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Infinite Dimensional Spaces and Cartesian Closedness

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Infinite dimensional spaces frequently appear in physics; there are several approaches to obtain a good categorical framework for this type of space, and cartesian closedness of some category, embedding smooth manifolds, is one of the most requested condition. In the first part of the paper, we start from the failures presented by the classical Banach manifolds approach and we will review the most studied approaches focusing on cartesian closedness: the convenient setting, diffeology and synthetic differential geometry. In the second part of the paper, we present a general settings to obtain cartesian closedness. Using this approach, we can also easily obtain the possibility to extend manifolds using nilpotent infinitesimal points, without any need to have a background in formal logic.

Key words: infinite dimensional spaces of smooth mappings, diffeology, synthetic differential geometry, cartesian closedness.

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1. Introduction and Mathematical Motivations for Cartesian Closedness

One of the aims of the present article is to review some of the most important, i.e. well-established, approaches used to define geometrical structures in infinite dimensional spaces. The review will be done with a particular focus on the property of cartesian closedness. One of the most important example we have in mind is the set $\mathbf{Man}(M, N)$ of all the smooth applications between two finite dimensional manifolds M and N. For the aims of the present article, we are interested to list some of the most studied structures on $\mathbf{Man}(M, N)$, and its subspaces, that permit to develop at least a tangency theory. Using this terminology we mean at least the notion of tangent functor and the notion of differentiability

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of maps between this type of infinite dimensional space, with sufficiently good categorical properties. This is not a trivial goal because, for example, an important example we can cite is the group $\mathrm{Diff}(M)$ of all the diffeomorphisms of a manifold M. Flows in a compact manifold M can be considered as 1-parameter subgroups of $\mathrm{Diff}(M)$, and it would seem useful to express the smoothness of a flow by means of a suitable differentiable structure on $\mathrm{Diff}(M)$, which should also behave like a classical Lie group with respect to this structure.

A typical restriction to distinguish among different approaches to infinite dimensional spaces is the hypotheses of compactness of the domain M, assumed to obtain some desired properties: is this a necessary hypotheses or are we forced to assume it due to some restrictions of the chosen approach?

Another interesting property is the possibility to extend the classical notion of manifold to a more general type of space, so as to get better categorical properties, like the existence of infinite products or co-products or a cartesian closed category.

Finally, several authors had to tackle the following problem: suppose we have a new notion of smooth space able to include the space $\mathbf{Man}(M, N)$, at least for M compact and finite dimensional, and to embed faithfully the category of smooth finite dimensional manifolds. Even if the extension of the notion of finite dimensional manifold is faithful, usually the category $\mathcal{C}$ of these new smooth spaces includes spaces which are too much general, so that it seems really hard to generalize for these spaces meaningful results of differential geometry of finite dimensional manifolds. For this reason, several authors (see, e.g., [1–4]) try to select, among all their new smooth spaces in $\mathcal{C}$, the best ones having some new more restrictive properties. In this way the category $\mathcal{C}$ acts as a universe, usually closed with respect to strong categorical operations (like arbitrary limits, colimits and cartesian closedness), and the restricted class of smooth spaces works as a true generalization of the notion of manifold.

For example, in [1] the category of Frölicher spaces acts as a universe, but indeed the monograph is about manifolds modelled in convenient vector spaces instead of classical Banach spaces (see Sect. 4). This permits to [1] to generalize, as far as possible, to infinite dimensional manifolds the results of finite dimensional spaces, but as a consequence, the class of manifolds modelled in convenient vector spaces loses some desired categorical properties.

Analogously, in synthetic differential geometry (SDG; see, e.g., [3–5]) the class of restricted smooth spaces is introduced using the notion of microlinear space and the universe is a suitable topos, i.e., a whole model for intuitionistic set theory. In this approach, the infinitesimals are used to study the properties of this class of restricted, better behaved, spaces.

Of course, this is not possible in theories that have not an explicit language of actual infinitesimals, like in the case of diffeological spaces (see [6]). For them, we can proceed either as in convenient vector spaces theory considering the notion of vector space in the category of smooth diffeological spaces (i.e., smooth diffeological spaces that are also vector spaces with smooth operations, see [6]) and considering manifolds modelled in diffeological vector spaces, or we can try to develop directly for a generic diffeological space some notion of differential geometry (see, e.g. [6–12]). In the following subsectio,ns we will return to this problem giving some more precise definitions.

To understand better some differences between the approaches we are going to describe shortly in this section, we want to motivate the notion of cartesian closure, because it is one of the basic choices shared by several authors like [1–5, 12–25]. We firstly fix the notations for the notions of *adjoint* of a map.

Definition 1. *If X, Y, Z are sets and $f : X \longrightarrow Z^Y$, $g : X \times Y \longrightarrow Z$ are maps, then*

$$\begin{aligned} \forall (x, y) \in X \times Y : \quad & f^\vee(x, y) := [f(x)]\,(y) \in Z, \\ \forall x \in X : \quad & g^\wedge(x) := g(x, -) \in Z^Y, \end{aligned}$$

hence

$$\begin{aligned} f^\vee :& X \times Y \longrightarrow Z, \\ g^\wedge :& X \longrightarrow Z^Y. \end{aligned}$$

The map $f^\vee$ is called the adjoint *of f and the map $g^\wedge$ is called the* adjoint* *of g.*

Let us note that $(f^\vee)^\wedge = f$ and $(g^\wedge)^\vee = g$, that is, the two applications

$$\begin{aligned} (-)^\vee :& \left(Z^Y\right)^X \longrightarrow Z^{X\times Y} \\ (-)^\wedge :& Z^{X\times Y} \longrightarrow \left(Z^Y\right)^X \end{aligned}$$

are one the inverse of the other and hence represent in explicit form the bijection of sets $\left(Z^Y\right)^X \simeq Z^{X\times Y}$, i.e., $\mathbf{Set}(X, \mathbf{Set}(Y, Z)) \simeq \mathbf{Set}(X \times Y, Z)$.

One of the main aim of the second part of the present work is to generalize the notions of smooth manifold and of smooth map between two manifolds so as to obtain a new category "with good properties" that will be denoted by $\mathcal{C}^\infty$; if we call *smooth maps* the morphisms of $\mathcal{C}^\infty$ and *smooth spaces* its objects, then this category must be *cartesian closed*, i.e., it has to verify the following properties for every pair of smooth space X, $Y \in \mathcal{C}^\infty$:

1. $\mathcal{C}^\infty(X, Y)$ is a smooth space, i.e. $\mathcal{C}^\infty(X, Y) \in \mathcal{C}^\infty$.

*Here we are using the notations of [26], but some authors, e.g. [1], used opposite notations for the adjoint maps.

2. The maps $(-)^\vee$ and $(-)^\wedge$ are smooth, i.e., they realize in the category $\boldsymbol{\mathcal{C}}^\infty$ the bijection $\boldsymbol{\mathcal{C}}^\infty(X, \boldsymbol{\mathcal{C}}^\infty(Y, Z)) \simeq \boldsymbol{\mathcal{C}}^\infty(X \times Y, Z)$.

Property 1 is another way to state that the category we want to construct must contain as objects the space of all the smooth maps between two generic objects X, $Y \in \boldsymbol{\mathcal{C}}^\infty$:

$$\begin{aligned}\boldsymbol{\mathcal{C}}^\infty(X, Y) &= \{f \mid X \xrightarrow{f} Y \text{ is smooth}\}\\ &= \{f \mid X \xrightarrow{f} Y \text{ is a morphism of } \boldsymbol{\mathcal{C}}^\infty.\end{aligned}$$

Moreover, let us note that as a consequence of property 2 we have that

$$X \xrightarrow{f} \boldsymbol{\mathcal{C}}^\infty(Y, Z) \text{ is smooth} \iff X \times Y X \xrightarrow{f^\vee} Z \text{ is smooth,} \tag{1}$$

$$X \times Y \xrightarrow{g} Z \text{ is smooth} \iff X \xrightarrow{g^\wedge} \boldsymbol{\mathcal{C}}^\infty(Y, Z) \text{ is smooth.} \tag{2}$$

The importance of (1) and (2) can be explained saying that if we want to study a smooth map having values in the space $\boldsymbol{\mathcal{C}}^\infty(Y, Z)$, then it suffices to study its adjoint map $f^\vee$. If, e.g., the spaces X, Y and Z are finite dimensional manifolds, then $\boldsymbol{\mathcal{C}}^\infty(Y, Z)$ is infinite-dimensional, but $f^\vee : X \times Y \longrightarrow Z$ is a standard smooth map between finite dimensional manifolds, and hence we have a strong simplification. Conversely, if $g : X \times Y \longrightarrow Z$ is a smooth map, then it generates a smooth map with values in $\boldsymbol{\mathcal{C}}^\infty(Y, Z)$, and all the smooth maps with values in this type of space can be generated in this way. Of course, this idea is frequently used, even if informally, in the calculus of variations. Let us note explicitly that the cartesian closure of the category $\boldsymbol{\mathcal{C}}^\infty$, i.e. properties 1 and 2, does not say anything about smooth maps with a *domain* of the form $\boldsymbol{\mathcal{C}}^\infty(Y, Z)$, but it reformulates in a convenient way the problem of smoothness of maps with *codomain* of this type. For a more abstract notion of cartesian closed category, see, e.g., [26–29].

We also want to see a different motivation drawn from [2]. Let us suppose to have a smooth function $g : \mathbb{R} \times I \longrightarrow \mathbb{R}$, where $I = [a, b]$, and define the integral function

$$f(t) := \int_a^b g(t, s)\, \mathrm{d}s \quad \forall t \in \mathbb{R}.$$

Then we can look at the function f as the composition of two applications

$$f : t \in \mathbb{R} \mapsto g(t, -) \mapsto \int_a^b g(t, -) \in \mathbb{R}.$$

Hence, if we denote by i the application

$$i : h \in \mathcal{C}^\infty(I,\mathbb{R}) \mapsto \int_a^b h \in \mathbb{R},$$

then

$$f = i \circ g^\wedge , \qquad \text{i.e.} \qquad f(t) = i\left(g^\wedge(t,-)\right) \quad \forall t \in \mathbb{R}.$$

In this way, it is natural to try a proof of the formula for the derivation under the integral sign in the following way:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(i \circ g^\wedge\right) = \mathrm{d}i\left(g^\wedge(t)\right)\left[\frac{\mathrm{d}g^\wedge}{\mathrm{d}t}(t)\right] = \\ = i\left[\partial_1 g(t,-)\right] = \int_a^b \partial_1 g(t,s)\,\mathrm{d}s. \quad (3)$$

Here we have supposed that the following properties hold:

- $g^\wedge : \mathbb{R} \longrightarrow \mathcal{C}^\infty(I,\mathbb{R})$ is smooth,
- $i : \mathcal{C}^\infty(I,\mathbb{R}) \longrightarrow \mathbb{R}$ is smooth,
- the chain rule for the derivative of the composition of two functions,
- the differential of the function i is given by $\mathrm{d}i(h) = i$ for every $h \in \mathcal{C}^\infty(I,\mathbb{R})$, because i is linear,
- $\frac{\mathrm{d}g^\wedge}{\mathrm{d}t}(t) = \partial_1 g(t,-)$.

