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Minimizing market impact of hedging insurance liabilities within risk appetite constraints

Abstract

More than 95% use of derivatives by insurance companies is empirically dedicated to hedge risks embedded within their long-dated liability financial guarantees that are characterized by their long term duration, large volumes and significant market risk exposure, with growing appetite (1885bn notional in 2014) given current persistent low interest rates and tail equity risks environment. As a matter of fact there has been significant evidence of illiquidity cost stemming from supply/demand imbalance for downside protection equity options, which Solvency II is expected to further strengthen, as insurers will be forced to hold sufficient capital to remain solvent during periods of market stress. As a result optimal put options buying strategies including explicit market impact of their transaction size are being devised, which are strongly dependent on the specific risk appetite of the insurance company, as illustrated through the use of a mean or a mean-variance P&L objective to maximize.

Keywords: liquidity, hedging, stochastic optimal control, risk appetite, market impact.

Introduction

Life insurance liabilities are characterized by three main features: long term duration, large volumes and significant market risk exposure. Given the persistent low interest rates environment across the curve since the 2008 financial crisis, the use of derivatives has enabled to hedge financial risks embedded within insurance liability guarantees, as illustrated by the significant recent increase in notionals from 786bn as of FY 2010 to 1885bn as of FY 2014.

As the guarantees embedded within the insurance liability hold a convex risk profile with respect to the underlying stock (the blue line in the chart below), insurance companies need to buy some convex hedge assets such as downside protection put options (in contrast with Futures which are linear hedge instruments and do not hold

convexity) in order to match the liability risk profile thus improve hedge effectiveness (translating into reducing the mismatch between the blue and red lines in the chart below).

However there has been already significant evidence of illiquidity cost stemming from supply/demand imbalance for options, which Solvency II is expected to further strengthen, as insurers will be forced to hold sufficient capital to remain solvent during periods of market stress consistent with the economic risks embedded within the long-dated liability guarantees. As a result, modeling the transaction of large hedge portfolios requires taking into account transaction size explicitly accounting for their market impact, in particular regarding equity derivatives that are highly sensitive to supply/demand balance.

