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Longevity risk evaluation in last survivor immediate annuities

Abstract

Insurance companies usually price annuities based on a life table with deterministic mortality improvement rates. Great uncertainty in future mortality improvement leads to a controversy over whether the current market has sufficiently allowed for longevity risk in the prices of annuity products. As to last-survivor annuity products, dependence between joint lives makes the situation even more complicated. This research incorporates stochastic Gompertz law to the semi-Markov joint-life mortality model, making a preliminary attempt at the dependent modeling of joint-life longevity risk for the pricing and risk management of annuities for joint lives. The proposed model is then applied to examine the market prices of longevity risk in last-survivor immediate annuities.

Keywords: annuity, longevity, joint lives, dependence, risk premium.

Introduction

During the past half-century, developed countries have witnessed remarkable mortality improvement leading to the growth of older population and increasing life expectancy. A large element of mortality improvement is driven by medical advances. Ongoing support for medical research will continue to lead to further mortality decline (Gallop, 2006). The US National Institute of Health Workshop Report on Aging has estimated that the 65- to 74-year-old age group in the US will be 36 million in 2030 compared with 21.5 million in 2010; the 75- to 84-year-old age group will be 25 million compared with 17 million in 2010; and the 85- to 99-year-old age group will be 5 million, compared with 2.1 million in 2010. Mortality improvement is anticipated in the foreseen future, while such a process is of great uncertainty in terms of the extent and pace of improvement. Uncertainty in mortality improvement puts enormous pressure on retirement funds and annuity insurance funds.

We refer to this higher-than-expected mortality improvement as longevity risk. Given life expectancy for the population as a whole, the idiosyncratic risk that a particular annuitant lives longer than expected could, in principle, be minimized by holding a sufficiently large portfolio of individual policies. Longevity risk is uncertainty about the life expectancy of the population as a whole. It is a systematic risk of the annuity business, having potentially significant impact on the annuity market. Insurance companies and private pension plan providers should incorporate longevity risk in their actuarial calculations. In the language of financial economics there should be a market price for the systematic longevity risk.

Currently, annuity prices in the private annuity market are usually based on the annuity life table projected to the current year and beyond. Deterministic, age specific mortality reduction factors are usually used for mortality projection. US insurance companies generally use the Annuity 2000 Basic Table and Scale G for pricing individual annuities (Doll et al., 2011). UK companies similarly use period mortality tables with some improvement projection methods. The UK Continuous Mortality Investigation Bureau (CMI, for short) used smoothed P-spline estimation of the annual mortality improvement rates in its mortality projection model. While deterministic modeling can provide best-estimate mortality scenarios, it is inadequate for some applications in practice. Stochastic models are better for risk evaluation and management where longevity and mortality risk constitute a significant risks, having the ability to deliver full probability distributions of the quantities of interest and allow us to quantify uncertainties and risks adequately for better risk management.

On the other hand, dependence between joint-life mortality has not been taken into account in the practice of pricing annuity products either. Joint-life longevity risk may be even more complicated than that in single life products, because of the dependence between the future lifetime of a husband and wife. The prices for joint-and-last survivor annuities in the current market are quite inconsistent. For example, prices for last survivor annuities are determined only by the younger age of a husband and wife. This paper is to examine to what extent the annuity market accounts for future improvements in mortality rates when pricing last survivor annuities. For this end, we propose a joint-life longevity risk model, incorporating stochastic mortality dynamics and the dependence between two lives in a couple.

Several stochastic mortality models have been proposed during the past two decades. The Lee-Carter model (Lee and Carter, 1992) and the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2006) are two popular ones among them. The Lee-Carter

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method models central mortality rates by two age-specific factors and a time-dependent factor. The model is valued for its parsimonious structure and easy interpretation. Various extensions and methodological improvements have been explored. Readers are referred to Wilmoth (1993), Brouhns et al. (2002), Renshaw and Haberman (2003 and 2006), Li et al. (2009), Delwarde et al. (2007), etc.

The CBD model forecasts the post-60 mortality using two factors that are measurable with time. The first factor affects mortality-rate dynamics at all ages, and its downward trend indicates general improvements in mortality over time. The second factor affects mortality-rate dynamics as a coefficient of age. Its increasing trend means mortality improvements have been relatively greater at younger old ages. This research makes a preliminary attempt to propose a stochastic joint-life longevity model, by applying dynamic Gompertz law to the semi-Markov joint-life model proposed by Ji et al. (2011). Stochastic Gompertz model mimics the CBD model but gives more flexibility of being applied to the semi-Markov mortality model.

1. It is well recognized that the Gompertz' law well describes adult mortality rates in a simple form.
2. The proposed stochastic Gompertz model has parsimonious and readily interpretable parameters.
3. It is convenient to be incorporated into the semi-Markov joint-life mortality model where bereavement effect is modeled by an exponential decay function applied to the Gompertz mortality law for transition intensities.

Applying Gompertz' law in stochastic modeling of mortality rates is not new. McNown and Rogers (1989) applied a univariate time series approach to the Heligman-Pollard model for forecasting the US mortality. Lockwood (2009) fitted univariate time series models to the parameters of a series of Gompertz-Makeham models of order (r, s), or GM (r, s) models, using CMI male assured lives data and the England and Wales population data from age 30 to 90. These motivate this research to propose a stochastic Gompertz model with time dependent parameters for the transition intensities in the semi-Markov joint-life mortality model.

The remainder of this paper is organized as follows: Section 1 specifies the semi-Markov joint-life longevity model, where transition intensities are stochastically modeled by Gompertz law with time-dependent parameters; the base mortality probabilities for two lives with semi-Markov dependence structure; and the method used for morality forecasting. Section 2 demonstrates the impact on the prices of last survivor annuities of joint-life longevity risk with dependence assumption for two lives. Section 3 compares the implied market prices of longevity risk in last survivor annuities written in the US and UK annuity market, based on the proposed joint-life longevity risk model. Section 4 concludes the paper.

1. A semi-Markov joint-life longevity model

1.1. Model specification. The semi-Markov mortality model for the dependent modeling of joint-life mortality was proposed by Ji et al. (2011), in which the force of mortality after bereavement is modeled as the product of a multiplicative function and the corresponding force of mortality when his/her spouse is still alive. Specifically, the force of mortality for widows, age x, s years after bereavement,

$$\mu^*(x,s) = F^f(s)(\mu_x^f + \lambda), \tag{1}$$

and for widowers, age y, s years after bereavement,

$$\mu^*(y,s) = F^m(s)(\mu_y^m + \lambda), \tag{2}$$

where μ_x^f and μ_y^m represent the force of mortality for married women and men, respectively, from all causes other than common shock, λ is the "common shock" parameter. The multiplicative functions are exponentially decreasing, in forms of $F^f(s) = 1 + a^f e^{-k^f s}$ and $F^m(t) = 1 + a^m e^{-k^m s}$, where $a^m, a^f, k^m, k^f > 0$.