Let us note explicitly that the space $\mathcal{C}^\infty(I,\mathbb{R})$ is infinite dimensional.

Really, the aim of (3) is not to suggest a new proof, but to hint that a theory where we can consider the previous properties seems to be very flexible and powerful.

2. Physical Motivations for Cartesian Closedness

The use of a cartesian closed category as a useful framework for physics can be motivated in four ways:

1. In physics, the necessity to use infinite dimensional spaces frequently appears. A classical example is the space $\mathbf{Man}(M,N)$ of all the smooth mappings between two smooth manifolds M and N, or some of its subspaces, e.g., the space of all the diffeomophisms of a smooth manifold. Typically, we are interested in infinite dimensional Lie groups, because they appear, for example, in the study

of both compressible and incompressible fluids, in magnetohydrodynamics, in plasma-dynamics or in electrodynamics (see, e.g., [30] and references therein). It is also sufficiently clear from the previous Sect. 1, and it will also be even more clear from the following Sect. 10, that cartesian closedness is also a desirable condition in the calculus of variations. Anyway, the most natural generalization of finite dimensional linear spaces theory, i.e. Banach spaces theory, is essentially incompatible with cartesian closedness and with an infinite dimensional theory of Lie groups. We will review these incompatibility results in the next Sect. 3.

2. There has been a great effort to obtain a theory of smooth spaces able to include cartesian closedness and smooth manifolds. The convenient setting ([1, 2]) is the more advanced theory of smooth spaces extending the theory of Banach manifolds. Some applications of this notion to classical field theory can be found in [31]. Anyway, other approaches to a new notion of smooth space appear as motivated also by problems of physics. For example, the notion of diffeological space has been used in [11, 12, 32], starting also from a variant of [18], to study quantization of coadjoint orbits in groups of diffeomorphisms of infinite dimension. Diffeological spaces form a cartesian closed, complete, co-complete quasi-topos ([6, 33–35]). On the one hand, it is easier to obtain and to study a diffeological space with respect to a manifold modelled in a convenient vector space; on the other hand, the category of diffeological spaces contains several pathological examples. In the present article, we will review both the convenient setting and the diffeology approach, focusing on some of their qualities and lacks.

3. A strong motivation toward cartesian closedness is surely the role of topos theory in foundational issues of quantum theory, quantum gravity and intuitionistic theory of general relativity. Of course, any topos is a cartesian closed category. The literature concerning this approach is vast. See, e.g., [36–45]. One of the basic ideas of this approach is to criticize some implicit hypothesis of every physical theory: the space time as a manifold, the logic as classical, the category of sets as a sufficiently reach framework for the right interpretation of physical theories, the implicit assumption of the real (or complex) field as the ring of scalars. This criticism permits to gain meaningful interpretations, e.g., in quantum theory or in general relativity (see, e.g., [46–50]). On the other hand, taking a smooth topos ([4, 5]) as a framework for physical theory permits to have at disposal the infinitesimal calculus of synthetic differential geometry ([3–5, 14]). In this article, we will see how it is possible to introduce, in a very simple way, nilpotent infinitesimals to the real line $\mathbb{R}$ obtaining a ring ${}^\bullet\mathbb{R}$, called a ring of Fermat reals. Suitably generalizing the definition of the category $\mathcal{C}^\infty$ of diffeological spaces, we will see how it is possible to add new infinitesimally closed points to every diffeological space (see Sect. 7), obtaining a new cartesian closed category ${}^\bullet\mathcal{C}^\infty$. This construction, even if it has several analogies with SDG, is so simple that it can be studied directly in classical logic. In other words, it can be studied directly "from

the outside" of ${}^{\bullet}\boldsymbol{\mathcal{C}}^{\infty}$, exactly as the models of SDG can in principle be studied classically, even if the internal logic of the corresponding topos is intuitionistic.

4. Lawvere [51] suggested to consider a cartesian closed categorical framework for the study of continuum mechanics. In fact, having cartesian closedness, it seems possible to study the mechanics of a continuum body without the strong limitations tied to manifolds theory. We will see some preliminary sketch of this program in the present article. We will show that we are not forced to assume that the configuration space of a continuum body B is necessarily a manifold but that, more generally, a diffeological space structure rises more naturally. The cartesian closedness of the category $\boldsymbol{\mathcal{C}}^{\infty}$ permits hence to define a motion without having charts and atlases. The possibility to add to any $B \in \boldsymbol{\mathcal{C}}^{\infty}$ new infinitesimal points obtaining the space ${}^{\bullet}B \in {}^{\bullet}\boldsymbol{\mathcal{C}}^{\infty}$ permits to study infinitesimal subbodies of B. This approach has been already used in [52] in some elementary examples.

3. Banach Manifolds and Locally Convex Vector Spaces

Banach manifolds are the more natural generalization of finite-dimensional manifolds if Banach spaces are taken as local model spaces. Even if, as we will see more precisely in this section, this theory does not satisfy our condition to present in this article only generalized notions of manifolds able to develop at least a tangency theory and having sufficiently good categorical properties, Banach manifolds are the most studied concept in infinite dimensional differential geometry. Some well- known references on Banach manifolds are [30, 53]. Among the most important theorems in this framework, we can cite the implicit and inverse function theorems and the existence and uniqueness of solutions of Lipschitz ordinary differential equations. The use of charts to prove these fundamental results is indispensable, so it is not easy to generalize them to the more general contexts where we cannot use the notion of chart having values in some modelling space with sufficiently good properties.

For the purposes of the present analysis, a typical example of infinite-dimensional Banach space is the space $\mathcal{C}^r(M,E)$ of $\mathcal{C}^r$-maps, where M is a compact manifold, E is a Banach space and $r \in \mathbb{N}$. The vector space $\mathcal{C}^r(M,E)$ is a Banach space with respect to the norm

$$\|f\|_r := \max_{0\le i\le r} \sup_{m\in M} \left\| \mathrm{d}^i f(m) \right\|, \tag{4}$$

but the theory fails for the space $\mathcal{C}^{\infty}(M,E) := \bigcap_{r=0}^{+\infty} \mathcal{C}^r(M,E)$ of smooth mappings defined in M and with values in E. On the one hand, the hypotheses of considering $r < +\infty$ and M compact in the previous definition (4) are not intrinsic to the problem, but are motivated solely by the limitations of the instrument we are trying to implement, i.e. a norm in the space $\mathcal{C}^r(M,E)$. Even if this is not a formal motivation, it remains very important in the real development of mathematics. On the other hand, more formally, any two different norms $\|-\|_r$ and $\|-\|_s$ are not equivalent, and hence the space $\mathcal{C}^\infty(M;E)$ is not normable with a norm generating the same topology generated by the family of norms $(\|-\|_r)_{r=0}^{+\infty}$ (for details, see, e.g., [54]). In the following, saying that the space $\mathcal{C}^\infty(M,E)$ is not normable, we will always mean with respect to this topology.

Moreover, $\mathcal{C}^\infty(M,E)$ is not a Banach manifold: indeed, it is separable and metric (see [54]), hence, if it were a Banach manifold, then it would be embeddable as an open subset of a Hilbert space (see [55]), and hence it would be normable.

Therefore, the category of Banach manifolds and smooth maps **Ban** is not cartesian closed because it is not closed with respect to exponential objects $\mathbf{Ban}(M,E)=\mathcal{C}^\infty(M,E)$, see condition 1 in the previous definition of cartesian closed category, Sect. 1.

This also proves that the category of Banach manifolds **Ban** and smooth maps does not have arbitrary limits: in fact if it had infinite products, then we would have

$$\prod_{m\in M} E=\mathbf{Ban}(M,E)=\mathcal{C}^\infty(M,E),$$

but we have already seen that this space is not a Banach manifold.

These important counter-examples can lead to the idea of considering the spaces equipped with a family of norms, like $(\|-\|_r)_{r=1}^{+\infty}$, or, more generally, of seminorms, i.e. toward the theory of locally convex vector spaces (see, e.g., [56]). But any locally convex topology on the space $\mathcal{C}^\infty(M,E)$ is incompatible with cartesian closure, as stated in the following

Theorem 2. *Let F be a first countable locally convex vector space contained in a cartesian closed subcategory $\mathcal{T}$ of the category* **Top** *of topological spaces and continuous functions such that $\mathcal{T}(F,\mathbb{R})$ always contains all the linear continuous functionals on the space F*

$$F^*:=\mathrm{Lin}(F,\mathbb{R})\subseteq\mathcal{T}(F,\mathbb{R}).$$

Moreover, let us suppose that for every $g\in F^*$ the application

$$\lambda\in\mathbb{R}\mapsto\lambda g\in F^* \tag{5}$$

is continuous with respect to the topology induced on F^* by the inclusion $F^*\subseteq\mathcal{T}(F,\mathbb{R})\in\mathcal{T}$. Then F is normable. Hence the category **Ban** *is not cartesian closed because the space*

$$F=\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$$

is not normable.

P r o o f. We can argue as in [1]: because $\mathcal{T}$ is cartesian closed, every evaluation

$$\mathrm{ev}_{XY}(x, f) := f(x) \quad \forall x \in X \, \forall f \in \mathcal{T}(X, Y)$$

is an arrow of $\mathcal{T}$ (this is a general result in every cartesian closed category, see, e.g., [27]) and hence it is also a continuous function, because $\mathcal{T}$ is a subcategory of **Top** by hypotheses. In this case, we also have that the restriction of $\mathrm{ev}_{F\mathbb{R}}$ to the subspace $F^* = \mathrm{Lin}(F, \mathbb{R}) \subseteq \mathcal{T}(F, \mathbb{R})$ of linear continuous functionals on the space F can also be (jointly) continuous:

$$\varepsilon := \mathrm{ev}_{F\mathbb{R}}|_{F \times F^*} : F \times F^* \longrightarrow \mathbb{R}.$$

Then we can find neighborhoods $U \subseteq F$ and $V \subseteq F^*$ of zero such that $\varepsilon(U \times V) \subseteq [-1, 1]$, that is

$$U \subseteq \{u \in F \,|\, \forall f \in V : \; |f(u)| \leq 1\} \, .$$

But then, because the map (5) is continuous, taking a generic functional $g \in F^*$, we can always find $\lambda \in \mathbb{R}_{\neq 0}$ such that $\lambda g \in V$, and hence $|g(u)| \leq 1/\lambda$ for every $u \in U$. Any continuous functional is thus bounded on U, so the neighborhood U itself is bounded (see, e.g., [1, 56]). But any locally convex vector space with a bounded neighborhood of zero is normable (see, e.g., [56, 57]). ∎

This theorem also asserts that notions like Fréchet manifolds (manifolds modelled in locally convex metrizable and complete vector spaces) are incompatible with cartesian closedness too.