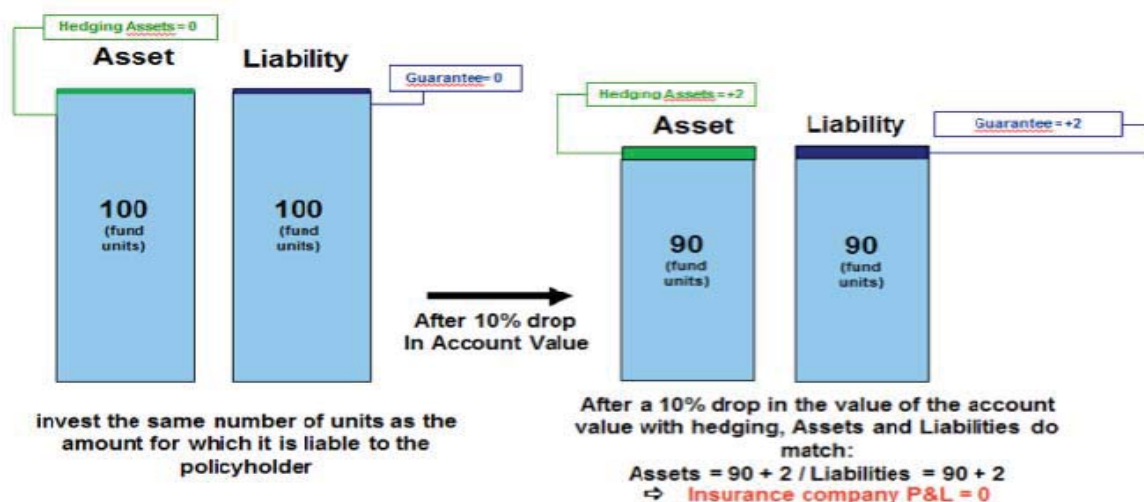


Fig. 1. Hedging liabilities with hedging assets principle

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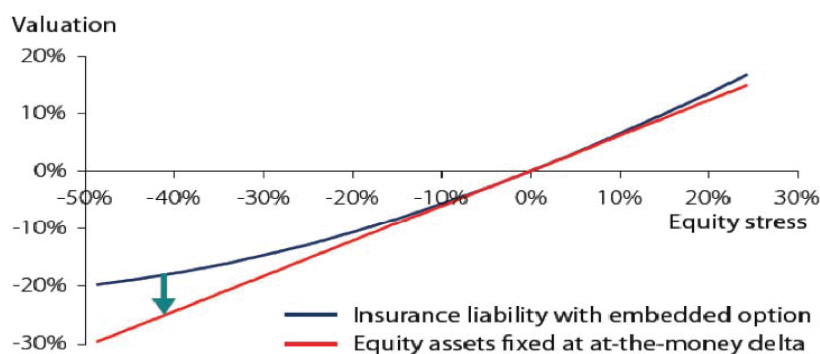


Fig. 2. Liability vs. hedging assets potential mismatch as market moves

Actually the primary tools used by insurers in 2014 were buying put options (44%, vs. 24% for calls), 90% of which were purchased, implying growing cost of hedging. As insurance companies have developed risk appetite framework since the 2008 risk crisis, we will devise optimal put options buying strategies including market impact with various risk appetite approaches:

- ◆ A mean approach focusing on minimizing the expected cost of the buying strategy which results in a stable quantity.
- ◆ A risk/reward approach more in line with a long term view consistent with the insurance industry time horizon, which adds the variance of the P&L into account to end up with a mean-variance like approach. This results in a more sensitive strategy to market conditions.

Our contribution is twofold:

- ◆ First, tailoring optimal trading strategies including market impact to concrete risk appetite framework.
- ◆ Second, illustrating significant differences in the corresponding optimal trading quantity, depending on the specific risk appetite, whether the agent maximizes the mean P&L only or also minimizes the dispersion of the P&L into consideration (mean-variance framework):
 - a. The optimal trading strategy with the objective to maximize the mean P&L is linear, translating into a rather static pace of trading and more trading activity as maturity gets closer, while minimizing the dispersion of the P&L in addition requires to trade more in the beginning of the trading program.
 - b. The mean approach is sensitive to significant single market events, while the optimal strategy in the mean-variance framework depends on the whole path, translating into more actively traded ITM options than ATM and OTM options.

The modeling framework is addressed in Section 1. The optimal transaction strategy is presented and illustrated in Section 2 where the objective is to

maximize the mean P&L, thereafter extended to a mean-variance P&L approach. We conclude in Final Section.

1. Integrating market impact into hedging modeling framework

As a result, an explicit modeling of such increasing cost of options is made through a market impact function, the influence of which the insurance company will try its best to minimize. In this context, best execution cannot be defined as a single number within a single trade. The market impact on the option price depends on a “temporary impact strength” that is proportional to the main empirically observed drivers: the speed of option trading (i.e. “number of options per unit of time”), the equity stock level, the option sensitivity to the equity stock.

Here the market impact on the option price is defined as follows:

$$\tilde{P}(t, S_t, \dot{x}_t) = P(t, S_t) - \eta \dot{x}_t S_t \Delta(t, S_t), \quad (1)$$

where:

- ◆ η is in $\$ \times \text{hour} / N \text{ shares}$ controls the temporary impact strength.
- ◆ \dot{x}_t is the speed of trading is in number of options per time unit.
- ◆ Δ is the put option sensitivity w.r.t the underlying asset. $\Delta(t, S_t) < 0$ for put options, therefore selling the option will tend to decrease its price.
- ◆ S is the underlying stock assumed here to have lognormal distribution: $dS_t = \sigma S_t dW_t$, $S_0 = s_0$.
- ◆ P is the unaffected put price at time t equal to the corresponding to the replicating cost from Black & Scholes.

We consider the problem of buying European put options over a finite time horizon $[0, T]$, where T is the end time and is greater than the option expiration date. The option position of the agent is described by a continuous and adapted curve x_t satisfying the boundary condition $x_0 = X$ and $x_T = 0$.

