for the multiplier at time t is:

$$m_t = \frac{1}{1 - \exp\left((\mu - r - \frac{1}{2}\sigma^2)(T-t) - z_p \sigma \sqrt{T-t} \right)}. \quad (25)$$

As the dynamic multiplier depends on both volatility and return estimates, in order to improve its efficiency, μ and σ can be made time dependent. However, since the estimation of the drift is hardly accurate for a short window, we will restraint the time dependency to the volatility. It will be re-estimated through a 3-month sliding window to take into account different market regimes.

The two approaches offer an interesting alternative to the constant multiplier which lacks flexibility depending on market condition. The comparison between these two approaches through a backtesting from 2006 to 2011 is illustrated in Figure 7. The focus on two periods (2006-2007 and post 2008 crisis) (Figure 8) illustrates that the VaR-based multiplier can perform better than the "optimal" one in bullish market and recovery (e.g 18% return Q2-2009 until Q1-2011 vs 11% in the post 2008 crisis). In contrast, during bear market, using the "optimal" multiplier (through $m < 1$) helps keep a relatively higher cushion but misses the recovery as it doesn't allow a high leverage.

In order to allow to participate in the market recovery to a greater extent, the multiplier is adjusted with a modified volatility estimator, either through a short-term exponentially weighted moving average (EWMA with $\lambda = 0.94$) realized volatility or an estimator based on implied volatility (of the strike consistent with the latest market returns). For example, if the underlying jumped 5% downward, the implied volatility with strike 95% will be chosen. For unavailable strikes, we use a linear interpolation. This strategy starts reinvesting into the risky asset as soon as Q3 2009, resulting in a higher performance by allowing the portfolio to capture more of the upside return when markets rebound. The backtesting in Figure 10 illustrates that the new multiplier is more reactive when adjusting with the implied volatility estimator. However, the 3-month realized volatility provides a higher multiplier and, when considering transaction costs, leads a lower cost of management.

Finally, the fixed frequency rebalancing is switched to a trigger rebalancing which occurs when the multiplier is out of a specific range chosen by the portfolio manager. In our case, on average the rebalancing frequency becomes every other day, which is consistent with the usual practice in CPPI asset management. At the same time, the cost of rebalancing is cut by half in comparison to a daily rebalancing (i.e. as low as a weekly or monthly rebalancing). Figure 11 illustrates the increasing performance specifically under a range-bound high volatility regime, e.g. Q1-Q3 2008.

Adjusting the multiplier dynamically allows it to be more reactive to market conditions and explicitly dependent on the investor's risk aversion. However, it does not totally annihilate the downside risk in case of sudden jumps, where options may be useful to hedge those gap risks

2.2.2. Hedging gap risks. The CPPI methodology will not necessarily protect the portfolio against a "black swan" event (such as a market crash of 20% in one day). To the extent that asset allocation shifts are implemented via underlying funds, the rebalancing trade can only occur at the end-of-day NAV. Even if futures are used to implement shifts intra-day, there can be gap movements in the future markets. This is where a small gap risk protection sleeve can add value to the portfolio. To protect against such a "black swan" event, it is important to already have put options on market indices in the portfolio.

2.3. Vanilla Put option. A simple hedging strategy for the CPPI through embedded option can be constructed using short maturity put options. Touching the bond floor is mathematically equivalent to the cushion becoming negative. Assuming the event hasn't occurred up to time t_k, using equation (1), we have:

$$C_{t_{k+1}} < 0 \Leftrightarrow m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{N}} < 0. \quad (26)$$

Hedging this risk is equivalent to forcing this quantity to be positive. This can be done by buying a put option at each of the CPPI rebalancing period with strike $(1-\frac{1}{m})e^{r\frac{T}{N}}S_{t_k}$ and as a maturity the CPPI rebalancing frequency. To hedge the whole portfolio the manager needs a number of $m\frac{C_{t_k}}{S_{t_k}}$ puts, which is the risky asset exposure. The discounted payoff in this case is $e^{-r\frac{T}{N}}C_{t_k}((m-1)e^{r\frac{T}{N}} - m\frac{S_{t_{k+1}}}{S_{t_k}})^+$. The hedging cost at time t_k can be written as:

$$\text{Cost}_{t_k} = m\frac{C_{t_k}}{S_{t_k}}\mathbb{E}^Q\left[\left((1-\frac{1}{m})e^{r\frac{T}{N}}S_{t_k} - S_{t_{k+1}}\right)^+\right]. \quad (27)$$

Two approaches can be considered:

- ♦ The hedging costs (put prices) are deducted only afterwards from the portfolio value (which allows an estimation of how much the hedge would cost). In this case, the cushion follows the recursive relation:

$$C_{t_{k+1}} = C_{t_k}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} + (1-m)e^{r\frac{T}{N}}\right)^+. \quad (28)$$

The cost of hedging can be computed as the sum of all put options prices necessary for the hedging:

$$C = \sum_{k=0}^{n-1} m\frac{C_{t_k}}{S_{t_k}}\mathbb{E}^Q\left[\left(\left(1-\frac{1}{m}\right)e^{r\frac{T}{N}}S_{t_k} - S_{t_{k+1}}\right)^+\right]. \quad (29)$$