For a more detailed study on the cartesian closedness and Banach manifolds, see [15–17]; for a more detailed study on the relationships between the topology on spaces of continuous linear functionals $\mathrm{Lin}(F, E)$ and normable spaces, see [58, 59].

Because one of our aim is to obtain a category $\boldsymbol{\mathcal{C}}^\infty$ of "smooth" (and hence topological) spaces embedding the category **Ban**, a direct consequence of Theorem 2 is that, in general, we will not have a locally convex topology on spaces of functions like $\boldsymbol{\mathcal{C}}^\infty(M, \mathbb{R})$. Nevertheless, we will see that in the category $\boldsymbol{\mathcal{C}}^\infty$ of diffeological spaces we always have that every arrow (i.e., every smooth function in a generalized sense) is also continuous and every evaluation is smooth.

The fundamental results of [60–62] show that a Banach Lie group G acting smoothly, transitively:

$$\forall x, y \in M \; \exists \, g \in G : \; g \cdot x = y$$

and effectively:

$$\forall g, h \in G : \; g \neq h \quad \Longrightarrow \quad \exists \, x \in M : \; g \cdot x \neq h \cdot x$$

on a compact manifold M must necessary be finite dimensional: $\dim(G) < +\infty$. This result strongly underscores that the space of all the diffeomorphisms $G = \mathrm{Diff}(M)$ of a compact manifold in itself cannot be a Banach Lie group.

We will see that the category **Ban** of smooth Banach manifolds is faithfully embedded in the category $\boldsymbol{\mathcal{C}}^\infty$ of diffeological spaces.

4. The Convenient Vector Spaces Settings

It is very interesting to note that the original idea to define the differential of functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ reducing it to the composition $f \circ c$ with differentiable curves $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ goes back (for didactic reasons!) to [63]. In this work a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ was called differentiable if all the compositions $f \circ c$ with differentiable curves $c : \mathbb{R} \longrightarrow \mathbb{R}^2$ are again differentiable and satisfy the chain rule. Later (see [64]) this notion has been extended to mapping $f : E \longrightarrow F$ between generic topological vector spaces: f is said differentiable at $x \in E$ if there exists a continuous linear mapping $l : E \longrightarrow F$ such that $f \circ c : \mathbb{R} \longrightarrow F$ is differentiable at 0 with derivative $(l \circ c')(0)$ for each everywhere differentiable curve $c : \mathbb{R} \longrightarrow E$ with $c(0) = x$. This notion of differentiable function is really more restrictive than the usual one, but it is equivalent to the standard notion of smooth function if in it we replace the word "differentiable" with "smooth". More generally, if we replace "differentiable" with "of class $\mathcal{C}^k$ and with locally Lipschitz kth derivative", we obtain an equivalence with the classical notion. These results have been proved by [65] and the whole theory of convenient vector spaces depends strongly on these nontrivial results.

Several theories which detach from the theory of Banach manifolds, like the convenient vector spaces setting or the following diffeological spaces, are grounded on the generalization of this idea (not necessarily knowing the cited article [63]). In particular, the theory of convenient vector spaces is probably the most developed theory of infinite dimensional manifolds able to overpass several problems of Banach manifolds. Presently, the most complete reference is [1], even if the theory started with [20] and [2].

Only to mention a few results, in the convenient vector spaces setting the hard implicit function theorem of Nash and Moser (see [1, 66]) can be proved, very good results can also be obtained for both holomorphic and real analytic calculus, the theorem of De Rham can be proved and the theory of infinite dimensional Lie groups can be well developed.

Definition 3. *We say that E is a* convenient vector space *iff E is a locally convex vector space where every smooth curve has a primitive, i.e.*

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E)\ \exists\, p \in \mathcal{C}^\infty(\mathbb{R}, E) :\ p' = c.$$

Considering the Cauchy–Bochner integral, any Banach space is hence a convenient vector space, but several nontrivial examples directly come from the cartesian closedness of the category of all the convenient vector spaces.

As mentioned in the previous section, what type of topology can be considered in a convenient vector space, due to the cartesian closedness of the related category, is a nontrivial point. The idea to reduce, as far as possible, any possible notion to the corresponding notion for smooth curves, can take us toward the natural idea to consider the final topology for which any smooth curve is also continuous, i.e. the following

Definition 4. *Let E be a convenient vector space, then we say that*

$$U \textit{ is } c^\infty\textit{-open in } E$$

iff

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E): \; c^{-1}(U) \textit{ is open in } \mathbb{R}.$$

The category of convenient vector spaces is cartesian closed so that, for example, $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is again a convenient vector space. We can now define as usual the notion of chart modelled in a c^∞-open set of a convenient vector space and hence the corresponding notion of smooth manifold and of smooth map between two manifolds. In the following, we will denote with $\mathcal{C}^\infty_{\text{cvs}}$ the category of smooth manifolds modelled in convenient vector spaces. Using suitable generalizations of Boman's theorem ([65]), it is hence possible to prove the following (see [1]).

Theorem 5. *Let M, N be manifolds modelled on convenient vector spaces, then we have that $f : M \longrightarrow N$ is smooth iff*

$$\forall c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, M): \quad f \circ c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, N).$$

Using the notion of c^∞-open subset of a convenient vector space and the notion of chart, it is possible to define a topology on every manifold considering the final topology in which every chart is continuous. We have hence the expected result: W is open in this topology on M if and only if $c^{-1}(W)$ is open in $\mathbb{R}$ for every smooth curve $c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, M)$.

The notion of Frölicher space provides the possibility to construct a category with very good properties, acting as a universe for the class of manifolds modelled in convenient vector spaces.

Definition 6. *A Frölicher space is a triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ consisting of a set X, a subset $\mathcal{C}_X \subseteq X^{\mathbb{R}}$ of curves on this set, and a subset $\mathcal{F}_X \subseteq \mathbb{R}^X$ of real valued functions defined on X, with the following properties:*

1. $\forall f \in \mathbb{R}^X : \quad f \in \mathcal{F}_X \iff [\forall c \in \mathcal{C}_X : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})]$,

2. $\forall c \in X^\mathbb{R} : \quad c \in \mathcal{C}_X \iff [\forall f \in \mathcal{F}_X : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})]$.

The category of Frölicher spaces is cartesian closed and it possesses arbitrary limits and colimits. A locally convex vector space E is a convenient vector space if and only if it is a Frölicher space with respect to the curves and functions defined as $\mathcal{C}_X := \mathcal{C}^\infty_{\mathrm{cvs}}(\mathbb{R}, E)$ and $\mathcal{F}_X := \mathcal{C}^\infty_{\mathrm{cvs}}(E, \mathbb{R})$. Finally, because of cartesian closedness, it is possible to define a unique structure of Frölicher space on the set $Y := \mathcal{C}^\infty(M, N)$ of all the smooth maps between two manifolds given by

$$\mathcal{C}_Y := \left\{ c : \mathbb{R} \longrightarrow N^M \,|\, c^\vee : \mathbb{R} \times N \longrightarrow M \text{ is smooth} \right\}$$

and

$$\mathcal{F}_Y := \left\{ f : N^M \longrightarrow \mathbb{R} \,|\, \forall c \in \mathcal{C}_Y : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R};, \mathbb{R}) \right\} .$$

In the following we will use again the symbol $\mathcal{C}^\infty(M, N)$ to indicate this structure of Frölicher space.

As mentioned at the beginning of this section, the notion of manifold modelled in convenient vector spaces permits to include several infinite dimensional spaces nonascribable to Banach manifold theory, but, at the same time, forces us to lose some good categorical property. In particular, the space of all smooth mappings $\mathcal{C}^\infty(M, N)$ between two manifolds has a manifold structure only for M and N finite dimensional (see [1], section IX). Moreover, if $\mathfrak{C}^\infty(M, N)$ is this manifold structure⋆ on the set $\mathcal{C}^\infty(M, N)$, then the exponential law

$$\mathcal{C}^\infty(M, \mathfrak{C}^\infty(N, P)) \simeq \mathcal{C}^\infty(M \times N, P)$$

holds if and only if N is compact (see [1], Theorem 42.14).

Using an intuitive interpretation introduced by [21], we can say that in the convenient vector spaces settings the fundamental figure of our spaces is the curve and every notion is reduced to a corresponding notion about curves. Later, we will use several times this intuitive, and fruitfully, interpretations also for other types of figures. In the notion of Frölicher space there is a particular emphasis on the symmetry between curves and functions, with the aim to obtain a category with less pathological spaces, but this symmetry has not been adopted by other authors, like in the following approach about diffeological spaces.

The possibility to use an infinitesimal language for diffeological spaces, has been also opened for SDG, because [67, 68] proved that the category of convenient spaces is embedded into the Cahier topos for SDG. A similar interesting approach

⋆Note that, e.g., if $M = N = \mathbb{R}$, this structure is different from the structure of convenient vector space (and Frölicher space) $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, i.e., it has other classes of curves and functions; for this reason the authors of [1] use a different symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$.

for the convenient setting is given by [69–71] with the study of microlinear spaces in the category of Frölicher spaces.

We will see that both Frölicher spaces and manifolds modelled in convenient vector spaces are embedded in the category $\mathcal{C}^\infty$ of diffeological spaces, so that our approach can supply a language of actual infinitesimals also to these settings. This is a problem posed by [4]:

> *In recent years, several alternative solutions to the problem of generalizing manifolds to include function spaces and spaces with singularities have been proposed in the literature. A particularly appealing one is the theory of convenient vector spaces* [...]. *These structures are in a way simpler than the sheaves considered in this book, but one should notice that the theory of convenient vector spaces does not include an attempt to develop an appropriate framework for infinitesimal structures, which is one of the main motivations of our approach.*

5. Diffeological Spaces

Using the language of the "fundamental figures" given on a general space X introduced by [21], we can describe diffeological spaces as a natural generalization of the previous idea to take as fundamental figures all the smooth curves $c : \mathbb{R} \longrightarrow X$ on the space X. To define the concept of diffeological space, we first denote with

$$\mathrm{Op} := \{U \mid \exists\, n \in \mathbb{N} : U \text{ is open in } \mathbb{R}^n\}$$

the set of all the domains of our new figures in the space X. In informal words, the idea of a diffeological space is to say that a smooth structure on the space X is given specifying all the smooth figures $p : U \longrightarrow X$, for $U \in \mathrm{Op}$. More formally, we have

Definition 7. *We say that* $(\mathcal{D}, X)$ *is a* diffeological space *iff* X *is a set and* $\mathcal{D} = \{\mathcal{D}_U\}_{U\in Op}$ *is a family of sets of functions*

$$\mathcal{D}_U \subseteq \mathbf{Set}(U, X) \quad \forall U \in Op.$$

The functions $p \in \mathcal{D}_U$ *are called* parametrizations *or* plots *or* figures on X of type U. *The family* $\mathcal{D}$ *has to satisfies the following conditions:*

1. *Every point of* X *is a figure, i.e., for every* $U \in Op$ *and every constant map* $p : U \longrightarrow X$*, we must have that* $p \in \mathcal{D}_U$.
2. *Every set of figures* $\mathcal{D}_U$ *is closed with respect to re-parametrization, i.e., if* $p : U \longrightarrow X$ *is a figure in* $\mathcal{D}_U$*, and* $f \in \mathcal{C}^\infty(V, U)$*, where* $V \in Op$*, then* $p \circ f \in \mathcal{D}_V$.