The cost arising from the strategy x including market impact is as follows:

$$C(x) = \int_0^T \tilde{P}_t \dot{x}_t dt = \int_0^T P_t \dot{x}_t dt + \eta \int_0^T \dot{x}_t S_t (-\Delta(t, S_t)) dt = -XP_0 - \int_0^T x_t dP_t + \eta \int_0^T \dot{x}_t^2 S_t (-\Delta(t, S_t)) dt \tag{2}$$

2. Optimal hedging transactions significantly depend on risk appetite

Here we consider a life insurance company minimizing the cost of buying a given large quantity of put options dedicated to hedge its liabilities. Such strategy will also depend on its specific risk appetite, such as a maximization of the mean P&L objective (or minimization of the mean cost of buying options), or a risk-reward objective including the minimization of the dispersion of the P&L. The standard procedure of the Hamilton-Jacobi-Bellman (HJB) framework in stochastic control problems is then applied, coupled with numerical schemes.

2.1. Mean P&L risk appetite. Under maximizing mean P&L, the insurance company needs to minimize the expected cost $E[C(x)]$. This is

equivalent to minimizing the last term on the right hand side of

$$\min_x E[C(x)] = -XP_0 + \eta \min_x E \left[\int_0^T \dot{x}_t^2 S_t (-\Delta(t, S_t)) dt \right] \tag{3}$$

One optimal strategy is characterized by the following trading rate:

$$\dot{x}_t = -\frac{X}{T},$$

which provides a rather stable pace of trading as illustrated below, depending only mildly on the stock price path as illustrated by the figure below right. This pace is rather constant at the beginning and increases as we get close to maturity, which is intuitive given the fixed quantity to buy within the fixed time period, implying the insurer must acquire faster as time passes (see figure below left).

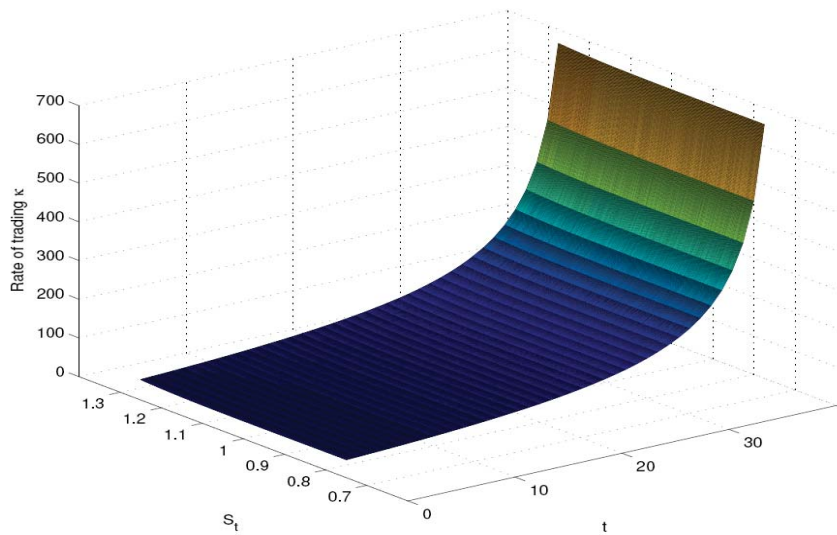


Fig. 3. Trading pace as a function of the stock level and time passing under maximizing mean P&L