In this paper, the deterministic force of mortality is extended to be time t and age x dependent. For simplicity, λ is assumed to be zero, that is, transitions from "common shock" events are not taken into account. The main reason for this assumption is that there is no historic mortality data for "common shocks". We can hardly calibrate a process for the instantaneous transition. If data permits, time-t dependent or independent common shock transition can easily be incorporated, and will not affect the current setting. Figure 1 specifies the proposed joint-life longevity model.

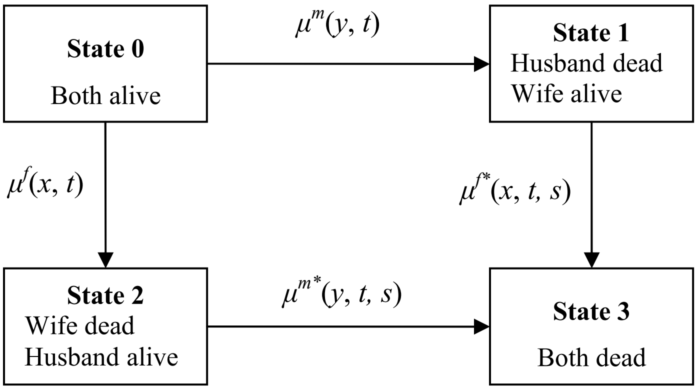

Fig. 1. Specification of the semi-Markov joint-life longevity model

The transition intensities are stochastically modeled and thereafter time-t dependent. Let $\mu^f(x, t)$ and $\mu^m(y, t)$ denote the force of mortality for a x-age wife and a y-age husband at time t respectively, in the married status; $\mu^{f*}(x, t, s)$ denotes the force of mortality for a x-age widow, at time t, s years after bereavement, while $\mu^{m*}(y, t, s)$ is the force of mortality for a y-age widower and widower, at time t and s years after bereavement. That is to say, the post-bereavement mortality rates are driven by three factors: age, chronological time, and time since bereavement. Specifically, the proposed semi-Markov joint-life longevity model will have:

$$\mu^{f*}(x,t,s) = (1 + a^f e^{-k^f s})\mu^f(x,t), \tag{3}$$

and,

$$\mu^{m*}(y,t,s) = (1 + a^m e^{-k^m s})\mu^m(y,t). \tag{4}$$

1.2. Stochastic transition intensities. The force of mortality for a husband and a wife is modeled by the Gompertz formula with two Gompertz parameters being time-t dependent. In a re-parametrization form, Gompertz law models the hazard function of a random lifetime variable for an individual as:

$$\mu_x = \xi \exp\{\xi(x - \gamma)\}, \tag{5}$$

where γ is the mode parameter and ξ denotes the force of mortality at the modal age, that is, $\mu_r = \xi$. The force of mortality at the modal age coincides with the Gompertz ageing parameter. Carriere (1992) showed that, the aging parameter also coincides with the measure of spread about the mode. The inverse of the aging parameter represents the spread of the Gompertz distribution. Allowing the Gompertz parameters to be time-t dependent, we express the forces of mortality $\mu^f(x, t)$ and $\mu^m(x, t)$ for a x-age wife and a y-age husband, at time t, in the married state, mathematically as:

$$\mu^f(x,t) = \xi_t^f \exp\{\xi_t^f(x - \gamma_t^f)\}, \tag{6}$$

and

$$\mu^m(y,t) = \xi_t^m \exp\{\xi_t^m(y - \gamma_t^m)\}, \tag{7}$$

where ξ_t^f and γ_t^f are the Gompertz mode parameter and aging parameter at time t for females, and ξ_t^m and γ_t^m are for males. Their values determine the time-t mortality profile of females and males in the married status.

The Human Mortality Database provides historical data of mortality rates, death counts and population size at detailed levels. We fit Gompertz' law to the US historic population period mortality data for age 60 to 109 during years from 1950 to 2007. Maximum likelihood estimation methodology is used to estimate the parameters of the Gompertz distribution. It is assumed that the number of deaths, which is a counting random variable, follows the Poisson distribution (see, e.g., Wilmoth, 1993; and Brouhns et al., 2002). Figure 2 depicts the estimated values of ξ_t and γ_t from year 1950 to 2007. The mortality improvement has occurred with increasing mode parameter γ_t and increasing aging parameter ξ_t. An increasing γ_t causes the left shift of the lifetime distribution. An increasing ξ_t indicates of the concentration about the modal age of the lifetime distribution.

We also fit Gompertz law to the England and Wales population data from year 1950 to 2009, a similar period as the US example. Figure 3 depicts the estimated values of ξ_t and γ_t. Generally increasing mode parameter γ_t and increasing aging parameter ξ_t indicate a trend of declining mortality; while, the estimated volatilities of the stochastic process for the UK mortality might be greater than the volatilities for the US mortality.

A vector stochastic process is thereafter proposed to model the Gompertz parameters ξ_t and γ_t, modeling their trends, in a correlated form, for the stochastic forecasting of $\mu^f(x, t)$ and $\mu^m(x, t)$. Denote $z^f(t)$ and $z^m(t)$

to be the vector stochastic process for the Gompertz parameters for females and males respectively, that is, $z^f(t) = [\xi_t^f, \gamma_t^f]'$ and $z^m(t) = [\xi_t^m, \gamma_t^m]'$. Define the drift vector ν^f and ν^m for females and males respectively by

$$\nu^f = \begin{pmatrix} \nu_1^f \\ \nu_2^f \end{pmatrix} \quad \text{and} \quad \nu^m = \begin{pmatrix} \nu_1^m \\ \nu_2^m \end{pmatrix},$$

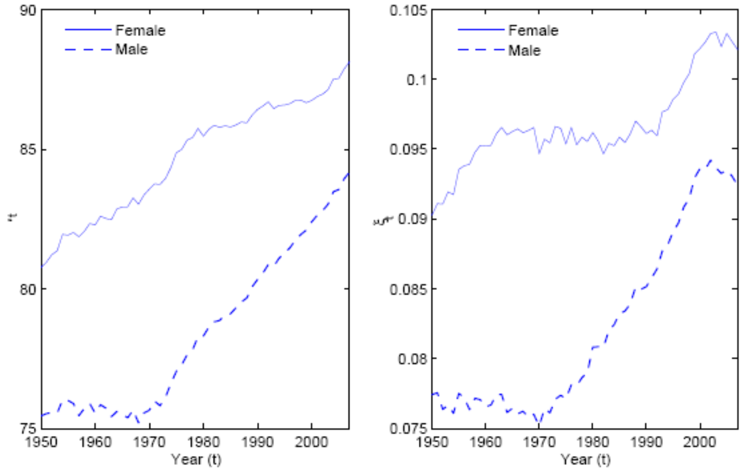

Fig. 2. Estimated values of ξ_t and γ_t for the US historic mortality data from 1950 to 2007