- ♦ In practice, the price of the puts used for the hedge will be deducted from the portfolio value at each step. This is translated in the second approach where the cushion dynamics follows the recursive equation:

$$\tilde{C}_{t_{k+1}} = e^{-r\frac{T}{N}}\tilde{C}_{t_k}\frac{\left(m\frac{S_{t_{k+1}}}{S_{t_k}} + (1-m)e^{r\frac{T}{N}}\right)^+}{\mathbb{E}^Q\left[\left(m\frac{S_{t_{k+1}}}{S_{t_k}} + (1-m)e^{r\frac{T}{N}}\right)^+ \mid \mathcal{F}_{t_k}\right]}. \quad (30)$$

We compare the effects of the put hedging in Figure 13. We can see the following remarks:

- ♦ The guarantee is ensured and the manager no longer holds the risk of breaching the floor. However, once the put is exercised and the floor recovered, the manager needs to monetize in order to keep the guarantee until maturity.
- ♦ In terms of distributions, the CPPI distribution with a put hedging is a truncation of the classical CPPI where losses are cut (left tail limited by the guarantee).

2.4. Gap put option. An alternative risk mitigating action lies in the use of Gap Options which allow for a protection against sudden significant and persistent downside market moves: if a gap event occurs between two consecutive dates, the buyer receives the difference between the performance of the risky asset at gap $r = \frac{S_t}{S_{t-1}} - 1$ and the threshold J. In case of the CPPI, the proposed solution is a gap put option whose notional is the risky exposure with strike $J = 1/m$, where m is the multiplier.

2.5. Definition and properties. (see Tankov, 2010 for details). Suppose that the time to maturity T of a gap option is subdivided onto N periods of length h (e.g. days): $h = \frac{T}{N}$. The return of the kth period will be denoted by $R_k^\Delta = S_{kh} / S_{(k-1)h}$. Let α denote the return level which triggers the gap event and k^*

be the time of first gap expressed in the units of h: $k^* := \inf\{k : R_k^h \le \alpha\}$. The gap option is an option which pays to its holder the amount $f(R_{k^*}^h)$ at time hk^*, if $k^* \le N$ and nothing otherwise.

Assuming a deterministic interest rate r and an i.i.d log returns $(R_k^h)_{k=1}^N$ and denote the distribution of $log(R_1^h)$ by $p_h(dx)$. Then the price of a gap option is given by:

$$G_h = e^{rh}\int_{-\infty}^{\beta} f(e^x)\,p_h(dx)\,\frac{1-e^{-rT}(\int_{\beta}^{\infty} p_h(dx))^N}{1-e^{-rT}\int_{\beta}^{\infty} p_h(dx)}, \quad (31)$$

with $\beta := log(\alpha) < 0$.

Obtaining numerical results using this formula is complicated in the general case. An approximate formula is used.

Assume $S_t = S_0 e^{X_t}$, where X is a Lévy process. Considering the hypothesis $rh \sim 10^{-4}$ and $h \to 0$, the following formula is obtained:

$$G_h \simeq \int_{-\infty}^{\beta} f(e^x)\,\frac{1-e^{-rT}(\int_{\beta}^{\infty} p_h(dx))^N}{1-e^{-rT}\int_{-\infty}^{\beta} \nu(dx)}, \quad (32)$$

Assuming a Kou model (for its tractability and simplicity in integration) and considering the put payoff (i.e $f(x) = (K-x)^+$). The price then becomes:

$$G_h \simeq \frac{\lambda p \eta^-}{1+\eta^-} K^{1+1/\eta^-}\,\frac{1-e^{-T(r+\lambda p e^{\beta/\eta^-})}}{r+\lambda p e^{\beta/\eta^-}}. \quad (33)$$

with p the probability that a given jump is negative, η^- its intensity and λ the Poisson process rate.

Moreover, for the CPPI we are interested in the payoff $((m-1)e^{rh} - m\frac{S_{kh}}{S_{(k-1)h}})^+$ which is equivalent to $(\frac{m-1}{m}e^{rh} - x)^+$ and thus, $K = (1 - 1/m)e^{-rh}$.

Table 3. Final value metrics: comparison between different hedging strategies

	Hedging strategies		
	Vanilla Put Hedge 1	Vanilla Put Hedge 2	Gap Option
Mean	136.97	133.35	134.98
% quantile	218.70	215.40	217.00
5% quantilee	277.21	273.53	275.22
Hedging cost	N.C	2.26	1.08

The gap put option allows to cut the loss compensates for the loss as the portfolio value breaches the bond floor. However, insurance investors holding a CPPI who want to hedge it with gap option may face the following issues:

- The price of the gap option is usually sold higher than its theoretical cost for several reasons:
- The cost of the hedging the gap option for the bank may be quite higher because of the illiquidity of deep out of the money options that replicate it.
- The replicating formula is tricky to implement and interpret, as significantly model dependent (jumps multiple parameters, lack of robustness).
- Actually, the gap option proposed by the bank might have a different design and payoff from the one considered for the hedge.
- The bank usually hedges the gap up to the first order only.
- The gap risk is borne by the bank only if there is some reconciliation by the insurance company within 24/48 hours, out of which the insurer bears oneself the gap risk. As a result, operational risks are significant and represent a major part of the economic capital requirements (e.g. under Solvency II framework).