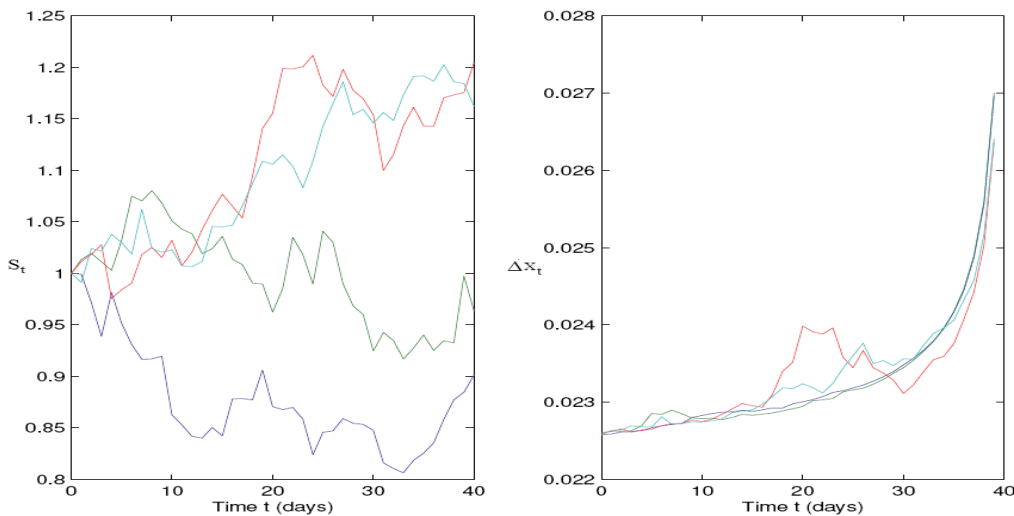


Fig. 4. The traded quantity Δx increases as time passes

It is important to mention that the last trade is the residual quantity and doesn't come out of our optimal control: the more penalty is given to the objective function at maturity, the less the agent would trade in this date.

2.1. Mean-variance P&L risk appetite. If the dispersion of the P&L is now added to the risk appetite framework, within the "Mean-variance" P&L approach, we are then interested in the variance of the P&L of the position. For each time t we define the P&L of the currently held option position x_t after an infinitesimal time span dt corresponding to $x_t dP_t$.

Using Ito's formula and the PDE verified by the put option we have:

$$x_t dP_t = \sigma S_t \Delta(t, S_t) dW_t,$$

where W is the brownian motion leading the underlying asset SDE.

When taking the variance of the P&L between 0 and T we have:

$$\mathbb{V} \left[\int_0^T x_t dP_t \right] = \mathbb{E} \left[\int_0^T \sigma^2(S_t) x_t^2 S_t^2 \partial_S P^2(t, S_t) dt \right] \quad (4)$$

$$U(t, S, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}_t \left[\int_t^T \alpha^2 S_u (-\Delta(u, S_u)) + \lambda x_u^{\alpha^2} S_u^2 \Delta(u, S_u)^2 du \right], \quad (7)$$

where the initial conditions are $S_0 = s_0$ and $x_0 = X$. The constraint $\int_0^T \alpha_t dt = X$ suggests that the value function U should satisfy a singular condition of the form

$$\lim_{t \rightarrow T} U(t, S, x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{if } x \neq 0 \end{cases} \quad (8)$$

The intuition for this terminal condition is that a state with a non zero asset position with no time left for its liquidation means that the liquidation task has not been performed, so that this state should receive an infinite penalty.

Now using the standard procedure of deriving the Hamilton-Jacobi-Bellman (HJB) equation in stochastic control problems, provides a non linear partial differential equation (PDE) with terminal conditions, specifically taking into account the constraint $\int_0^T \alpha_t dt = X$. The governing equation satisfies the following dynamic programming equation (DPE):

$$\partial_t U + \mathcal{L}U + \lambda x^2 S^2 \Delta(t, S)^2 + \inf_{\alpha} h(\alpha) = 0 \quad (9)$$

where

$$\mathcal{L}U = \frac{1}{2} \sigma^2 S^2 \partial_{SS} U \text{ and } h(\alpha) = -\alpha \partial_x U + S \alpha^2 (-\Delta(t, S)).$$

As a result we aim at minimizing the following objective function:

$$\mathbb{E}[\mathcal{C}(x)] + \tilde{\lambda} \mathbb{E} \left[\int_0^T x_t^2 \sigma^2(S_t) S_t^2 \Delta^2(t, S_t) dt \right] = -XP_0 + \mathbb{E} \left[\eta \int_0^T x_t^2 S_t (-\Delta_{\alpha_j}(t, S_t)) dt + \tilde{\lambda} \int_0^T x_t^2 \sigma^2 S_t^2 \Delta^2(t, S_t) dt \right] \quad (5)$$

In contrast with the mean approach, this minimization problem doesn't have an explicit solution. We will reformulate our problem within the framework of stochastic control.