3. *The family $\mathcal{D} = \{\mathcal{D}_U\}_{U \in Op}$ verifies a sheaf property, i.e., let $V \in Op$, $(U_i)_{i\in I}$ be an open cover of V and $p : V \longrightarrow X$ a map such that $p|_{U_i} \in \mathcal{D}_{U_i}$, then $p \in \mathcal{D}_V$. In other words, a figure to be locally implies a figure to be globally too.*

Finally a map $f : X \longrightarrow Y$ between two diffeological spaces $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ is said to be smooth *if it takes figures of the domain space in figures of the codomain space, i.e., if*

$$\forall U \in Op \; \forall p \in \mathcal{D}^X_U : \quad f \circ p \in \mathcal{D}^Y_U.$$

The category of all the diffeological spaces will be denoted with $\boldsymbol{\mathcal{C}}^\infty$.

Of course, a diffeological space with support set X is a subsheaf of the presheaf $\mathbf{Set}(-, X)$. If compared with Frölicher spaces, in Diffeology (i.e. the study of diffeological spaces, see [6]) the principal differences are in the generalization of the types of figures, in the losing of the symmetry between figures and corresponding functions (i.e., maps of type $f : X \longrightarrow U$ for $U \in \mathrm{Op}$) and in the fundamental sheaf property. For example, the generalization to figures of arbitrary dimension instead of curves only, permits to prove the cartesian closure of the category of diffeological spaces very easily and without the use of the nontrivial Boman's theorem (see [1, 2, 65]). The original idea to consider figures of general dimension instead of curves only, and the fundamental sheaf condition date back to [18, 72]; the definition of diffeological space, essentially in the form given above, is originally of [11, 12].

The category of diffeological spaces has very good categorical properties, with arbitrary limits (subspaces, products, pullbacks, etc.) and colimits (quotient spaces, sums, pushforwards, etc.), cartesian closedness (so that the set theoretical compositions and evaluations are always smooth) and indeed is a quasi-topos ([33, 35]). Classical Fréchet manifolds are fully and faithfully embedded in this category (see [73]).

We can now define a *diffeological vector space* (over $\mathbb{R}$) as any diffeological space $(E, \mathcal{D})$, where E is a vector space (over $\mathbb{R}$), and such that the addition and the multiplication by a scalar

$$(u, v) \in E \times E \mapsto u + v \in E \quad \text{and} \quad (r, u) \in \mathbb{R} \times E \mapsto ru \in E$$

are smooth (with respect to the suitable product diffeologies on the domains) and, as usual, the notion of *smooth manifolds modelled on diffeological vector spaces*.

Differential geometry on generic diffeological spaces can be developed surprisingly far as showed, e.g., by [6]: homotopy theory, exterior differential calculus, differential forms, Lie derivatives, integration on chains and Stokes formula, de

Rham cohomology, Cartan formula, generalization of symplectic geometry to diffeological spaces. As said in [6]:

> *Thanks to the strong stability of diffeology under the most important categorical operations* [...] *every general construction relating to this theory applies to spaces of functions, differential forms, fiber bundles, homotopy, etc. without leaving the strict framework of diffeology. This makes the development of differential geometry much more easier, much more natural, than usually.*

It is also interesting to note that some of these generalizations (like Stokes formula) are general consequences of this type of extension of the notion of manifolds, as proved by [74], and hence are not special for Diffeology.

From our point of view, Diffeology is surely formally clear, but sometimes lacks from the point of view of the intuitive geometrical interpretation. To illustrate this assertion, we can consider the notion of tangent vector as formulated in [6]. In the following we will assume that $(X, \mathcal{D})$ is a diffeological space and $x \in X$ is a point in the space X. The first idea is that the figures $q : U \longrightarrow X$ of type $U \subseteq \mathbb{R}^u$ of the space X permit to define the notion of smooth p-form without having the notion of tangent vector, but abstracting the properties of the pullback q^* of the figure $q \in \mathcal{D}_U$. In other words, let us suppose that we have already defined what a differential p-form on X is, then we would be able to define the pullback q^* of q as a map that associates to each differential form $d \in \Omega^p(X)$ and to each point $u \in U \subseteq \mathbb{R}^u$ a p-form in $\Lambda^p(\mathbb{R}^u)$. The idea is hence to define directly a p-form as this action on figures through pullback, and asking the natural condition of composition of pullbacks in case we take a parametrization $f \in \mathcal{C}^\infty(V, U)$ of the domain of the figure q:

Definition 8. *A* differential p-form defined on X *is a family of maps of the form* $(\alpha_U)_{U \in \mathrm{Op}}$. *Each* α_U, *for* U *open in* $\mathbb{R}^u$, *associates to each figure* $q \in \mathcal{D}_U$ *a smooth* p*-form* $\alpha_U(q) : U \longrightarrow \Lambda^p(\mathbb{R}^n)$, *i.e.,*

$$\alpha_U : \mathcal{D}_U \longrightarrow \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^u)),$$

and it must satisfy the condition

$$\alpha_V(q \circ f) = f^*(\alpha_U(q))$$

for every plot $q \in \mathcal{D}_U$ *and for every smooth parametrization* $f \in \mathcal{C}^\infty(V, U)$ *defined on the open set* $V \in Op$. *The set of all the differential* p*-forms defined on* X *will be denoted by* $\Omega^p(X)$.

The method used to arrive at this definition is the (frequently used in mathematics) "inversion of the effect with the cause" in case of bijection between effects

and causes. Indeed, if $X =$ is an open set of $\mathbb{R}^d$, then it is possible to prove that we have a natural isomorphism between the new definition and the classical notion of smooth p-form, i.e., $\Omega^p(U) \simeq \mathcal{C}^\infty(U, \Lambda^p(U))$, in other words, the pullbacks of p-forms uniquely determine the p-forms themselves.
The previous definition satisfies all the properties one needs from it, like the possibility to define a diffeology on $\Omega^p(X)$, vector space structure, pullbacks, exterior differential, exterior product, a natural notion of germ generated by a p-form so that two forms are equal if and only if they generate the same germ (that if they are "locally" equal), etc.

The first intuitive drawback of the definition of $\Omega^p(X)$ is that there is no mention to spaces $\Lambda^p_x(X)$ of p-forms associated to each point $x \in X$ and of the relationships between these spaces and the whole $\Omega^p(X)$. Therefore, to understand better the following definitions, we introduce the following

Definition 9. *We say that two forms α, $\beta \in \Omega^p(X)$* have the same value at x, *and we write $\alpha \sim_x \beta$, if and only if for every figure $q \in \mathcal{D}_U$ such that*

$$0 \in U \quad \textit{and} \quad q(0) = x$$

(*in this case we will say that q* is centered at x) *we have that*

$$\alpha(q)(0) = \beta(q)(0).$$

Equivalence classes of p-forms by means of the equivalence relation $\sim_x$ are called values of α at x *and we will denote with $\Lambda^p_x(X) := \Omega^p(X)/\sim_x$ this quotient set.*

Using these values of 1-forms we can define tangent vectors. Firstly, we introduce the paths on X and the values of a 1-form on each path with the following

Definition 10. *Let us introduce the space of all the paths on X, i.e.,*

$$\mathit{Paths}(X) := \mathcal{C}^\infty(\mathbb{R}, X),$$

and for each path $q \in \mathit{Paths}(X)$, the map $j(q) : \Omega^1(X) \longrightarrow \mathbb{R}$ evaluating each 1-form at zero

$$j(q) : \alpha \in \Omega^1(X) \mapsto \alpha(q)(0) \in \mathbb{R}.$$

The map $j(q)$ is linear and smooth (because it is an evaluation in a cartesian closed category), hence

$$j : \mathit{Paths}(X) \longrightarrow L^\infty(\Omega^1(X), \mathbb{R}),$$

where $L^\infty(\Omega^1(X), \mathbb{R})$ is the space of all the linear smooth functionals defined on the space of 1-forms of X.

Secondly, we say that the set of all these values $j(q)$ generates the whole tangent space. The set of these generators is introduced in the following

Definition 11. *The space $C^\wedge_x(X)$ is the image of all the paths passing through x under the map j:*

$$C^\wedge_x(X) := \{j(q) \mid q \in \mathit{Paths}(X) \text{ and } q(0) = x\} \subseteq L^\infty(\Omega^1(X), \mathbb{R}).$$

In the space $C^\wedge_x(X)$ one can naturally define a multiplication by a scalar $r \in \mathbb{R}$ that formalizes the idea to increase the speed of going through a given path $q \in \mathrm{Paths}(X)$:

$$r \cdot j(q) = j\left[q(r \cdot (-))\right],$$

where $q(r \cdot (-))$ is the path $q(r \cdot (-)) : s \in \mathbb{R} \longrightarrow q(r \cdot s) \in X$. But the space $C^\wedge_x(X)$ is not necessarily a vector space because is not closed with respect to addition of these values $j(q)$ of 1-forms on paths q centered at x, hence we finally define

Definition 12. *A tangent vector $v \in T_x(X)$ is a linear combination of elements of $C^\wedge_x(X)$, i.e.,*

$$v = \sum_{i=1}^{n} s_i v_i$$

for some

$$\begin{aligned} &n \in \mathbb{N}, \\ &(v_i)_{i=1}^n \text{ sequence of } C^\wedge_x(X), \\ &(s_i)_{i=1}^n \text{ sequence of } \mathbb{R}. \end{aligned}$$

As we said, even if the definitions we have just introduced are formally correct, in our opinion their intuitive geometric meaning remains obscure. Probably, this partial lack of a clear and intuitive geometrical meaning is due to the searching for the greatest generality. In classical manifolds theory, the definition of tangent vector through 1-forms is not geometrically intrinsic unless of Riemannian manifolds, so it is not clear why passing to a more general space we are able to obtain this identification in an intrinsic way. Secondly, diffeological spaces include also spaces with singular points, like $X = \left\{(x, y) \in \mathbb{R}^2 \mid x \cdot y = 0\right\}$. At the origin $x = (0, 0) \in X$, there is no way to define in a geometrically meaningful way the sum of the two tangent vectors corresponding to $\boldsymbol{i} = (1, 0)$ and $\boldsymbol{j} = (0, 1)$ (without using the superspace $\mathbb{R}^2$). This is the principal motivation that conducts SDG to introduce the notion of *microlinear space* as the spaces where to each pair of tangent vectors it is possible to associate an infinitesimal parallelogram, fully contained in the space itself, whose diagonal represents the sum of these two tangent vectors. The previous space X is not microlinear exactly at the origin.