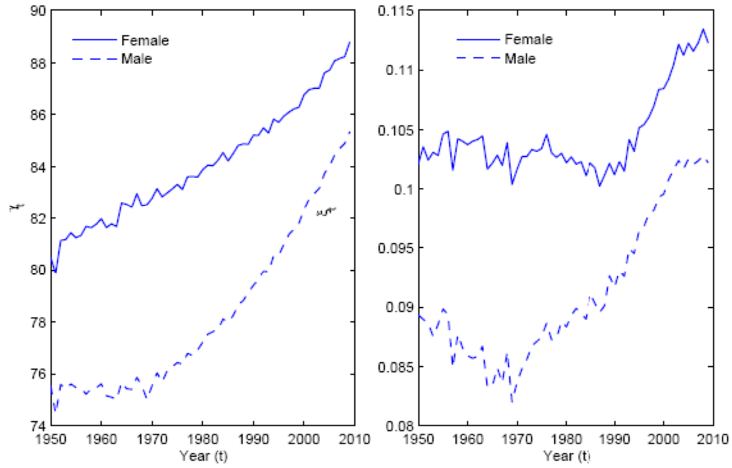

Fig. 3. Estimated values of ξ_t and γ_t for the England and Wales population mortality data from 1950 to 2009

the 2×2 – dimensional covariance matrix V^f and V^m for females and males respectively by

$$V^f = \sigma^f \sigma^{f'} = \begin{pmatrix} V_{11}^f & V_{12}^f \\ V_{21}^f & V_{22}^f \end{pmatrix} \quad \text{and} \quad V^f = \sigma^m \sigma^{m'} = \begin{pmatrix} V_{11}^m & V_{12}^m \\ V_{21}^m & V_{22}^m \end{pmatrix},$$

and the 2-dimensional standard normal random variable by $Z(t)$. Symbol " ′ " means the transpose of a matrix.

In discrete time, the vector processes $z^f(t)$ and $z^m(t)$ are modeled as a two-dimensional random walk with drift.

Specifically:

$$z^f(t+1) = z^f(t) + \nu^f + \sigma^f Z(t+1), \tag{8}$$

and

$$z^m(t+1) = z^m(t) + \nu^m + \sigma^m Z(t+1). \tag{9}$$

The choice of diffusion matrix σ^f and σ^m is not unique but will not make any difference to our analysis, as stated in Cairns et al. (2006). Following Cairns et al. (2006), $\sigma^{f(m)}$ is chosen to be the Cholesky decomposition of $V^{f(m)}$.

Fitting the vector processes $z^{f(t)}$ and $z^{m(t)}$ to the estimated historic Gompertz parameters from the US population data for 60 to 109 ages during year 1950 to 2007 (58 observations), we have:

$$v^f = \begin{pmatrix} 0.1292 \\ 2.083\times10^{-4} \end{pmatrix},\ V^f = \sigma^f \sigma^{f'} = \begin{pmatrix} 0.0324 & -7.2589\times10^{-5} \\ -7.2589\times10^{-5} & 5.3877\times10^{-7} \end{pmatrix},$$

for females, and

$$v^m = \begin{pmatrix} 0.1533 \\ 2.632\times10^{-4} \end{pmatrix},\ V^m = \sigma^m \sigma^{m'} = \begin{pmatrix} 0.0475 & 1.6330\times10^{-5} \\ 1.6330\times10^{-5} & 4.7453\times10^{-7} \end{pmatrix},$$

for males.

Similarly, fitting the vector processes $z^f(t)$ and $z^m(t)$ to the estimated historic Gompertz parameters from the England and Wales historic population mortality data for 60 to 109 ages from year 1950 to 2009 (60 observations), we get:

$$v^f = \begin{pmatrix} 0.1407 \\ 1.7225\times10^{-4} \end{pmatrix},\ V^f = \sigma^f \sigma^{f'} = \begin{pmatrix} 0.0945 & -2.3514\times10^{-4} \\ -2.3514\times10^{-4} & 1.587\times10^{-6} \end{pmatrix},$$

for females, and

$$v^m = \begin{pmatrix} 0.1659 \\ 2.1766\times10^{-4} \end{pmatrix},\ V^m = \sigma^m \sigma^{m'} = \begin{pmatrix} 0.1137 & 8.6000\times10^{-6} \\ 8.6000\times10^{-6} & 2.0164\times10^{-6} \end{pmatrix},$$

for males.

The estimated results indicate some information about the trend of mortality during the past half-century. Firstly, there are upward trends in the Gompertz modal parameter and aging parameter, which means that human lifetime distribution is increasingly concentrated around the increasing modal age.

Secondly, the pace of mortality improvement is faster but more volatile for males than for females. However, this feature comes from a short period of data. The conclusion may be reversed in a long run of human mortality evolvement.

Finally, using the historic mortality data as of 1950, the correlation between the two Gompertz parameters is negative for female mortality but positive for male mortality. When we fit the stochastic processes to different period of data, the sign of correlation changes for males. This might be due to the high volatility in male mortality evolvement, or it might be the stochastic Gompertz model is not robust to the data. We acknowledge that the validity and robustness of the model needs to be tested; however, we would like to leave it to future work, for not been distracted from the main purpose of this study.

Other mortality forecasting models, like the Lee-Carter model or other more complicated stochastic parameter models, may also play the role. When more data become available in the future, further research work is expected in the area of joint-life mortality forecasting, choosing suitable models and testing the robustness of each method in the joint lives context.

1.3. Base rates of mortality. This research is to examine joint-life longevity risk. Base rates of mortality should in principle be related to the mortality experience of annuitants. Therefore, the currently used annuity life tables in the US and UK annuity market are used as base tables. A life table gives mortality probabilities at each age for an individual. It is the aggregate mortality for an individual in the status of being married, single, divorced, or widowed with any period of time after bereavement.

Using the semi-Markov joint-life model, without mortality projection at this stage, we can derive marginal mortality probabilities for the married and the widowed. Let h_x be the percentage of population in the married status, and g_x be the proportion in the widowed status; $1 - h_x - g_x$ is for the others, whose mortality rates are assumed to be the same as the aggregate rates. Then, the aggregate mortality rate is approximately represented by the equation:

$$\mu_x^{aggregate} = \frac{h_x}{h_x + g_x}\mu_x^{married} + \frac{g_x}{h_x + g_x}\mu_x^{widowed}.$$

Borrowing information from relevant census study on age-specific marital status, we can approximately estimate the parameters for the semi-Markov mortality model, by equating the approximately mixed single-life mortality probabilities to those in the referred life table for both males and females.

Meanwhile, we can also estimate the values of the Gompertz parameters for the aggregate mortality probabilities in the referred life table for both males and females.

US insurance companies generally use the Annuity 2000 Basic Mortality Table (A2000, for short) as the base mortality for annuity pricing. Using the marital status of the population in 2000 studied by Kreider and Simmons (2003), and the individual mortality rates in A2000, we estimate the parameters for the base rates of mortality in the semi-Markov joint-life longevity model, by fitting the mixed marginal mortality distribution from the semi-Markov joint-life mortality model to the mortality probabilities at ages beyond 59 in the A2000 life table. The fitting approach is based on a least squares minimization. Table 1 summarizes the estimated parameters for the semi-Markov joint-life mortality model and individual single-life Gompertz mortality model.