Conclusion

In this article we have presented a study of the CPPI as an insurance contract, a review of its theory and practice as well as its modeling and hedging issues for a risk/return/cost perspective. The main conclusions are:

- Continuous CPPI is only theoretical: given market frictions and the probability of not ensuring the guarantee, all the more that jumps occur more than not.
- As a result, jump processes are a valuable input for the CPPI modeling: they allow to catch a probability of breaching the floor different than zero (even in the continuous-time framework; Garcia and Goosens, 2009 and Garcia et al., 2008) came up with the same conclusion) and therefore, detect, define and hedge gap risk.
- Correctly choosing and adjusting the multiplier dynamically significantly reduce the downside risk according to a Value-At-Risk indicator: The multiplier decreases in period of turmoils reducing the risky exposure and increases back during market recovery.
- Hedging the gap risk is possible through two types of options: vanilla puts and gap put options. The first one is more common due to liquid assets, but the hedging cost may turn out to be too expensive and the maturity too limited. The second type of options is less liquid (bought only through an agreement) but is cheap.

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Appendix

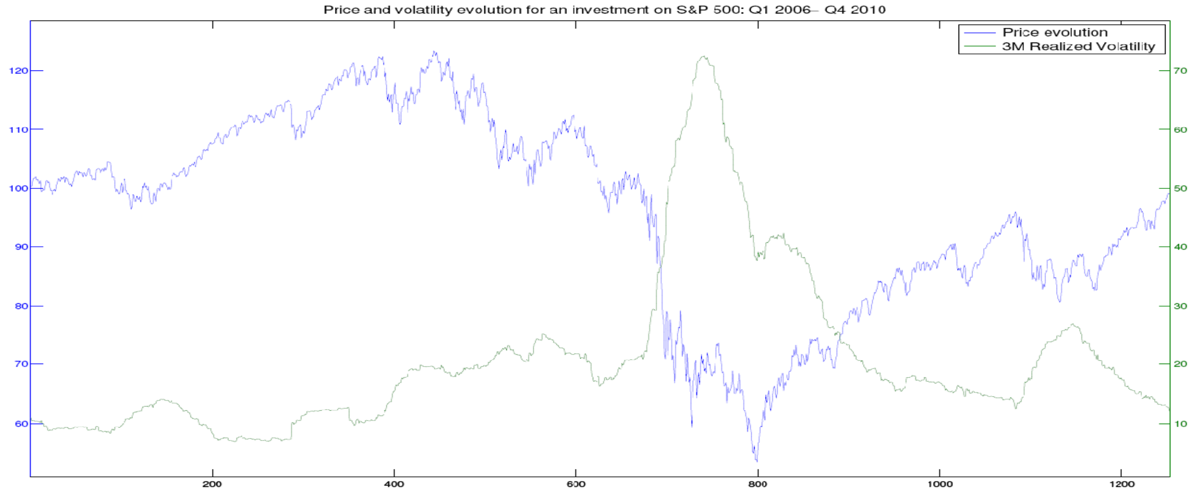

Fig. 1. Evolution of an investment in the S&P500 for the period Q1 2006 to Q4 2010