As we will see later in the present work, it is possible to add, in a meaningful and simple way, new infinitesimally closed points to every diffeological space. This provides for these spaces, hence, a possible language of infinitesimals. The use of these infinitesimals opens the possibility to simplify and clarify some concepts already developed in the framework of diffeological spaces, e.g., gaining a more clear geometrical meaning. Almost surely, this gain could be done only for a suitable class of diffeological spaces. For a preliminary development in this direction, in particular for the class of inf-linear spaces, corresponding formally to infinitesimally linear spaces of SDG, see [34].

6. Synthetic Differential Geometry

The fundamental ideas upon which SGD[*] was born, originate from the works of Ehresmann [75], Weil [76] and A. Grothendieck (see [77]). The first step was the introduction, by [75], of the concept of k-jet at a point p in a manifold M. This important geometric structure is determined by the kth order Taylor's formula of real valued functions f defined in a neighborhood of the point $p \in M$. As said by [78]:

> [...] *the study of jets can be seen as a development of the earlier idea of studying the infinitely nearby points on algebraic curves on manifolds. Presumably it was Ehresmann's initiative which stimulated the paper by* [76].

In this latter work A. Weil introduced the idea to formalize nilpotent infinitesimals using algebraic methods, more precisely using quotient rings like $\mathbb{R}[x]/(x^2)$ or $\mathbb{R}[x,y]/(x^2,y^2)$. In general, the idea is to consider formal power series in n variables $\mathbb{R}[[x_1,\dots,x_n]]$ modulo the $(k+1)$-th power of a given ideal $I=(i_1,\dots,i_m)$ of series $i_1,\dots,i_m \in \mathbb{R}[[x_1,\dots,x_n]]$ with zeros constant term, i.e., such that $i_j(\underline{0})=0$ for every $j=1,\dots,m$. These types of objects are now called *Weil algebras*, and C. Ehresmann's jets are also special cases of Weil algebras. Very roughly, we can guess the fundamental idea of A. Weil saying that, e.g., an element $p \in \mathbb{R}[x]/(x^2)$ can be written as $p=a+x\cdot b$, with a, $b \in \mathbb{R}$; addiction in this space is computed in the more obvious way, and multiplication is defined by $(a+x\cdot b)\cdot(\alpha+x\cdot\beta)=a\alpha+x\cdot(a\beta+b\alpha)$. We arrive at the same result if we multiply the two polynomials $p=a+x\cdot b$ and $q=\alpha+x\cdot\beta$ using the formal rules $x^2=0$. At the end, with a construction as simple as the definition of the field of complex numbers, we have extended the real field into a ring with a nonzero element x having zero square, i.e., a first order infinitesimal (let us note

[*]Frequently SDG is also called smooth infinitesimal analysis.

that in this ring there are not infinitesimals of greater order). Using the same idea, we can see that with the Weil algebra $\mathbb{R}[x,y]/(x^2,y^2)$ we can extend the real field with two first order infinitesimals x, y whose product is not zero, because $x \cdot y \neq 0$. Suitably generalized to algebras of germs of smooth functions defined on manifolds, these two examples, i.e. $\mathbb{R}[x]/(x^2)$ and $\mathbb{R}[x,y]/(x^2,y^2)$, correspond isomorphically to the first and second tangent bundles, respectively (see, e.g., [1, 3–5, 76, 79, 80] for more details). The next fundamental step to obtain a single framework, where all these types of nilpotent infinitesimals are available, has been performed by A. Grothendieck. His first aim was to use nilpotent infinitesimals to treat infinitesimal structures in algebraic geometry. The basic idea was to study an algebraic locus like $S^1 = \left\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2=1\right\}$, not only as a subset of points in the plane, but as the functor $S^1_{\mathrm{F}} : \mathbf{CRing} \longrightarrow \mathbf{Set}$, from the category $\mathbf{CRing}$ of commutative rings with 1 to the category of sets, defined as

$$S^1_{\mathrm{F}}(A) := \left\{(a,b) \in A^2 \,|\, a^2+b^2=1\right\},$$
$$S^1_{\mathrm{F}}(A \xrightarrow{f} B) := (f \times f)|_{S^1_{\mathrm{F}(A)}} : S^1_{\mathrm{F}}(A) \longrightarrow S^1_{\mathrm{F}}(B)$$

(where $f : A \longrightarrow B$ is a ring homomorphism and $f \times f : (a,b) \in A^2 \mapsto (f(a), f(b)) \in B^2$). Using this approach, algebraic geometers started to understand that the functor corresponding to the trivial locus $\{x \in \mathbb{R} \,|\, x = x\} = \mathbb{R}$, i.e., the functor $R(A) := \{a \in A \,|\, a = a\} = A =$ the underlying set of the ring A, behaves like a set of scalars containing infinitesimals. For example, $D(A) := \left\{a \in A \,|\, a^2 = 0\right\}$ is a subfunctor of this functor R, and plays the role of the space of first order infinitesimals. Being a subfunctor, D "behaves" like a subset* of R. These ideas spring into the notion of Grothendieck topos. Lawvere found that in the Grothendieck topos, and in other similar categories that later will originate the general notion of topos (see [81]), an intuitionistic set-theoretic language can be directly interpreted. In [21], he proposes a way to generalize these construction of algebraic geometry to smooth manifolds, and to use this generalization as a foundation for infinitesimal reasonings, with a single formalism valid both for finite and infinite dimensional manifolds. This proposal is a part of a bigger project, whose objective is to establish an intrinsic axiomatizaton for continuum mechanics. The inclusion of infinite dimensional spaces, like functions spaces, is a natural consequence of the cartesian closedness of every topos.

The construction of a model for SDG which embeds the category of smooth finite dimensional manifolds is not a simple task. Classical references are [4, 5]. Here we only want to sketch some of the fundamental ideas, first of all,

*In the sense that each Topos is a model of intuitionistic set theory, so that it is possible to define a formal language for intuitionistic set theory where sentences like $D \subseteq R$ are rigorous and true in the model (see [4, 5] for more details).

to underline the conceptual differences between SDG and the above mentioned approaches to infinite dimensional differential geometry.

The first idea to generalize from the context of algebraic geometry to manifolds theory is to find a corresponding of the category of **CRing** of commutative rings, i.e., to pass from a context of polynomial operations to more general smooth functions. Indeed, that category is replaced by that of $\mathcal{C}^\infty$-rings:

Definition 13. *A $\mathcal{C}^\infty$-ring $(A, +, \cdot, \iota)$ is a ring $(A, +, \cdot)$ together with an interpretation $\iota(f)$ of each possible smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, that is a map*

$$i(f) : A^n \longrightarrow A^m$$

such that ι preserves projections, compositions and identity maps, i.e.:

1. *If $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a projection, then $\iota(p) : A^m \longrightarrow A$ is a projection.*
2. *If $\mathbb{R}^d \longrightarrow g\mathbb{R}^n \longrightarrow f\mathbb{R}^m$ are smooth, then $\iota(f \circ g) = \iota(f) \circ \iota(g)$.*
3. *If $1_{\mathbb{R}^n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map, then $\iota(1_{\mathbb{R}^n}) = 1_{A^n}$.*

Frequently, we will use also the notation $A(f) := i(f)$. A homomorphism of $\mathcal{C}^\infty$-rings *$\varphi : A \longrightarrow B$ is a ring homomorphism which preserves the interpretation of smooth maps, that is such that the following diagram commutes*

$$\begin{tikzcd}
A^n \arrow[r, "\varphi^n"] \arrow[d, "A(f)"'] & B^n \arrow[d, "B(f)"] \\
A^m \arrow[r, "\varphi^m"] & B^m
\end{tikzcd}$$

for every smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

We may define a $\mathcal{C}^\infty$-ring in an equivalent but more concise way: let C^∞ denote the category whose objects are the spaces $\mathbb{R}^d$, $d \geq 0$, and with smooth functions as arrows, then a $\mathcal{C}^\infty$-ring is a finite product preserving functor $A : C^\infty \longrightarrow$ **Set**, and a $\mathcal{C}^\infty$-homomorphism is just a natural transformation $\varphi : A \longrightarrow B$. Indeed, given such a functor, the set $A(\mathbb{R})$ has the structure of a commutative ring $(A(\mathbb{R}), +_A, \cdot_A)$ given by $+_A := A(\mathbb{R} \times \mathbb{R} \longrightarrow +\mathbb{R})$ and $\cdot_A := A(\mathbb{R} \times \mathbb{R} \longrightarrow C{\cdot}\mathbb{R})$, where $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the ring operations on $\mathbb{R}$.

Here are some examples of $\mathcal{C}^\infty$-rings

Example 14. *The ring $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ of real valued smooth functions $a : \mathbb{R}^d \longrightarrow \mathbb{R}$, with pointwise ring operations, is a $\mathcal{C}^\infty$-ring. Usually, it is denoted simply with $\mathcal{C}^\infty(\mathbb{R}^d)$. The smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is interpreted in the following* *way. Let* $(h_1, \ldots, h_n) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})^n$ *be* n *elements of the ring* $\mathcal{C}^\infty(\mathbb{R}^d)$*. Their product*

$$(h_1, \ldots, h_n) : x \in \mathbb{R}^d \mapsto (h_1(x), \ldots, h_n(x)) \in \mathbb{R}^n$$

can be composed with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *and projected into its* m *components obtaining*

$$\iota(f) := (p_1 \circ f \circ (h_1, \ldots, h_n), \ldots, p_m \circ f \circ (h_1, \ldots, h_n)) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}),$$

where $p_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ are the projections.

Example 15. *If* M *is a smooth manifold, the ring of real valued functions defined on* M*, i.e.,* $\mathcal{C}^\infty(M, \mathbb{R})$*, is a* $\mathcal{C}^\infty$*-ring. Here, a smooth function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is interpreted using composition, similarly to the previous example. This ring is also denoted by* $\mathcal{C}^\infty(M)$*. Moreover, it is well known that*

$$\mathcal{C}^\infty(M) = \mathcal{C}^\infty(N) \quad \Longrightarrow \quad M = N$$

for second countable Hausdorff manifolds. If $g : N \longrightarrow M$ *is a smooth map between manifolds, then the* $\mathcal{C}^\infty$*-homomorphism given by*

$$\mathcal{C}^\infty(g) : a \in \mathcal{C}^\infty(M, \mathbb{R}) \mapsto a \circ g \in \mathcal{C}^\infty(N, \mathbb{R})$$

verifies the analogous embedding property:

$$\mathcal{C}^\infty(g) = \mathcal{C}^\infty(h) \quad \Longrightarrow \quad g = h.$$

This means that manifolds can be faithfully considered as $\mathcal{C}^\infty$-rings.