Such problems usually parameterize the strategies x by their speed of trading and define the control α such that $\alpha_t := -\dot{x}_t$. The parameterized strategy x^α is defined by:

$$x_t^\alpha := X - \int_0^t \alpha_s ds, \quad 0 \leq t \leq T, \quad (6)$$

where the control variable α is a function of the current time t , the current stock price S_t and the position x_t . We define our value function $U(t, S, x)$:

The function $h(\alpha)$ is a quadratic function with a second degree positive coefficient (since $\Delta(t, S) < 0$ for a put option). It is a convex function and attains its minimum for $h'(\alpha) = 0$. This gives:

$$\alpha^*(t, S, x) = \frac{\partial_x U}{2S(-\Delta(t, S))}. \quad (10)$$

$$\text{and } h(\alpha^*) = -\frac{(\partial_x U)^2}{4S(-\Delta(t, S))}.$$

and upon substituting back into Eq. (6), the value function U then solves the non-linear partial differential equation (PDE):

$$\partial_t U + \mathcal{L}U + \lambda x^2 S^2 \Delta(t, S)^2 - \frac{(\partial_x U)^2}{4S(-\Delta(t, S))} = 0 \quad (11)$$

Since no closed solution can be found, the resolution is purely numerical using the separation of variables and dimension reduction methods then the implicit finite differences scheme. In order to prevent instability of the solution. Since S doesn't depend on the control, the problem will be simplified by giving a more convenient form for U by choosing a more convenient parameterization for the state variable x . That is, a multiplicative parameterization allows to write $dx_t = -\kappa_t x_t dt$ and the value function U as:

$$\begin{aligned}
 U(t, S, x) &= \inf_{\kappa \in \mathcal{K}} \mathbb{E}_t \left[\int_t^T \left\{ \kappa_u^2 (x_u^\kappa)^2 S_u (-\Delta_{adj}(u, S_u)) + \lambda (x_u^\kappa)^2 \sigma^2(S_u) S_u^2 \Delta^2(u, S_u) \right\} du \right] = \\
 &= \inf_{\kappa \in \mathcal{K}} \mathbb{E}_t \left[(x_t^\kappa)^2 \int_t^T e^{-\int_t^u 2\kappa_s ds} \left\{ \kappa_u^2 S_u (-\Delta_{adj}(u, S_u)) + \lambda \sigma^2(S_u) S_u^2 \Delta^2(u, S_u) \right\} du \right] = \\
 &= x^2 \inf_{\kappa \in \mathcal{K}} \mathbb{E}_t \left[\int_t^T e^{-\int_t^u 2\kappa_s ds} \left\{ \kappa_u^2 S_u (-\Delta_{adj}(u, S_u)) + \lambda \sigma^2(S_u) S_u^2 \Delta^2(u, S_u) \right\} du \right] = x^2 u(t, S).
 \end{aligned} \tag{12}$$

The constraint $\int_0^T \alpha_t dt = X$ becomes $\int_0^T e^{-\kappa_t dt} = 0$. We can therefore solve the reduced value function u :

$$u(t, S) = \inf_{\kappa \in \mathcal{K}} E_t \left[\int_t^T e^{-\int_t^u 2\kappa_s ds} \left\{ \kappa_u^2 S_u (-\Delta_{adj}(u, S_u)) + \lambda \sigma^2(S_u) S_u^2 \Delta^2(u, S_u) \right\} du \right] \tag{13}$$

This leads to the the following non-linear PDE for u

$$\partial_t u + \mathcal{L}u + \lambda \sigma^2(S) S^2 \Delta^2(t, S) - \frac{1}{S(-\Delta(t, S))} u^2 = 0, \tag{14}$$

where

$$\mathcal{L}u = \frac{1}{2} \sigma^2 S^2 \partial_{SS} u. \tag{15}$$

With the boundary conditions:

$$\begin{aligned}
 u(T, S) &= +\infty, \\
 u(t, 0) &\rightarrow u_0(t), \\
 u(t, s_{max}) &\rightarrow u_1(t),
 \end{aligned} \tag{16}$$

where u_0 and u_1 has the following singular PDEs:

$$\partial_t u_0 + \inf_{\kappa} \{-2\kappa u_0\} = 0.$$

$$\partial_t u_1 + \mathcal{L}u_1 + \inf_{\kappa} \{-2\kappa u_1\} = 0.$$

We solve this problem numerically using the implicit finite differences scheme.