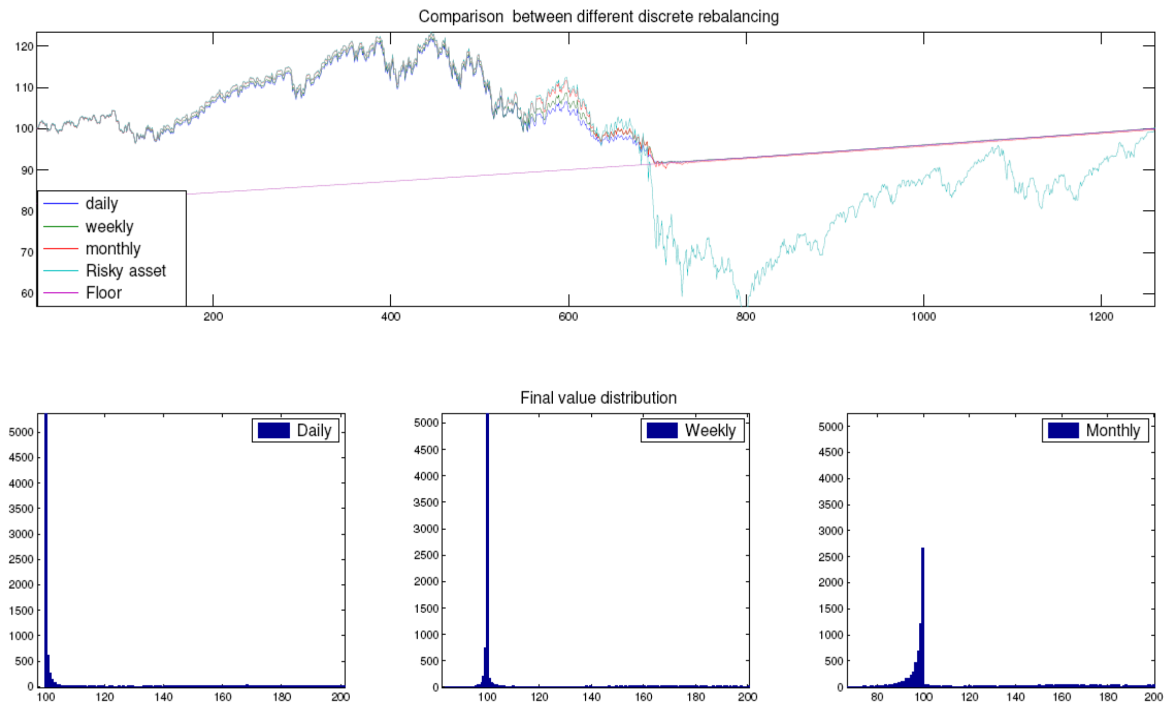

Fig. 2. Backtesting and distribution of the three various rebalancing frequencies under B&S model.

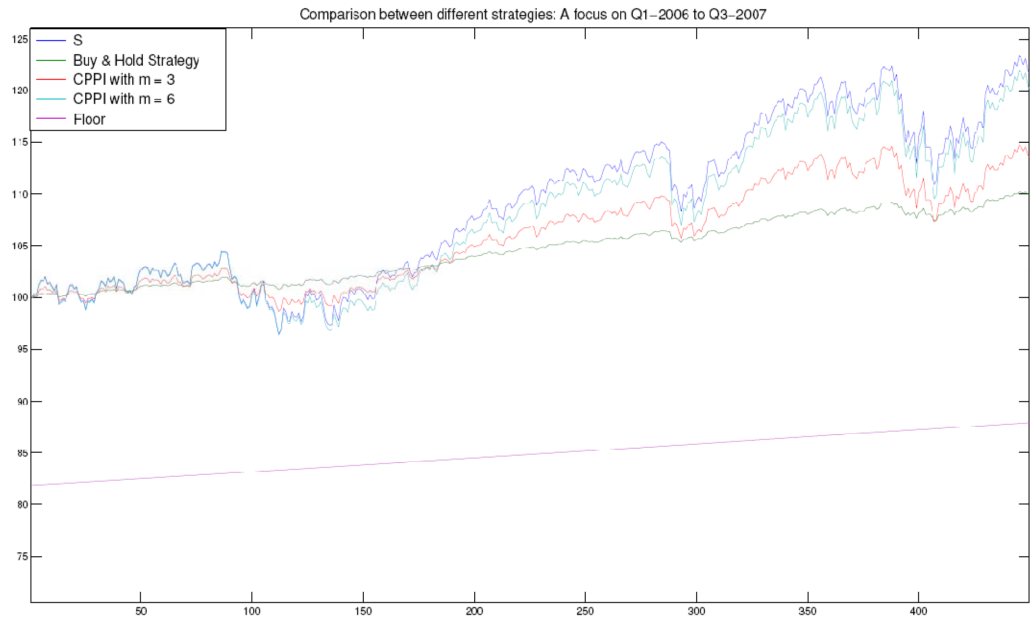

Fig. 3. Comparison between the Buy & Hold strategy, CPPI with *m* = 3 and CPPI with = 6 through backtesting (S&P500)

Fig. 4. Calibration of the Kou model using 1-month maturity call options price on the S&P500

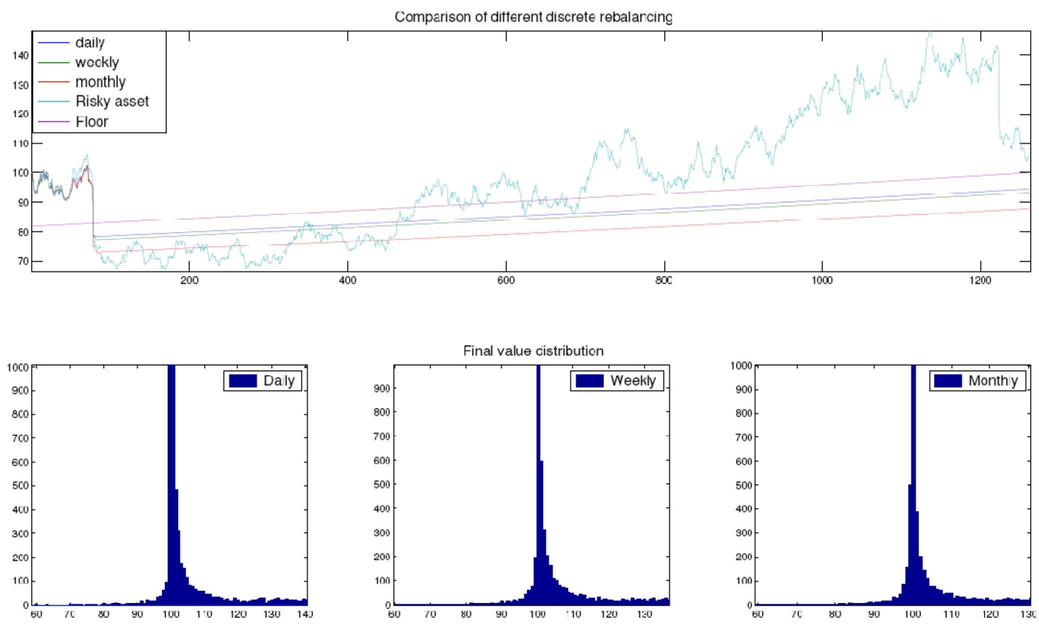

Fig. 5. Simulation and distribution of the three various rebalancing frequencies under Kou model.