Table 1. Parameter values for base mortality in the semi-Markov joint-life longevity model and individual single-life mortality model, for the US

The semi-Markov model		Single-life model	
Parameters	Values	Parameters	Values
γ^f	91.9541	$\gamma^{f,\,Inde}$	90.4699
ξ^f	0.1096	$\xi^{f,\,Inde}$	0.1138
γ^m	89.1318	$\gamma^{m,\,Inde}$	87.1418
ξ^m	0.0870	$\xi^{m,\,Inde}$	0.0972
a^f	3.7804		
k^f	0.3901		
a^m	10.4253		
k^m	0.7754		

The parameters displayed in Table 1 indicate several important results. Firstly, the force of mortality in the married status is generally lower than the marginal, or independent, force of mortality of the same age, which represents the combined rate of mortality of the married and the widowed. Secondly, the effect of bereavement will increase the force of mortality after bereavement by a higher level for males than for females, however males recover from bereavement faster than females. This estimated result is consistent to the result in Ji et al. (2011).

Similarly, using the population marital status information provided by the UK Government Actuary's Department and the UK CMI Series 00 Immediate Annuity Life tables, we can estimate the parameters for the base mortality rates in the semi-Markov joint-life longevity mortality model and single-life Gompertz model applied to the UK annuitants. Table 2 summarizes the estimated parameters, which are based on the mortality rates at ages beyond 59 in the CMI Series 00 Immediate Annuity Life tables.

Table 2. Parameter values for base mortality in the semi-Markov joint-life longevity model and individual single-life mortality model, for the UK

The semi-Markov model		Single-life model	
Parameters	Values	Parameters	Values
γ^f	92.8059	$\gamma^{f,\,Inde}$	90.0127
ξ^f	0.1296	$\xi^{f,\,Inde}$	0.1383
γ^m	88.7920	$\gamma^{m,\,Inde}$	86.6564
ξ^m	0.1044	$\xi^{m,\,Inde}$	0.1202
a^f	8.4790		
k^f	0.3921		
a^m	8.6868		
k^m	0.3603		

The estimated parameter values for the UK are slightly different from the values fitted for the US. The difference lies in the parameters for the semi-Markov property, that is, the selection effect of bereavement. From the values for the UK, males and females are subject to a nearly same broken heart effect shortly after bereavement, and they recover from bereavement at a similar speed. Here, we just state the results from the data. The reasons underlying this difference between the US and UK are beyond the scope of our study.

1.4. Joint-life mortality projection. Based on the proposed model, we can stochastically project the mortality rates in the married state and the marginal, independent, or aggregate mortality rates at the same time, for females and males respectively. In this way, we stochastically model time-t dependent forces of mortality $\mu^f(x,t)$ and $\mu^m(y,t)$, and the corresponding time-t dependent aggregate forces of mortality. The select effect of bereavement, which may evolve with time, is implied by a set of forecasted time-t mortality rates.

The probability that the last survivor of a currently x-age wife and y-age husband at time t_0 = 2011 will survive t years from now can be computed as:

$${}_t p_{\overline{xy}}(t_0) = {}_t p_x^f(t_0) + {}_t p_y^m(t_0) - {}_t p_{x:y}^{00}(t_0), \tag{10}$$

where

$${}_t p_x^f(t_0) = \prod_{j=0}^{t-1} \exp\left\{-\int_0^1 \left(\mu^{f,\,Inde}(x+j+s, t_0+j)\right)ds\right\},$$

$$ {}_t p_y^m(t_0) = \prod_{j=0}^{t-1} \exp\left\{ -\int_0^1 \left(\mu^{m,\,Inde}(y+j+s, t_0+j) \right) ds \right\}, $$

$$ {}_t p_{x:y}^{00}(t_0) = \prod_{j=0}^{t-1} \exp\left\{ -\int_0^1 \left(\mu^f(x+j+s, t_0+j) + \mu^m(y+j+s, t_0+j) \right) ds \right\}. $$

This approach calls for less computer resources in joint-life mortality projection. Alternatively, we can only stochastically model time-t forces of mortality $\mu^f(x, t)$ and $\mu^m(y, t)$ in the married state, and assume the two exponentially decreasing multiplicative functions for the select effect of bereavement keep unchanged with time, that is, the dependence structure is time-invariant.

When projecting mortality in the married status and the aggregate mortality, we assume that their Gompertz parameters, for a husband and a wife respectively, follow the same vector random walk process. It may be argued that mortality in the married status and the marginal (or aggregate) mortality may evolve differently. At the current stage, we have no historic mortality data for married couples to support or test this argument. If better data become available in the future, we can explore this topic further.

Meanwhile, forces of mortality are projected by the vector random walk process that has been fitted using the population mortality database. It is acknowledged that this database is not perfect for calibrating the vector stochastic processes for annuitants' mortality improvement. However, without more suitable data, we use the population data to calibrate the process, at least approximately. Furthermore, correlation between the improvements of mortality for males and females has not been examined. We leave it to future work.

2. Implication for last survivor annuity values

Joint and last survivor annuities are typical products associated with joint-life longevity risk in the current annuity market. They provide benefits to the annuitant and his/her spouse until both of them have passed away. These products are not only offered by insurance companies but are also an important benefit to pension plan retirees. If annuitants live longer than expected because of unexpected mortality improvement, the financial soundness of annuity portfolios could be at risk.

In this section, we examine how stochastic joint-life mortality improvement will affect the cost of a last survivor annuity using the proposed methodology. The impact of joint-life longevity risk can be measured by the increase in the cost of a last survivor annuity due to the allowance of mortality improvement compared with the corresponding cost when mortality improvement is not allowed.

The quantity of interest, the cost of a last survivor annuity, is a non-linear function of the fitted Gompertz parameters in the current year and the parameters of the vector stochastic Gompertz processes. It is not possible to analytically derive the distribution of the last survivor annuity net premiums and relevant confidence intervals. Monte-Carlo simulation is used here.

2.1. Simulation method. We simulate the realization of future Gompertz parameters for 5,000 times. From each realization of future Gompertz parameters, we have a surface of mortality for individuals and married couples. The present value of annuity payments can be estimated for each realization of future mortality surface. Specifically, the cost of a last survivor annuity is simulated in the following steps:

1. Simulate N trajectories of the Gompertz parameters for individual mortality and joint survival mortality from the year, in which the base mortality is applied to, to the current year and beyond. Each trajectory is simulated based on the Gompertz parameters for the base mortality and the calibrated vector stochastic processes from the historic population mortality data.
2. From each trajectory, compute individual survival probabilities and joint survival probabilities, and estimate the expected present value of annuity payments.
3. From step (2), get an empirical distribution of the cost of a last survivor annuity.

In calibrating the stochastic processes for the Gompertz parameters, there are generally two sources of parameter uncertainty: sampling errors in the historic Gompertz parameters estimated from the Poisson models, and parameter uncertainty in calibrating the stochastic mortality models to the historic Gompertz parameters. A parametric bootstrap simulation technique can be used to allow for parameter uncertainties.