As a result the optimal execution strategy under the mean-variance P&L approach is more dependent on the stock price, as illustrated in the figure below left by faster traded when the stock level is low than when the stock level is high: as the stock decreases, the put option value increases and the impact arising from trading it too (more “In-The-Money”), which puts further pressure to trade as soon as possible to prevent from growing market impact thus cost of the put option; the opposite as the stock market rises. The mean-variance P&L framework also prevents the insurance company from waiting too long, but instead favors a decreasing trading pace.

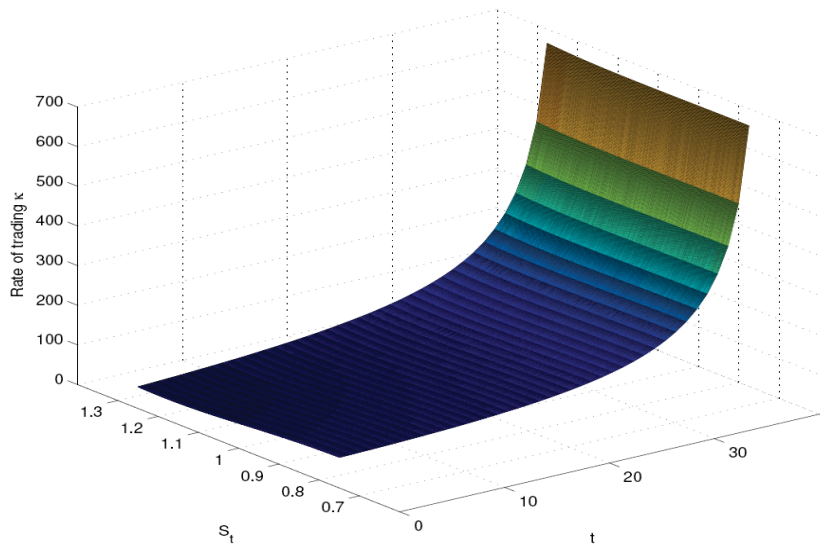


Fig. 5. Trading pace as a function of the stock level and time passing under maximizing mean-variance P&L

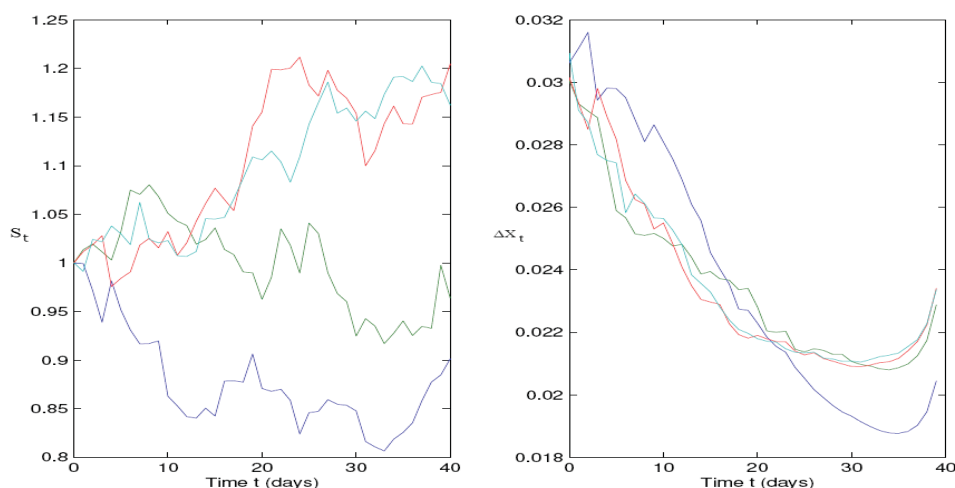


Fig. 6. Traded quantity strongly decreases as time passes under maximizing mean-variance P&L