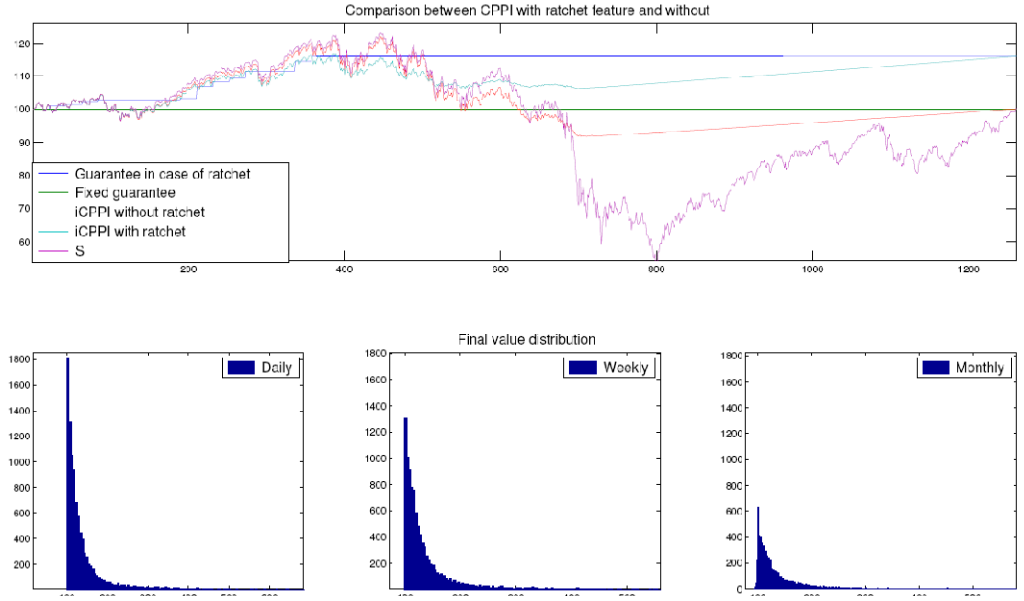

Fig. 6. The figure on the top is a backtesting on the previous set of data to compare a classical iCPPI and one with the ratchet feature. The three histograms on the bottom are those of the final value distribution for the three different rebalancing frequencies on the iCPPI with ratchet

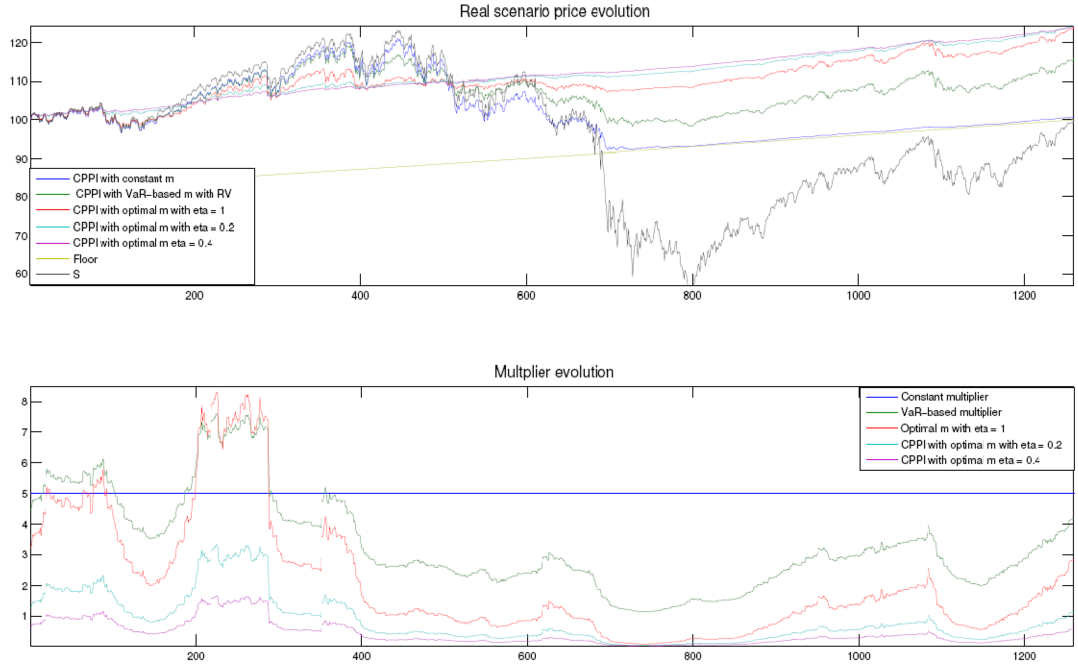

Fig. 7. Comparison between different multipliers (VaR-based with p = 99.5% and the optimal one with risk tolerance η = 0.2, 0.4 and 1) based on Realized Volatility