We conduct two simulation methods, with and without allowance for parameter uncertainty. Allowance for parameter uncertainty in the model is more time-consuming but does not make much difference in the simulated annuity values. So, we do not use a parametric bootstrap in the simulation and ignore the trivial impact of parameter uncertainty.

2.2. The results. Allowance for mortality improvement will increase the expected present value of annuity payments. We use a last survivor annuity issued to a 65 year old husband and a 65 year old wife, with annual payments of \$1 (or £1) paid monthly in advance, as an example to illustrate the extent of increase in the annuity value due to mortality improvement. Annuities with more frequent payments in a year are approximately calculated using the Woolhouse's formula, which is discussed in Chapter 5 of Dickson et al. (2009). The interest rate is assumed to be 4.25%, which is an approximate average interest rate on the 20-year US treasure bills and 20-year UK government bonds during April 2011.

Based on the proposed semi-Markov joint-life longevity model, we can simulate a surface of survival probabilities for last survival status. For each scenario, we compute the cost of the annuity with and without allowance for mortality improvement. With no allowance for future mortality improvement, the Gompertz parameters in the future will be the same as the parameters in the current year ($t_0 = 2011$).

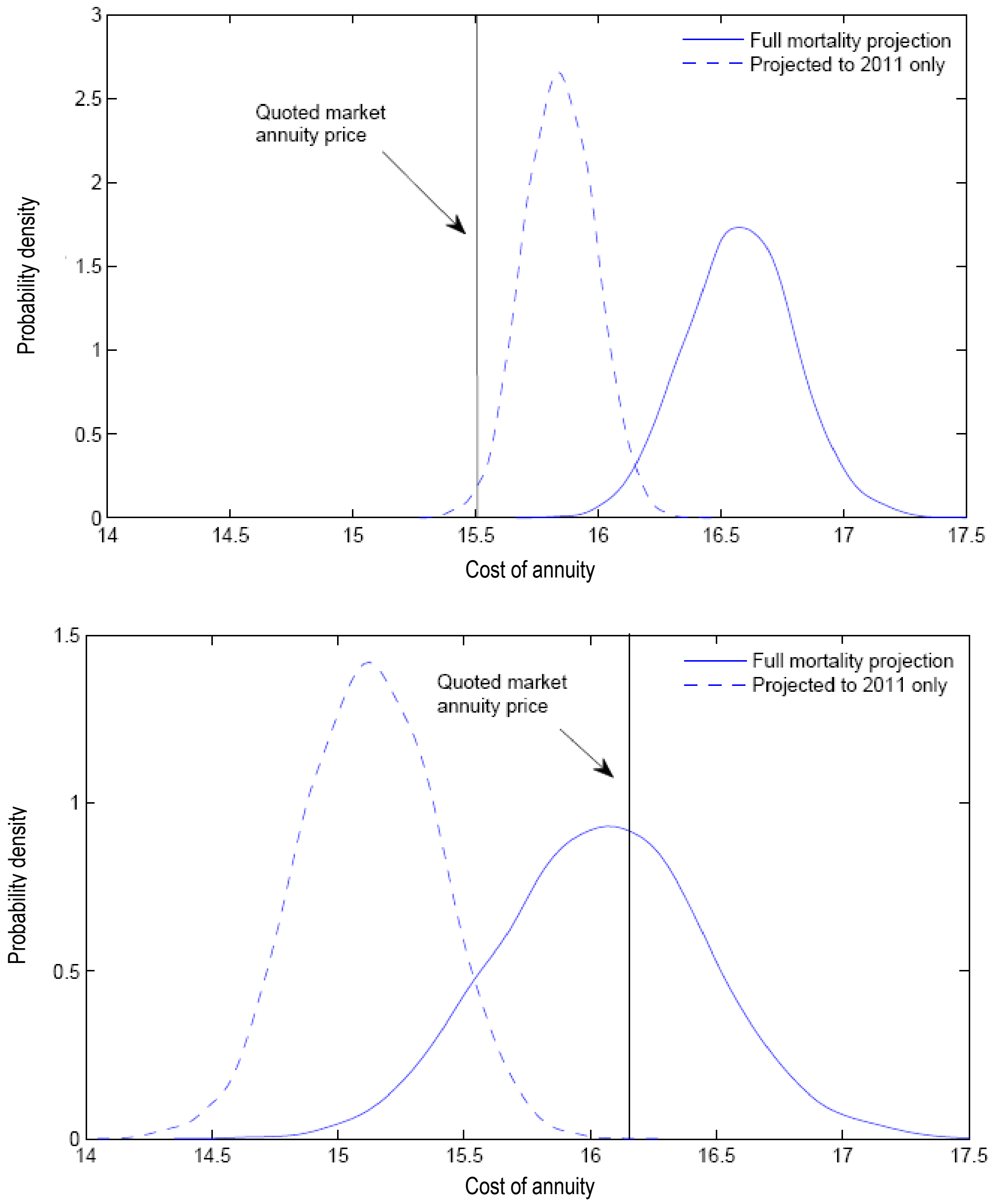

Fig. 4. Distribution of the cost of last survivor annuity in year 2011 with and without allowance for future mortality improvement, for the US market (Top) and the UK market (bottom), female age $x = 65$ and male age $y = 65$, interest rate 4.25%

We depict in Figure 4 the simulated cost of the last survivor annuity in the current year 2011 with and without allowance for future mortality improvement, for US market (Top) and the UK market (Bottom) respectively. The simulated empirical distribution is smoothed using a kernel density estimation method. The quoted market price and simulated cost is for an annuity per unit annual benefit paid monthly in advance. No allowance for future mortality improvement means the base rates of mortality have been projected to the current year only.

The information from Figure 4 can be summarized as follows. Firstly, allowance for mortality improvement dramatically increases the cost of last survivor annuities. Systematic longevity risk has significant impact on the annuity cost. For pricing annuities, a mortality projection model is critical. In fact, this is a fundamental problem in the annuity market.

Secondly, the modeled joint-life longevity risk from the proposed semi-Markov model is more significant in the US market than in the UK market. From the US model, there is a very short overlap between the distribution of simulated annuity value with and without allowance for mortality improvement.

Thirdly, based on the same interest rate, the annuity value in the UK market may generally be lower than the value in the US market, while the simulated annuity value is more volatile in the UK market. The underlying reason for the annuity volatility is the more volatile mortality improvement calibrated from the historic England and Wales population mortality data.

Finally, the quoted annuity rate in the US market is lower than the simulated annuity rates, which are based on a risk-free interest rate. We would expect that the market annuity rate should be higher than the average of the simulated annuity rate with full mortality projection, if the market sufficiently allows for mortality improvement in their pricing. It appears that, the US annuities are underestimated. The underpricing problem is less in the UK market. In the next section, the proposed method is applied to identify how joint-life longevity risk has been taken into account in the practice of pricing last survivor annuities.