Example 16. *Let* A *be a* $\mathcal{C}^\infty$*-ring and* I *an ideal of* A*, then the quotient ring* A/I *is also a* $\mathcal{C}^\infty$*-ring. Indeed, if* $A(f) : A^n \longrightarrow A^m$ *is the interpretation of the map* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$*, we can define the interpretation* $(A/I)(f) : (A/I)^n \longrightarrow (A/I)^m$ *as*

$$\begin{aligned}&(A/I)(f)([a_1]_I, \ldots, [a_n]_I) := \\ &\qquad = ([p_1(A(f)(a_1, \ldots a_n))]_I, [p_m(A(f)(a_1, \ldots a_n))]_I),\end{aligned}$$

where $[a_i]_I \in A/I$ denotes the equivalent classes of the quotient ring, and $p_j : A^m \longrightarrow A$ are the projections (see, e.g., [4] for more details). Examples included in this case are the analogous of the above mentioned $D_k := \mathcal{C}^\infty(\mathbb{R})/(x^{k+1})$ and $D(2) := \mathcal{C}^\infty(\mathbb{R})/(x^2, y^2)$, or the ring $\triangle := \mathcal{C}^\infty_0(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)/m^g_{\{0\}}$, where $m^g_{\{0\}}$ is the ideal of smooth functions generating the zero germ at $0 \in \mathbb{R}^n$ and finally $\mathbb{I} := \mathcal{C}^\infty_0(\mathbb{R}^n \setminus \{0\})$. These $\mathcal{C}^\infty$-rings will play the role, in the final model, of *infinitesimals of* kth *order* D_k, of *pairs of infinitesimals of first order whose product is not necessarily zero* $D(2)$, of the set of all the infinitesimals $\triangle$, and of the set of all the *invertible infinitesimals* $\mathbb{I}$, *respectively.*

For each subset $X \subseteq \mathbb{R}^n$, a function $f : X \longrightarrow \mathbb{R}$ is said to be *smooth* if there is an open superset $U \supseteq X$ and a smooth function $g : U \longrightarrow \mathbb{R}$ which extends f, i.e., such that $g|_X = f$. We can proceed as in the previous example using composition to define $\mathcal{C}^\infty(X)$, the $\mathcal{C}^\infty$-ring of real valued functions defined on X. An important example, that uses this generalization and the previous example, is $\mathcal{C}^\infty(\mathbb{N})/K$, where $\mathcal{C}^\infty(\mathbb{N})$ is the ring of smooth functions on the natural numbers, and K is the ideal of eventually vanishing functions. In the final model, this ring will act as a set of *infinitely large natural numbers.*

Example 17. *A* $\mathcal{C}^\infty$*-ring* A *is called* finitely generated *if it is isomorphic to one of the form* $\mathcal{C}^\infty(\mathbb{R}^n)/I$*, for some* $n \in \mathbb{N}$ *and some finitely generated ideal* $I = (i_1, \dots, i_m)$*. For example, given an open subset* $U \subseteq \mathbb{R}^n$*, we can find a smooth function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that* $f(x) \neq 0$ *if and only if* $x \in U$*. So* U *is diffeomorphic to the closed set* $\hat{U} = \{(x, y) \mid y \cdot f(x) = 1\} \subseteq \mathbb{R}^{n+1}$*. Hence we have the isomorphism of* $\mathcal{C}^\infty$*-rings*

$$\mathcal{C}^\infty(U) \simeq \mathcal{C}^\infty(\mathbb{R}^{n+1})/(y \cdot f(x) - 1).$$

This proves that the $\mathcal{C}^\infty$*-ring* $\mathcal{C}^\infty(U)$ *is finitely generated. Using this result and Whitney's embedding theorem it is possible to prove that for a finite dimensional second countable Hausdorff manifold* M*, the* $\mathcal{C}^\infty$*-ring* $\mathcal{C}^\infty(M)$ *is finitely generated too* (*see* [*4*, *5*]).

Therefore, the category $\mathbb{L}$ of finitely generated $\mathcal{C}^\infty$-rings seems a good step toward the goal to embed finite dimensional manifolds in a category with infinitesimal objects. However, in general function spaces can not be constructed in $\mathbb{L}$. In order to have these function spaces, the first step is to extend the category $\mathbb{L}$ in the category $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ of presheaves on $\mathbb{L}$, i.e., of functors $F : \mathbb{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$:

$$\mathbf{Man} \subseteq \mathbb{L} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}.$$

This is a natural step in this context because the embedding $\mathbb{L} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is a well-know result in the category theory (Yoneda embedding), and because the category $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is a topos. So, we can concretely see the possibility to embed the category of smooth manifolds in a topos containing infinitesimal objects. Let us note that manifolds are directly embedded in $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ without "an extension with new infinitesimal points", so the approach is very different with respect to nonstandard analysis ([82]).

So, what is the ring of scalars representing the geometric line in the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$? If A, $B \in \mathbb{L}$ are finitely generated $\mathcal{C}^\infty$-rings, and $f : A \longrightarrow B$ is a $\mathcal{C}^\infty$-homomorphism, then this geometric line is represented by the functor

$$R(A) = \mathbb{L}(A, \mathcal{C}^\infty(\mathbb{R})), \tag{6}$$

$$R(A \xrightarrow{f} B) : g \in R(A) \mapsto g \circ f \in R(B), \tag{7}$$

corresponding, via the Yoneda embedding, to the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R})$. Analogously, the set of first order infinitesimal D corresponds in the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ to the functor

$$D(A) = \mathbb{L}(A, \mathcal{C}^\infty(\mathbb{R})/(x^2)), \tag{8}$$

$$D(A \xrightarrow{f} B) : g \in D(A) \mapsto g \circ f \in D(B). \tag{9}$$

Really, the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is not a final model of SDG for several reasons. Among these, we can mention that the properties like $1 \neq 0$ or $\forall r \in \mathbb{R}$ (x is invertible $\vee$ $(1-x)$ is invertible) are nonprovable in $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$, and this is essentially because the embedding $\mathbf{Man} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ does not preserve open covers. A description of the final models is outside the scopes of the present article. For more details, see, e.g., [4] and references therein.

It is in the opinion of important researchers in SDG ([4], see citation at the end of Sect. 4) that these topos models are not sufficiently simple, even if, at the same time, they are very rich and formally powerful. For these reasons, SDG is usually presented in an "axiomatic" way, in the framework of a naive intuitionistic set theory*, but with explicit introduction of particular axioms useful to deal with smooth spaces (i.e., objects of $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ or a better model) and smooth functions (i.e., arrows of $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ or a better model). This possibility is due to the above mentioned internal language of a topos (that represents its intuitionistic semantics). For example, a basic assumption is the so-called Kock–Lawvere axiom:

Axiom *R is a ring and we define $D := \left\{h \in R \,|\, h^2 = 0\right\}$, called the set of first order infinitesimal. They satisfy*

$$\forall f : D \longrightarrow R \; \exists! \, m \in R : \; \forall h \in D : \; f(h) = f(0) + h \cdot m. \tag{10}$$

The universal quantifier "for every function $f : D \longrightarrow R$" really means "for every set theoretical function from D to R", but definable using intuitionistic

*Exactly as almost every mathematician works in naive (classical) set theory. On the other hand, to work in SDG, one has to learn to work in intuitionistic logic, i.e., avoiding the law of the excluded middle, the proofs by reduction ad absurdum ending with a double negation, the full De Morgan laws, the equivalence between double negation and affirmation, the full equivalence between universal and existential quantifiers through negation, the axiom of choice, etc.

logic (hence, classically it is a subset of $\mathcal{C}^\infty(\mathbb{R})$). In semantical terms, this corresponds to "for every arrow in the model $\mathbf{Set}^{\mathbb{L}^{\text{op}}}$", i.e., for every smooth natural transformation between the functor D (see (8) and (9)) and the functor R (see (6) and (7)). It is not surprising to assert that (10) is incompatible with classical logic: applying the Kock–Lawvere axiom (10) with the function

$$f(h) = \begin{cases} 1 \text{ if } & h \neq 0 \\ 0 \text{ if } & h = 0 \end{cases} \tag{11}$$

and considering the hypothesis $\exists\, h_0 \in D : \; h_0 \neq 0$, we obtain

$$1 = 0 + h_0 \cdot m.$$

Squaring this equality, we obtain $1 = 0$. This incompatibility with classical logic is a natural motivation to consider intuitionistic logic, only in a context of topos theory and only if one is already thinking on the existence of models like $\mathbf{Set}^{\mathbb{L}^{\text{op}}}$. In another context, we think that the more natural idea is to criticize (10) asking some kind of limitation on the class of functions to which it can be really applied.

Finally we cite that the work of [76] has been the base for several other researches tempting to formalize nilpotent infinitesimal methods. In this direction, we can cite Weil functors (see [1, 79, 83]) and the recent [80].

7. The Cartesian Closure of a Category of Figures: Motivations and Basic Hypotheses

The ideas used in this section arise from analogous ideas about diffeological spaces and Frölicher spaces (see Sect. 4). In particular, our first references are [2, 18]. For these reasons, in this section we will not present the proofs of the most elementary facts; these can be easily generalized from the analogous proofs of [1, 2, 18] or [6].

We present the definition of cartesian closure starting from a concrete category $\mathcal{F}$ of topological spaces (satisfying few conditions), and embedding it in a cartesian closed category $\bar{\mathcal{F}}$. We will call $\bar{\mathcal{F}}$ the *cartesian closure of* $\mathcal{F}$.
In this section we will assume the following hypotheses on the category $\mathcal{F}$:

1. $\mathcal{F}$ is a subcategory of the category of topological spaces **Top**, and contains all the constant maps $c : H \longrightarrow X$ and all the open subspaces $U \subseteq H$ (with the induced topology) of every object $H \in \mathcal{F}$. The corresponding inclusion $i : U \hookrightarrow H$ is also an arrow of $\mathcal{F}$, i.e. $i \in \mathcal{F}_{UH} := \mathcal{F}(U, H)$.

In the following, we will denote by $|-| \;:\; \mathcal{F} \longrightarrow \mathbf{Set}$ the forgetful functor, which associates to any $H \in F$ its support set $|H| \in \mathbf{Set}$. Moreover, with τ_H we will denote the topology of H, and with $(U \prec H)$ the topological subspace of

H induced on the open set $U \in \mathcal{T}_H$. The remaining assumptions on $\mathcal{F}$ are the following:

2. The category $\mathcal{F}$ is closed with respect to restrictions to open sets, that is if $f \in \mathcal{F}_{HK}$ and U, V are open sets in H, K, respectively, and finally $f(U) \subseteq V$, then $f|_U \in \mathcal{F}(U \prec H, V \prec K)$;

3. Every topological space $H \in \mathcal{F}$ has the following "sheaf property": let H, $K \in \mathcal{F}$ be two objects of $\mathcal{F}$, $(H_i)_{i\in I}$ an open cover of H and $f : |H| \longrightarrow |K|$ a map such that
$$\forall i \in I : f|_{H_i} \in \mathcal{F}(H_i \prec H, K),$$
then $f \in \mathcal{F}_{HK}$.