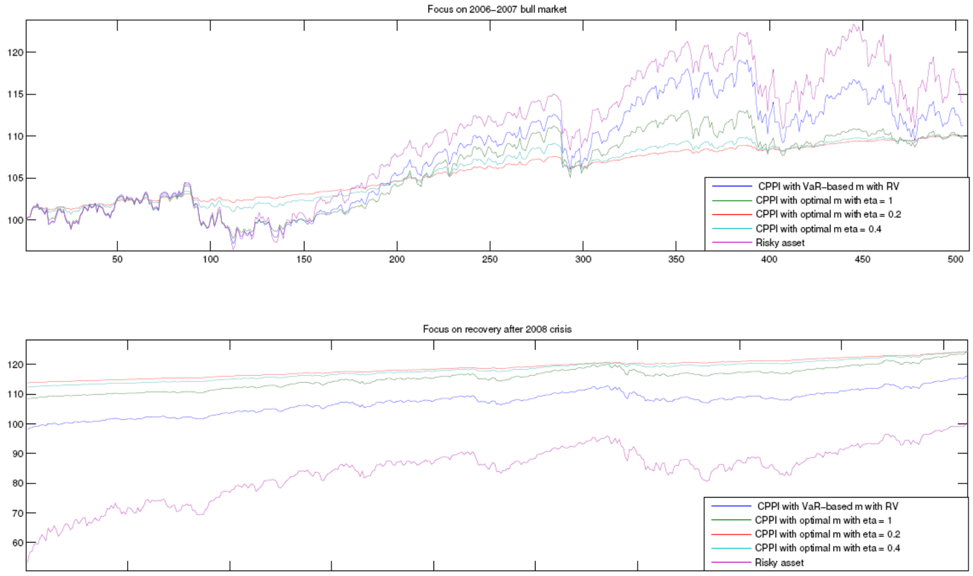

Fig. 8. Focus on two bullish market periods where the CPPI with the VaR-based m performs better than the optimal one

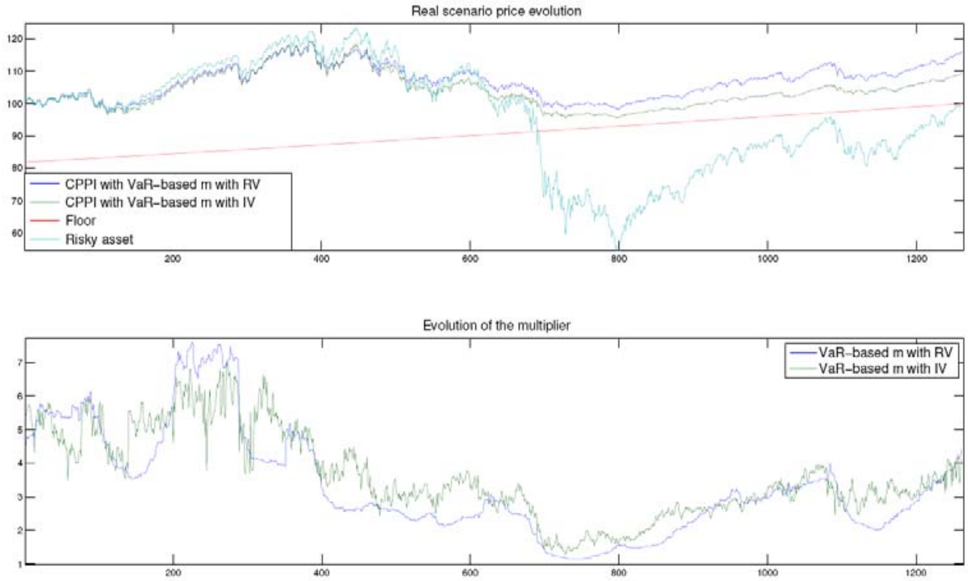

Fig. 9. Comparison between dynamic multiplier based on RV and on IV through backtesting