3. The market prices of longevity risk

3.1. Pricing method. In the recent literature, several pricing methods have been developed for pricing the longevity/mortality risk. Cairns et al. (2006), Dahl and Møller (2006), and Dahl et al. (2008) use a risk-neutral pricing theory; Wang (1996, 2000, 2001, 2002) has developed a method that uses a one-factor risk distortion operator to drive a risk-distorted measure for universally pricing financial and insurance. Lin and Cox (2005) and Denuit et al. (2007) have applied the Wang transform to pricing mortality risk. Other methods include the utility maximization principle, the principle of equivalent utility, and the Sharpe ratio approach. Chen et al. (2010) investigated connections and differences among the risk-neutral method, the Wang transform and the Sharpe ratio rule. They stated that the Wang Transform is stable for large probabilities whereas it is highly unstable for small probabilities, and robustness of the Wang transform becomes worse as the maturity becomes longer. In addition, the Wang transform is unable to deliver a risk-adjusted dynamics. Readers are referred to Chen et al. (2010) and references therein for a review of these methods.

Risk neutral pricing theory is well established. Financial economic theory states that, if the market is arbitrage-free, there exists a risk-neutral measure such that the price of an asset equals the expected discounted payments under the risk-neutral measure. If the market is complete, there exists a unique risk-neutral measure, while, in an incomplete market many risk-neutral risk measures might exist. As pointed out in Cairns et al. (2006), we are far from having a complete market in which all contingent claims can be replicated by self-financed portfolio. There is no liquid market for systematic longevity risk. A further assumption is needed, that is, market players act in an equilibrium setting and this equilibrium selects a market consistent risk-neutral measure.

In this research, we follow the method proposed in Cairns et al. (2006) to define such a market-consistent Q-measure. In their method, the risk-adjusted pricing measure $Q(\eta)$ is modeled using an adjustment to the dynamics of the stochastic process of mortality rates. Specifically, under the risk-neutral measure $Q(\eta)$,

$$z^{f(m)}(t+1) = z^{f(m)}(t) + \nu^{f(m)} + \sigma^{f(m)}(\tilde{Z}(t+1) + \eta^{f(m)})$$

$$= z^{f(m)}(t) + \tilde{\nu}^{f(m)} + \sigma^{f(m)}\tilde{Z}(t+1),$$

where $\tilde{\nu}^{f(m)} = \nu^{f(m)} + \sigma^{f(m)}\eta^{f(m)}$. $\tilde{Z}(t+1)$ is a standard two dimensional normal random variable under Q-measure.

The vector $\eta^{f(m)}$ is the market prices of longevity risk associated with the stochastic processes for Gompertz parameters $\gamma_t^{f(m)}$ and $\xi_t^{f(m)}$. $\eta_1^{f(m)}$ is the market price of longevity risk associated the stochastic process of the Gompertz modal parameter, γ_t, representing left shift in mortality distribution; while $\eta_2^{f(m)}$ is the market price of longevity risk associated with the stochastic process of the Gompertz aging parameter, ξ_t, representing dispersion in mortality distribution. The implied risk premium for systematic longevity risk can be derived if the market prices of annuities reflect the uncertainty of longevity risk.

Let us denote $P(s, \tau)$ to be the price of a zero-coupon bond issued at time s, which pays one dollar at maturity time τ ($\tau \geq s$). Define $\delta(t)$ to be the risk-free interest rate at time t. In the risk neutral measure Q,

$$P(s,\tau) = E_Q[\exp(-\int_s^\tau \delta(t)dt) \mid \mathcal{F}_s],$$

where $\{\mathcal{F}_s, s = 0,1,\ldots\}$ is the natural filtration for the process.

Assuming that the longevity and interest rate risks are independent, the cost of an annuity is the present value of contingent payments, discounted by the risk-free interest rate, using the Q-measure. Using risk-neutral survival probabilities, we then derive the price of last survivor immediate annuity issued to a y-year old husband and x-year old wife by the following equation:

$$\ddot{a}_{\overline{xy}}^{market}(2011) = 1 + \sum_{\tau \geq 1} P(0,\tau) E_{Q(\eta^f, \eta^m)}[{}_\tau p_{\overline{xy}} \mid \mathcal{G}_0], \quad (11)$$

where $\ddot{a}_{\overline{xy}}^{market}(2011)$ is the market price of a last survivor immediate annuity with \$1 per year paid in advance in year 2011. For annuities with more frequent payments in a year, we approximate $\ddot{a}^{(m)}$, where payment is made 1/m-thly, using the Woolhouse's formula.

3.2. The US market. We use the prices of last survivor immediate annuities to derive the market price of joint-life longevity risk, since the market of immediate annuities is larger and more transparent than the market of deferred annuities. The risk-free interest rate is assumed to be constant and equal to 4.25%, which is the average interest rate on the US 20-year Treasure bill in April 2011. The prices of immediate annuities in the US market at the same time are quoted from the ImmediateAnnuity.com[1]. The quoted prices are for annuities per \$1 annual benefit paid in advance, in monthly instalment. The ImmediateAnnuity.com claims that the quoted price from its web site is close to the lowest price in the current market. We assume the quoted prices are net of expense.

Theoretically, if the market is consistent, there will be unique market prices of longevity risk. However, the market for annuities is not consistently priced. We actually derive a series of the market prices of joint-life longevity risk using the quoted annuity prices for different age combinations.

For the convenience of comparison, we assume $\eta_1^f = \eta_2^f$ and $\eta_1^m = \eta_2^m$. That is interpreted as assuming that the market prices of the two elements of risk are same. Our aim is more to demonstrate to what extent the market prices of joint-life longevity risk are reflected in the current market prices of last survivor annuities, than to calculate the exact values of market prices of risk η_1 and η_2, which would require more data and more assumptions.

We quote a series of market prices of last survivor annuities for \$1 paid monthly in advance, without guarantee. From the quoted prices, we know that, in the US market, the price of a last survivor annuity depends on the age of younger annuitant only. The age of elder annuitant will not change the quoted prices. This phenomenon is not actuarially sound. Annuity payments to a 65-year-old husband and 65-year-old wife are expected to be greater than the payments to a 75-year-old husband and 65-year-old wife. However, the quoted immediate annuity prices for these two couples are same according to the current pricing practice.

Table 3. Market prices of joint-life longevity risk in the US market, η^f and η^m, calibrated from the quoted market prices of immediate last survivor annuities with equal inception ages from 65 to 75

Female age	Male age	Market annuity price	η^f	η^m	$\eta^f = \eta^m$
65	65	15.56	-0.1714	-0.8509	-0.4154
66	66	15.23	-0.1238	-0.9116	-0.4096
67	67	14.90	0.0038	-0.9323	-0.3996
68	68	14.81	-10.1144	1.0747	-0.2296
69	69	14.38	0.2181	-1.3702	-0.2682
70	70	14.06	-2.4832	0.7675	-0.2377
71	71	13.65	-0.2085	-0.3234	-0.2520
72	72	13.26	0.3759	-1.5999	-0.2530
73	73	13.16	0.5207	-1.6741	-0.0668
74	74	12.77	0.5484	-1.7395	-0.0532
75	75	12.41			-0.0241

Table 3 displays the quoted prices of last survivor annuities where the age of the female, who is younger, ranges from 65 to 75. According to the current market pricing practice, the quoted annuity price for each age combination applies to all the cases that the female is at the specified age and her spouse is the same age or older. For calibrating the market prices of longevity risk, we assume that the husband and wife are the same age. Each pair of values of η^f and η^m for an age combination is estimated using the quoted annuity values for that age combination and the next one.