If we want to generalize the definition of diffeological space to the regularity class $\mathcal{C}^n$, $n \leq +\infty$, embedding finite dimensional $\mathcal{C}^n$-manifolds, we can set $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, the category having as objects open sets $U \subseteq \mathbb{R}^u$ (with the induced topology), for some $u \in \mathbb{N}$ depending on U, and with hom-set the usual space $\mathcal{C}^n(U, V)$ of $\mathcal{C}^n$ functions between the open sets $U \subseteq \mathbb{R}^u$ and $V \subseteq \mathbb{R}^v$. Thus, $\boldsymbol{\mathcal{C}}^n := \overline{\mathbf{O}\mathbb{R}^n}$, the category of $\mathcal{C}^n$-diffeological spaces, is the cartesian closure of the category $\mathbf{O}\mathbb{R}^n$.

In general, what type of category $\mathcal{F}$ we have to choose depends on the setting we need: e.g., in case we want to consider manifolds with boundary, we have to take the analogous of the above mentioned category $\mathbf{O}\mathbb{R}^n$, but having as objects sets of type $U \subseteq \mathbb{R}^u_+ = \{x \in \mathbb{R}^u \,|\, x_u \geq 0\}$.

7.1. The cartesian closure and its first properties

The basic idea to define a space X of regularity $\mathcal{C}^n$, which faithfully generalizes the notion of manifold, is to substitute the notion of chart by a family of mappings $d : H \longrightarrow X$ of type $H \in \mathcal{F}$. Indeed, for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$ these mappings are of type $d : U \longrightarrow X$ with U open in some $\mathbb{R}^u$, thus they can be thought of as u-dimensional figures on X (see also Sects. 5 and 4). The idea is that a $\boldsymbol{\mathcal{C}}^n$ space can be thought as a support set together with the specification of all the finite-dimensional figures on the space itself. Generally speaking, we can think of $\mathcal{F}$ as a category of *types of figures* ([21]). Always considering the case $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, we can also think $\mathcal{F}$ as a category which represents *a well-known notion of regular space and regular function*: with the cartesian closure $\bar{\mathcal{F}}$, we want to extend this notion to a more general type of space (e.g., spaces of mappings). These are the ideas we have already seen in Sect. 5 in the case of diffeological spaces, only suitably generalized to a category of topological spaces $\mathcal{F}$ instead of $\mathcal{F} = \mathbf{O}\mathbb{R}^infty$, which is the case of diffeology. We will see that this generalization permits to obtain, for a suitable choice of the category of figures $\mathcal{F}$, a category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ where every diffeological space $X \in \boldsymbol{\mathcal{C}}^\infty$ can be extended adding new infinitely close points ${}^\bullet X \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$. It is possible to see that the corresponding extension functor ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ has very good properties ([34]).

Definition 18. *In the sequel we will frequently use the notation $f \cdot g := g \circ f$ for the composition of maps so as to facilitate the lecture of diagrams, but we will continue to evaluate functions "on the right" hence $(f \cdot g)(x) = g(f(x))$.*

Objects and arrows of $\bar{\mathcal{F}}$ generalize the same notions of the diffeological setting (see Sect. 5.).

Definition 19. *If X is a set, then we say that $(\mathcal{D}, X)$ is an object of $\bar{\mathcal{F}}$ (or simply an $\bar{\mathcal{F}}$-object) if $\mathcal{D} = \{\mathcal{D}_H\}_{H \in \mathcal{F}}$ is a family with*

$$\mathcal{D}_H \subseteq \mathbf{Set}(|H|, X) \quad \forall H \in \mathcal{F}.$$

We indicate by the notation $\mathcal{F}_{JH} \cdot \mathcal{D}_H$ the set of all the compositions $f \cdot d$ of functions $f \in \mathcal{F}_{JH}$ and $d \in \mathcal{D}_H$. The family $\mathcal{D}$ has finally to satisfy the following conditions:

1. $\mathcal{F}_{JH} \cdot \mathcal{D}_H \subseteq \mathcal{D}_J$.
2. *$\mathcal{D}_H$ contains all the constant maps $d : |H| \longrightarrow X$.*
3. *Let $H \in \mathcal{F}$, $(H_i)_{i \in I}$ an open cover of H and $d : |H| \longrightarrow X$ a map such that $d|_{H_i} \in \mathcal{D}_{(H_i \prec H)}$, then $d \in \mathcal{D}_H$.*

Finally, we set $|(\mathcal{D}, X)| := X$ to denote the underlying set of the space $(\mathcal{D}, X)$.

Because of condition 1, we can think of $\mathcal{D}_H$ as the set of *all* the regular functions defined on the "well-known" object $H \in \mathcal{F}$ and with values in the new space X; in fact, this condition says that the set of figures $\mathcal{D}_H$ is closed with respect to re-parametrizations with a generic $f \in \mathcal{F}_{JH}$. Condition 3 is the above mentioned sheaf property, and asserts that the property of being a figure $d \in \mathcal{D}_H$ has a local character depending on $\mathcal{F}$.

We will frequently write $d \in_H X$ to indicate that $d \in \mathcal{D}_H$, and we can read it[⋆] saying that *d is a figure of X of type H* or *d belongs to X at the level H* or *d is a generalized element of X of type H*.

The definition of arrow $f : X \longrightarrow Y$ between two spaces X, $Y \in \bar{\mathcal{F}}$ is the usual one for diffeological spaces, that is f takes, through composition, generalized elements $d \in_H X$ of type H in the domain X to generalized elements of the same type in the codomain Y:

[⋆]The following are common terminologies used in topos theory, see [4, 5, 21]

Definition 20. *Let X, Y be $\bar{\mathcal{F}}$-objects, then we will write*

$$f : X \longrightarrow Y$$

or, more precisely if needed*

$$\bar{\mathcal{F}} \vDash f : X \longrightarrow Y$$

iff f maps the support set of X into the support set of Y:

$$f : |X| \longrightarrow |Y|$$

and

$$d \cdot f \in_H Y$$

for every type of figure $H \in \mathcal{F}$ and for every figure d of X of that type, i.e. $d \in_H X$. In this case, we will also use the notation $f(d) := d \cdot f$.

Note that we have $f : X \longrightarrow Y$ in $\bar{\mathcal{F}}$ iff

$$\forall H \in \mathcal{F}\ \forall x \in_H X : \quad f(x) \in_H Y,$$

moreover $X = Y$ iff

$$\forall H \in \mathcal{F}\ \forall d : \quad d \in_H X \iff d \in_H Y.$$

These and many other properties justify the notation $\in_H$ and the name "generalized elements".

With these definitions $\bar{\mathcal{F}}$ becomes a category. Note that it is, in general, in the second Grothendieck universe (see [26, 77]) because $\mathcal{D}$ is a family indexed in the set of objects of $\mathcal{F}$ (this is not the case for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, which is a set and not a class).

The simplest $\bar{\mathcal{F}}$-object is $\bar{K} := (\mathcal{F}_{(-)K}, |K|)$ for $K \in \mathcal{F}$, where we recall that $\mathcal{F}_{HK} = \mathcal{F}(H, K) = \left\{ f \mid H \xrightarrow{f} K \text{ in } \mathcal{F} \right\}$. For the space $\bar{K} \in \bar{\mathcal{F}}$ we have that

$$\bar{\mathcal{F}} \vDash \bar{K} \xrightarrow{d} X \quad \iff \quad d \in_K X.$$

Moreover, $\mathcal{F}(H, K) = \bar{\mathcal{F}}(\bar{H}, \bar{K})$. Therefore, $\mathcal{F}$ is fully embedded in $\bar{\mathcal{F}}$ if $\bar{H} = \bar{K}$ implies $H = K$; e.g., this is true if the given category of figures $\mathcal{F}$ verifies the following condition:

$$|H| = |K| = S \ \text{ and } \ H \xrightarrow{1_S} K \xrightarrow{1_S} H \quad \Longrightarrow \quad H = K.$$

*We shall frequently use notations of type $\mathbb{C} \vDash f : A \longrightarrow B$ if we need to specify better the category $\mathbb{C}$ we are considering.

For example, this is true for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$.

Moreover, let us note that the composition of two smooth functions in $\bar{\mathcal{F}}$ of type $d : \bar{H} \longrightarrow X$ and $f : X \longrightarrow \bar{K}$ for H, $K \in \mathcal{F}$, gives $d \cdot f \in \bar{\mathcal{F}}(\bar{H}, \bar{K}) = \mathcal{F}(H, K)$, which is an arrow in the old category of types of figures $\mathcal{F}$.

Another way to construct an object of $\bar{\mathcal{F}}$ on a given support set X is to generate it starting from a given family $\mathcal{D}^0 = (\mathcal{D}^0_H)_H$, with $\mathcal{D}^0_H \subseteq \mathbf{Set}(|H|, X)$ for any $H \in \mathcal{F}$, closed with respect to constant functions, i.e., such that

$$\forall H \in \mathcal{F}\ \forall d : |H| \longrightarrow X \text{ is constant} \quad \Longrightarrow \quad d \in \mathcal{D}^0_H.$$

We will indicate this space by $(\mathcal{F} \cdot \mathcal{D}^0, X)$. Its figures are, locally, compositions $f \cdot d$ with $f \in \mathcal{F}_{HK}$ and $d \in \mathcal{D}^0_K$. More precisely, $\delta \in_H (\mathcal{F} \cdot \mathcal{D}^0, X)$ iff $\delta : |H| \longrightarrow X$ and for every $h \in |H|$ there exists an open neighborhood U of h in H, a space $K \in \mathcal{F}$, a figure $d \in \mathcal{D}^0_K$ and $f : (U \prec H) \longrightarrow K$ in $\mathcal{F}$ such that $\delta|_U = f \cdot d$. Diagrammatically we have

$$\begin{tikzcd}
H \arrow[r, "\delta"] & X \\
h \in U \arrow[u, hook] \arrow[ur, "\delta|_U"] \arrow[r, "f"'] & K \arrow[u, "d"']
\end{tikzcd}$$

On each space $X \in \bar{\mathcal{F}}$ we can put the final topology $\mathcal{T}_X$ for which any figure $d \in_H X$ is continuous, that is

Definition 21. *If $X \in \bar{\mathcal{F}}$, then we say that a subset $U \subseteq |X|$ is* open *in X, and we will write $U \in \mathcal{T}_X$ iff $d^{-1}(U) \in \mathcal{T}_H$ for any $H \in \mathcal{F}$ and any $d \in_H X$.*

With respect to this topology any arrow of $\bar{\mathcal{F}}$ is continuous, and we still have the initial $\mathcal{T}_H$ in the space $\bar{H}$, that is $\mathcal{T}_H = \mathcal{T}_{\bar{H}}$ (recall that, because of the fundamental hypotheses on the category of types of figures $\mathcal{F}$ fixed in Sect. 7, every type of figure $H \in \mathcal{F}$ is a topological space).