Conclusion

Within the context of Solvency II mostly characterized by a Mark-to-Market valuation framework, the size of such transactions may put significant constraints on the insurance company through higher cost of hedging liabilities stemming from their market impact. Introducing a market impact function and using

stochastic optimal control theory with respect to a specific criterion (mean P&L or mean-variance P&L) we have devised an optimal path for the subsequent expected transaction size, where the Risk Appetite of the insurance company has significant influence on the optimal transaction execution path, not only in terms of pattern of the pace of trading over the period but also with respect to the tock level and path.

References

1. Frederic Abergel and Grégoire Loeper (2013). *Pricing and hedging contingent claims with liquidity costs and market impact*.
2. Robert Almgren (2012). Optimal trading with stochastic liquidity and volatility, *Siam J. Financial Math*, 3, pp. 163-181.
3. Robert Almgren and Neil Chriss (2000). *Optimal execution of portfolio transactions*, December.
4. Robert Almgren, Chee Thum, Emmanuel Hauptmann, and Hong Li (2005). Direct estimation of equity market impact, May.
5. Robert F. Almgren (2001). *Optimal execution with nonlinear impact functions and trading-enhanced risk*. October.
6. Oleg Bondarenko (2003). Why are put options so expensive? *Quarterly Journal of Finance*, 4.
7. Jean-Philippe Bouchaud (2009). *Price impact*. March.
8. Jean-Philippe Bouchaud, J. Doyne Farmer, and Fabrizio Lillo (2008). How markets slowly digest changes in supply and demand, September.
9. L. Chan and J. Lakonishok (1995). The behavior of stock prices around institutional trades, *Journal of Finance*, 50, pp.1147–1174.
10. P.A. Forsyth, J.S. Jennedy, S.T. Tse, and H. Windcliff (2011). *Optimal trade execution: A mean–quadratic-variation approach*, September.
11. Peter A. Forsyth (2010). A Hamilton Jacobi Bellman approach to optimal trade execution, August.
12. Rudiger Frey (2007). Market volatility and feedback effects from dynamic hedging.
13. Jim Gatheral and Alexander Schied (2011). Optimal trade execution under geometric brownian motion in the almgren and chriss framework, *International Journal of Theoretical and Applied Finance*, 14(3), pp. 353-368.
14. Sanford J. Grossman and Merton H. Miller (1988). Liquidity and market structure, *The journal of Finance*, 43(3), pp. 617-633, July.
15. Jens Carsten Jackwerth and Mark Rubinstein (1998). Recovering stochastic processes from option prices, Working paper.
16. S. Jaimungal and D. Kinzebulatov (2014). Optimal execution with a price limiter, *Risk*, July.
17. Aymeric Kalife (2004). *Portfolio insurance strategies by a large player*, December.
18. Donald B. Keim and Ananth Madhavan (1995). Execution costs and investment performance: An empirical analysis of institutional equity trades. Working paper, School of Business Administration, University of Southern California.
19. Idris Kharroubi and Huyen Pham (2013). *Optimal portfolio liquidation with execution cost and risk*, February.
20. Tianhui Michael Li and Robert Almgren. A fully-dynamic closed-form solution for delta-hedging with market impact. Submitted to Operations Research.
21. Grégoire Loeper (2013). Option pricing with market impact and non-linear black and scholes pdes's.

22. Esteban Moro, Javier Vicente, Luis G. Moyano, Austin Gerig, J. Doyne Farmer, Gabriella Vaglica, Fabrizio Lillo, and Rosario N. Mantegna (2009). Market impact and trading profile of large trading orders in stock markets. Working paper, June.
23. Jiongmin Yong and Xun Yu Zhou (1999). *Stochastic controls: Hamiltonian Systems and HJB Equations*. Springer.