The average value of the market price of longevity risk is -1.1434 for females and is -0.7560 for males. The market price of risk for each element of longevity risk, γ_t and ξ_t, is negative for both female and male mortality. It means that the market's view about the left shift in future lifetime distribution is smaller than modeled by the proposed joint-life longevity model; the marked is also less worried about the concentration of future lifetime distribution about the modal age than modeled. This

[1] Available at: http://www.immediateannuities.com/

result may indicate an underpricing problem with last survivor annuities in the US market.

The estimated values of η^f and η^m fluctuate dramatically. In addition, it seems that the value of one parameter is reflecting the value of the other. It may be due to that the stochastic processes for female and male mortality are uncorrelated. An increase in the market price of risk for the elements of longevity risk in female mortality rates leading to a decrease in the price for the elements of longevity risk in male mortality rates, and vice versa. A model that allows for correlation between the future mortality improvements in female and male mortality rates may give more reasonable results than the current setting.

From the modeling results, we believe that the US annuity market underprices last survivor annuities. We acknowledge that the constraints $\eta_1^f = \eta_2^f = \eta^f$ and $\eta_1^m = \eta_2^m = \eta^m$ will not exactly reflect the market's view about longevity risk. Dramatic fluctuation in η^f and η^m makes it hard to tell a general level of the market's view about longevity risk.

We further assume $\eta^f = \eta^m$ to determine a general extent of the underpricing. Negative market prices of longevity risk at all ages indicate that the market underestimates longevity risk in last survivor annuities. The extent of underpricing is more severe for younger old annuitants. This may be due to cross subsidy or natural hedge between younger annuitants and older annuitants in pricing.

From the estimated prices in the last column in Table 3, the average level of the market price of longevity risk is -0.2372. It appears that last survivor annuities are underpriced according to our joint-life longevity model. The market does not allow adequately for longevity risk is supported here in the case of last survivor annuities.

The market is aggressive in pricing last survivor immediate annuities, perhaps due to very competitive pricing strategy, with low rate of voluntary annuitization. However, unexpected mortality improvements in joint-life mortality could jeopardize the financial solvency of an annuity fund that has not adequately anticipated the possible impact of longevity risk.

3.3. The UK market. The UK annuity market is bigger and more developed than the US market, because of the legal obligation to annuitize substantial proportion of retirement funds. Meanwhile, the UK market was aware of longevity risk earlier than the US market. In addition, it is more liquid because of the availability of longevity risk securitization instruments.

More information can be gleaned by comparing these two markets. The prices of immediate annuities in the UK annuity market are quoted from the Annuity Online[1], which gives an indication of an averaged annuity price from a number of annuity providers during February 2011. We quote for last survivor annuities for annuitants in good health and non-smoking. We assume again these quoted prices are net of expense.

Table 4 displays the quoted market prices of unit annuities, and the calibrated market prices of longevity risk. The quoted prices are for annuities per £1 annual benefit paid in advance, in monthly instalment, without guarantee, same as the US example above. We use the same age combinations as the US example, and calibrate the market prices of longevity risk assuming that the husband and wife are the same age.

Assuming $\eta_1^f = \eta_2^f$ and $\eta_1^m = \eta_2^m$, the average value of η^f is 0.0647 and of η^m is -0.3437. The market price of risk is positive for the elements of longevity risk in female mortality rates, and negative for the elements in male mortality rates. Underpricing last survivor annuities also appears to the UK annuity market, though to a lesser extent than in the US market. This point is confirmed, as a positive market price of longevity risk is derived if $\eta^f = \eta^m$ is assumed. It could be interpreted that the joint-life longevity risk appears to be more adequately allowed in the UK annuity market than in the US annuity market; in addition, the US market considered the longevity risk of joint lives less consistently than the UK market, because of the wider range of market prices of longevity risk in the US market.

Table 4. Market prices of joint-life longevity risk in the UK market, η^f and η^m, calibrated from the quoted market prices of immediate last survivor annuities with equal inception ages from 65 to 75

Female age	Male age	Market annuity price	η^f	η^m	$\eta^f = \eta^m$
65	65	16.17	-0.9635	0.5396	0.0397
66	66	15.77	0.1398	-0.1652	0.0306
67	67	15.40	0.1605	-0.2002	0.0356
68	68	15.01	0.1743	-0.2198	0.0396
69	69	14.62	0.2029	-0.2759	0.0459
70	70	14.20	0.4180	-1.2110	0.0494
71	71	13.78	0.4168	-1.1955	0.0513
72	72	13.34	0.0244	0.0887	0.0494
73	73	12.88	0.4341	-1.2509	0.0484
74	74	12.47	-0.3605	0.4534	0.0695
75	75	11.98			0.0611

As in the US market, the UK market has a similar pricing practice for last survivor annuities. The male's age will not be a pricing factor unless he is at least two years younger than his wife. The quoted

[1] Available at: http://www.annuities-online.com/

last survivor annuity price for a 65-year old wife and 65-year old husband is applied to all the cases that the wife is aged 65 and the husband is aged 63 or older. That is to say, the last survivor annuity price only depends on the female's age (x) once her spouse's age (y) satisfies $y \geq x - 2$. In that case, the age of the male is ignored in determining the expected annuity payments.

The prices of last survivor annuities in both the US and the UK market do not reflect the difference between single-life mortality and joint-life mortality. They are not based on a joint-life mortality model. Dependence between joint lives, including the "broken heart" effect, has not been considered. The irrational last survivor annuity pricing structure in the US and UK market implicitly affect the calibrated market price of longevity risk of joint lives, to some extent.

3.4. Joint-life vs. single-life. For further understanding of the results in the previous section, we also examined the market prices of single-life longevity risk in the US and UK annuity market. Recall that, in projecting joint-life mortality rates, the single-life mortality is projected as the marginal mortality distribution. It enables a meaningful comparison between the market prices of longevity risk in joint-life products and in single-life products.

The market prices of single-life longevity risk are calibrated based on the projected marginal single-life mortality, using quotes of single-life immediate annuities in the US and UK market. Table 5 lists the quoted market prices of single-life immediate annuities per unit annual benefit in monthly instalments, paid in advance, without guarantee, in the US and UK market. The quoted market prices of single-life annuities in the UK are generally higher than the quoted prices in the US, while the difference narrows with the inception age of annuity.