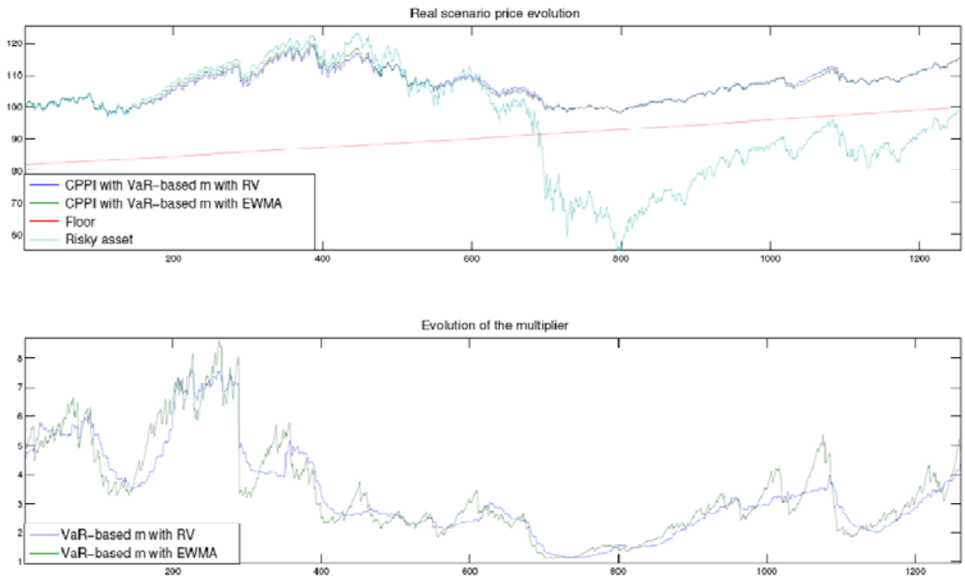

Fig. 10. Comparison between dynamic multiplier based on RV and on EWMA through backtesting

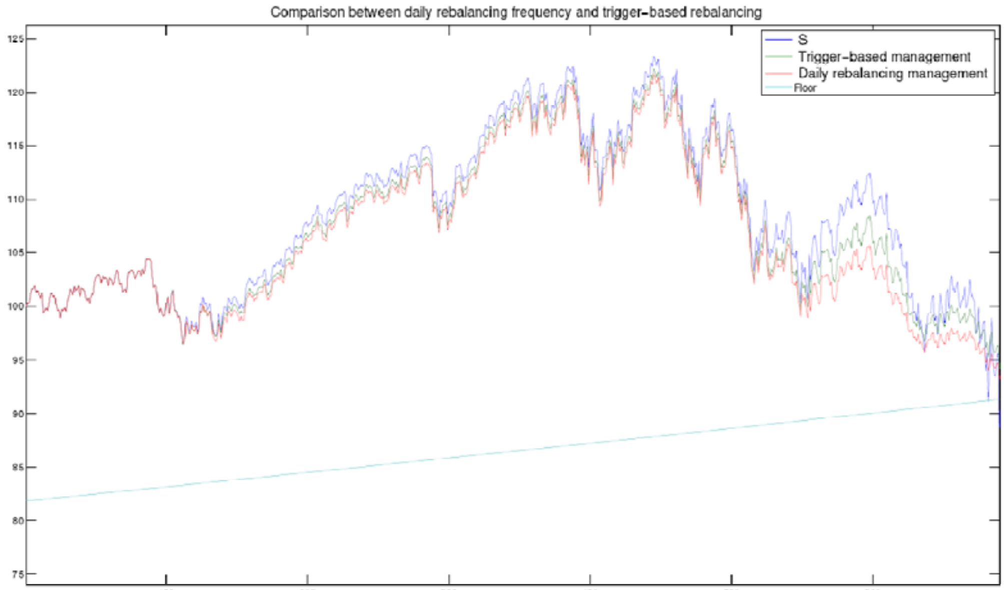

Fig. 11. Comparison between trigger rebalancing vs fixed frequency rebalancing

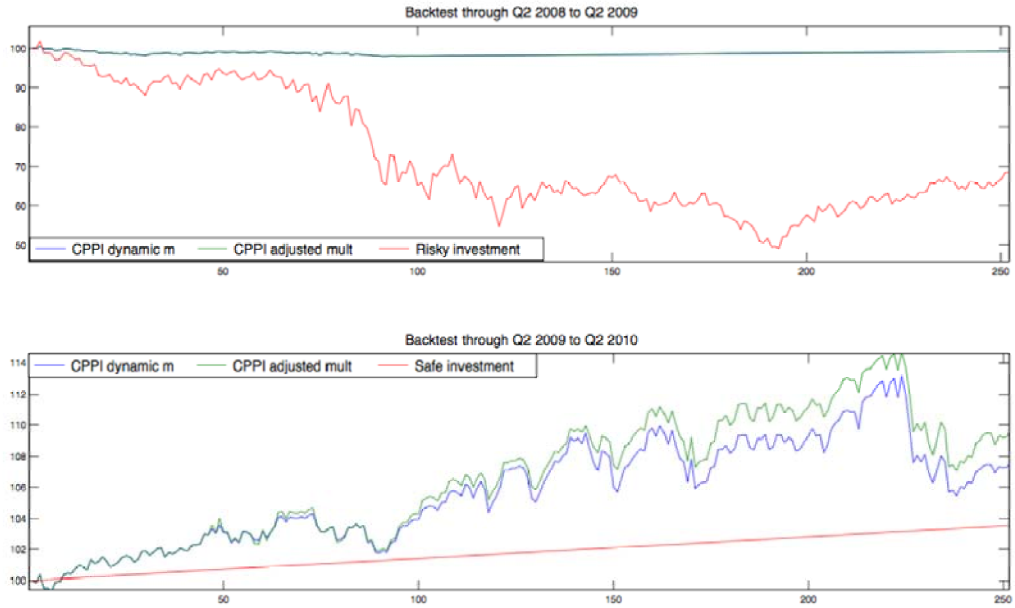

Fig. 12. Comparison between the dynamic multiplier and an adjusted one based on a manager decision depending on market recovery