Table 5. Quoted market prices per unit annual benefit paid monthly in advance for single-life immediate annuities in the US and UK market for different inception ages

	The US market		The UK market	
Inception age	Males	Females	Males	Females
65	13.17	14.16	14.97	16.09
66	12.86	13.87	14.62	15.68
67	12.53	13.56	14.21	15.31
68	12.21	13.37	13.78	14.92
69	11.92	13.09	13.40	14.52
70	11.63	12.80	13.00	14.10
71	11.28	12.42	12.59	13.68
72	10.93	12.07	12.11	13.23
73	10.71	11.72	11.67	12.77
74	10.39	11.37	11.23	12.35
75	10.11	11.05	10.76	11.86

Figure 5 compares the calibrated market prices of longevity risk in last survivor annuities (top) and single-life annuities (bottom) in the US and the UK market. The risk-free interest rate is 4.25%. The market prices of longevity risk of joint lives are taken for the last column in Table 3 and 4, which are calibrated assuming $\eta^f = \eta^m$.

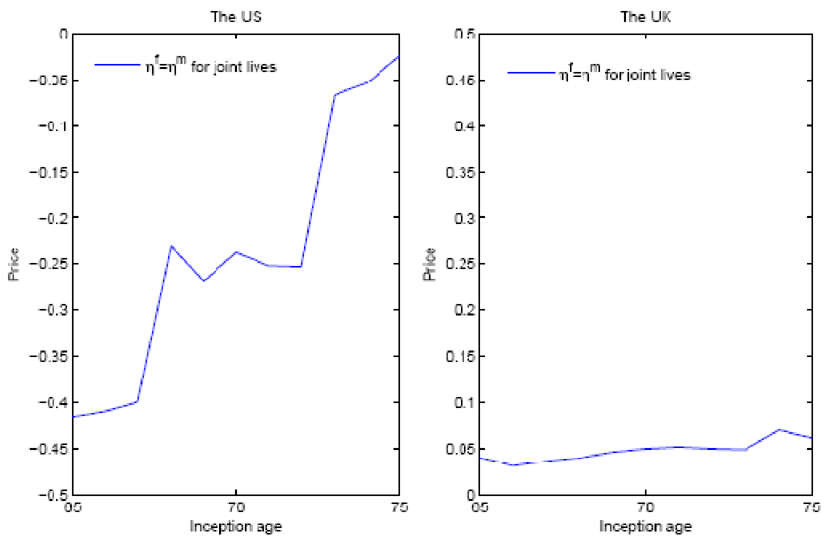

Fig. 5A. The estimated market prices of longevity risk in last survivor annuities (top) and single-life annuities (bottom) in the US market (left) and the UK market (right)

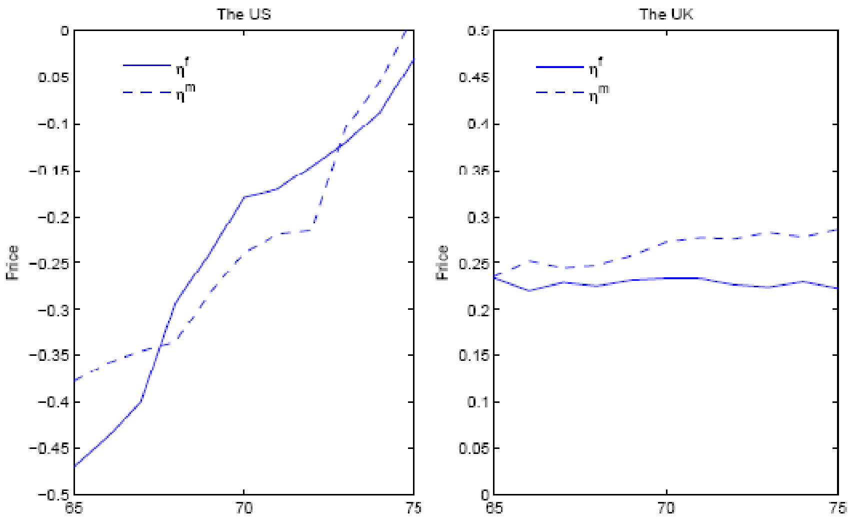

Fig. 5B. The estimated market prices of longevity risk in last survivor annuities (top) and single-life annuities (bottom) in the US market (left) and the UK market (right)

In the US, the line of market prices of joint-life longevity risk is roughly between the two lines of market prices of single-life longevity risk. However, the market prices of longevity risk, either joint-life or single-life, are generally negative.

In the UK, all the calibrated market prices of longevity risk are positive, although the market prices of longevity risk in last survivor annuities is much lower than the corresponding market price of single-life longevity risk. The market prices of longevity risk are positive in the UK annuity market, but negative in the US market. It indicates that the UK annuity market appears to more adequately allow for longevity risk when pricing immediate annuities than the US annuity market. Furthermore, the market prices of longevity risk in the US are far below zero. The US market does not correctly estimate the future improvements in mortality rates. Underpricing appears to be prevalent in the US annuity market. Mortality assumptions for pricing annuities needs to be reviewed. Further study in fair pricing annuities is required.

Concluding remarks

In this paper, we have mainly focussed on the sustainability and reasonability of the prices of last survivor annuities in the private market. For this end, we propose a semi-Markov joint-life longevity risk model, and investigate the market prices of joint-life longevity risk in the US and UK, using the risk-neutral pricing theory.

The effect of mortality improvement has substantial impact on last survivor annuities. However, market prices of longevity risk in last survivor annuities for two components of mortality processes are quite volatile. Negative market prices of longevity risk calibrated from the prices of last survivor annuities in both the US and the UK market indicate that last survivor annuities may not be well priced currently.

We compare the market prices of joint-life longevity risk against the market prices of single-life longevity risk. The results indicate that the US market systematically underprices joint-life annuities and single-life annuities. The UK annuity market has more conservative allowance for longevity risk when pricing single-life annuities. Unfortunately we do not see consistent pricing of joint-life annuities.

Joint-life pricing structures are irrational in both the US and the UK annuity market. The impact could be destructive for the development of annuity market. Last survivor annuities are likely to become more critical following the European Union ban on gender-specific annuity rates, which will take effect in 2012. Careful attention is called for to avoid underpricing these products. Further study in fare pricing this type of annuities is required.

The aforementioned remarks are based on the proposed semi-Markov joint-life longevity model, which is built up upon a Gompertz distribution with stochastic parameters for the stochastic modeling of force of mortality. The stochastic Gompertz mortality model is a natural extension of the Gompertz law. It is most easily incorporated into the semi-Markov joint-life model. However, we acknowledge that mortality forecasting using models with stochastic parameter relies on the accuracy of the underlying parametric model.

Generally speaking, the proposed model is a preliminary step in the modeling and risk management of joint-life longevity risk. Currently, a little research has been done in this area. What we have done here is only a lead-in to future analysis.

The correlation in the improvement of mortality for males and females has not been reflected. If there exists some correlation, it may have non-negligible impact on joint-life longevity risk. This should be investigated further, especially if suitable data become available.

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