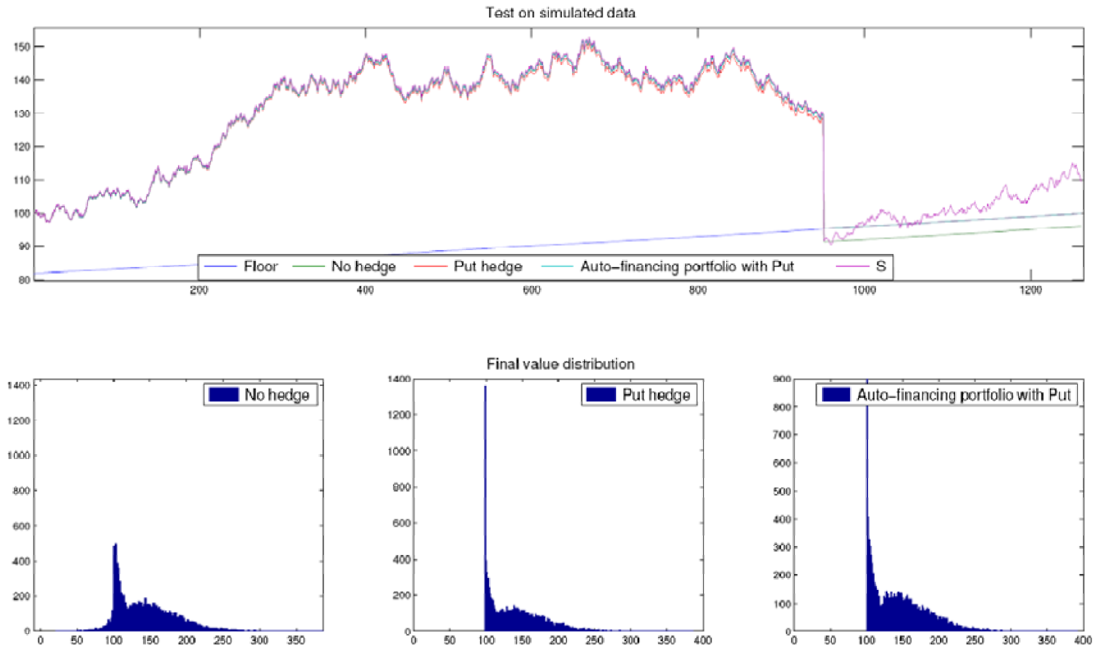

Fig. 13. Comparison between no hedging and put hedging in its two approaches

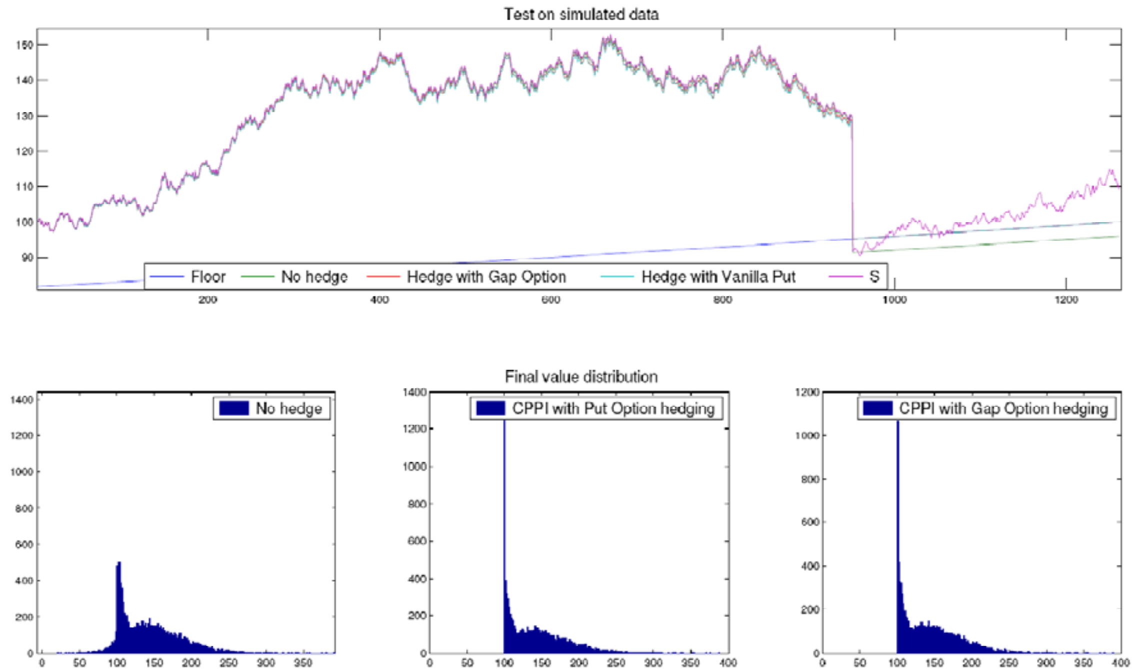

Fig. 14. Comparison between a vanilla and a gap option hedging