Recalling that in the case $\mathcal{F} = \mathbf{O}\mathbb{R}^inf ty$ we obtain that the cartesian closure $\bar{\mathcal{F}}$ is the category of diffeological spaces, it can be useful to cite here [6]:

> *Even if diffeology is a theory which avoids topology on purpose, topology is not completely absent from its content. But, in contrary to some approach of standard differential geometry, here the topology is a byproduct of the main structure, that is diffeology. Locality, through local smooth maps, or local diffeomorphisms, is introduced without referring to any topology* a priori *but will suggest the definition of a topology* a posteriori *[i.e., $\mathcal{T}_X$].*

Ultimately, this choice is due to the necessity to obtain a cartesian closed category. In fact, if we do not start from a primitive notion of topology in the definition of $\bar{\mathcal{F}}$-space, we can obtain cartesian closedness without having the problem to define a topology in the set of maps $\bar{\mathcal{F}}(X,Y)$. Indeed, this is not an easy problem, and classical solutions like the compact-open topology (see, e.g., [1, 84] and references therein) is not applicable to the smooth case. In fact, the compact-open topology, which essentially coincides with the topology of uniform convergence, is well suited for continuous maps $f : X \longrightarrow Y$ between locally compact Hausdorff topological spaces X and Y (indeed, the category of these topological spaces is cartesian closed, see [27]). It can be generalized to the case of $\mathcal{C}^k$-regularity using k-jets ($k \in \mathbb{N}_{>0}$), i.e., using Taylor's formulae up to k-th order (see, e.g., [1]), but a generalization including the smooth case $\mathcal{C}^\infty$ even for a compact domain X fails. In fact, for X compact and Y a Banach space, the space $\mathcal{C}^k(X,Y)$ with the $\mathcal{C}^k$ compact-open topology is normable, but the space $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is not normable, so its topology cannot be the compact-open one (see also Sect. 3 and Theorem 2 for more details).

The study of the relationships between different topologies on the space of maps $\mathcal{C}^\infty(M,N)$ for M, N manifolds, is not completely solved (see again [1] for some results in this direction).

7.2. Categorical properties of the cartesian closure

We shall now examine subobjects in $\bar{\mathcal{F}}$ and their relationships with restrictions of functions; after this we will analyze completeness, co-completeness and cartesian closure of $\bar{\mathcal{F}}$.

Definition 22. *Let $X \in \bar{\mathcal{F}}$ be a space in the cartesian closure of $\mathcal{F}$, and $S \subseteq |X|$ a subset, then we define*

$$(S \prec X) := (\mathcal{D}, S),$$

where, for every type of figure $H \in \mathcal{F}$, we have set

$$d \in \mathcal{D}_H \quad :\Longleftrightarrow \quad d : |H| \longrightarrow S \quad \text{and} \quad d \cdot i \in_H X.$$

Here $i : S \hookrightarrow |X|$ is the inclusion map. In other words, we have a figure d of type H in the subspace S iff composing d with the inclusion map i we obtain a figure of the same type in the superspace X. We will call $(S \prec X)$ the subspace induced on S by X.

Using this definition only it is very easy to prove that $(S \prec X) \in \bar{\mathcal{F}}$ and that its topology $\mathcal{T}_{(S \prec X)}$ contains the topology induced by $\mathcal{T}_X$ on the subset S.

Moreover, we have that $\mathcal{T}_{(S\prec X)} \subseteq \mathcal{T}_X$ if S is open in X, hence in this case we have on $(S \prec X)$ exactly the induced topology.

Finally we can prove that these subspaces have good relationships with restrictions of maps:

Theorem 23. *Let $f : X \longrightarrow Y$ be an arrow of $\bar{\mathcal{F}}$ and U, V be subsets of $|X|$ and $|Y|$, respectively, such that $f(U) \subseteq V$, then*

$$(U \prec X) \longrightarrow f|_U (V \prec Y) \quad in \quad \bar{\mathcal{F}}.$$

Using our notation for subobjects we can prove the following useful and natural properties, directly from Definition 22:

- $(U \prec \bar{H}) = \overline{(U \prec H)}$ for U open in $H \in \mathcal{F}$ (recall the definition of $\bar{H} \in \bar{\mathcal{F}}$, for $H \in \mathcal{F}$, given in Sect. 7.1. and also recall that, because of the hypotheses of Sect. 7. on the category $\mathcal{F}$, the subspace $(U \prec H)$ is a type of figure, i.e. $(U \prec H) \in \mathcal{F}$, and we can thus apply the operator $\bar{(-)} : \mathcal{F} \longrightarrow \bar{\mathcal{F}}$ of inclusion of the types of figures $\mathcal{F}$ into the cartesian closure $\bar{\mathcal{F}}$);
- $i : (S \prec X) \hookrightarrow X$ is the lifting of the inclusion $i : S \hookrightarrow |X|$ from **Set** to $\bar{\mathcal{F}}$;
- $(|X| \prec X) = X$
- $(S \prec (T \prec X)) = (S \prec X) \quad \text{if} \quad S \subseteq T \subseteq |X|$;
- $(S \prec X) \times (T \prec Y) = (S \times T \prec X \times Y)$.

These properties imply that the relation $X \subseteq Y$ iff $|X| \subseteq |Y|$ and $(|X| \prec Y) = X$ is a partial order. Note that this relation is stronger than saying that the inclusion is an arrow, because it asserts that X and the inclusion verify the universal property of $(|X| \prec Y)$, that is X is a subobject of Y. A trivial but useful property of this subobjects notation is the following

Corollary 24. *Let $S \subseteq |X'|$ and $X' \subseteq X$ in $\bar{\mathcal{F}}$, then*

$$(S \prec X') = (S \prec X),$$

that is, in the operator $(S \prec -)$ we can change the superspace X with any one of its subspaces $X' \subseteq X$ containing S.

P r o o f. In fact $X' \subseteq X$ means $X' = (|X'| \prec X)$ and hence $(S \prec X') = (S \prec (|X'| \prec X)) = (S \prec X)$ because of the previous properties of the operator $(- \prec -)$. ∎

An expected property that transfers from $\mathcal{F}$ to $\bar{\mathcal{F}}$ is the sheaf property; in other words, it states that the property of being an arrow of the cartesian closure $\bar{\mathcal{F}}$ is a local property.

Theorem 25. *Let X, $Y \in \bar{\mathcal{F}}$ be spaces in the cartesian closure, $(U_i)_{i\in I}$ an open cover of X and $f : |X| \longrightarrow |Y|$ a map from the support set of X to that of Y such that*

$$\bar{\mathcal{F}} \models (U_i \prec X) \xrightarrow{f|_{U_i}} Y \quad \forall i \in I.$$

Then

$$\bar{\mathcal{F}} \models X \xrightarrow{f} Y.$$

Completeness and co-completeness are analyzed in the following theorem. For its standard proof see, e.g., [2] for a similar theorem.

Theorem 26. *Let $(X_i)_{i\in I}$ be a family of objects in $\bar{\mathcal{F}}$ and $p_i : |X| \longrightarrow |X_i|$ be a map, for every $i \in I$. Let us define*

$$d \in_H X \quad :\Longleftrightarrow \quad d : |H| \longrightarrow |X| \quad \text{and} \quad \forall\, i \in I : \quad d \cdot p_i \in_H X_i$$

then $(X \xrightarrow{p_i} X_i)_{i\in I}$ is a lifting of $(|X| \xrightarrow{p_i} |X_i|)_{i\in I}$ in $\bar{\mathcal{F}}$.

Moreover, let $j_i : |X_i| \longrightarrow |X|$ be a map, for every $i \in I$, and let us suppose that

$$\forall\, x \in |X|\; \exists\, i \in I\, \exists\, x_i \in X_i : \; x = j_i(x_i).$$

Let us define $d \in_H X$ iff $d : |H| \longrightarrow |X|$ and for every $h \in |H|$ there exists an open neighborhood U of h in H, an index $i \in I$ and a figure $\delta \in_U X_i$ such that $d|_U = \delta \cdot j_i$; then we have that $(X_i \xrightarrow{j_i} X)_{i\in I}$ is a co-lifting of $(|X_i| \xrightarrow{j_i} |X|)_{i\in I}$ in $\bar{\mathcal{F}}$.

The category of $\bar{\mathcal{F}}$ spaces is thus complete and co-complete and we can hence consider spaces like quotient spaces $X/\sim$, disjoint sums $\sum_{i\in I} X_i$, arbitrary products $\prod_{i\in I} X_i$, equalizers, etc.

Directly from the definitions of lifting and co-lifting, it is easy to prove that on quotient spaces we exactly have the quotient topology and that on any product we have a topology stronger than the product topology. We can write these assertions in the following symbolic way:

$$\mathcal{T}_{X/\sim} = \mathcal{T}_X / \sim, \tag{12}$$

$$\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}_{X\times Y}, \tag{13}$$

where X and Y are $\bar{\mathcal{F}}$ spaces, $\sim$ is an equivalence relation on $|X|$, $(X/\sim) \in \bar{\mathcal{F}}$ is the quotient space, $\mathcal{T}_X/\sim$ is the quotient topology, and $\mathcal{T}_X \times \mathcal{T}_Y$ is the product topology. Analogously, let $j_i : X_i \longrightarrow \sum_{i\in I} X_i$ be the canonical injections in the disjoint sum of the family of $\bar{\mathcal{F}}$-spaces $(X_i)_{i\in I}$, i.e. $j_i(x) = (x, i)$. Then, we can prove that A is open in $\sum_{i\in I} X_i$ if and only if

$$\forall i \in I : \quad j_i^{-1}(A) \in \mathcal{T}_{X_i}, \tag{14}$$

that is on the disjoint sum we have exactly the colimit topology. Because any colimit can be obtained as a lifting from **Set** of quotient spaces and disjoint sums (see [27]), we have the general result that the topology on the colimit of $\bar{\mathcal{F}}$-spaces is exactly the colimit topology. In symbolic notations we can write

$$\mathcal{T}\left(\operatorname*{colim}_{i\in I} X_i\right) = \operatorname*{colim}_{i\in I} \mathcal{T}_{X_i}.$$

Finally, if we define

$$\mathcal{D}_H := \{d : |H| \longrightarrow \bar{\mathcal{F}}(X,Y) \mid \bar{H}\times X \xrightarrow{d^\vee} Y \quad \text{in} \quad \bar{\mathcal{F}}\} \quad \forall H \in \mathcal{F}$$

(we recall that we use the notations $d^\vee(h,x) := d(h)(x)$ and $\mu^\wedge(x)(y) := \mu(x,y)$, see Sect. 1), then $\langle \mathcal{D}, \bar{\mathcal{F}}(X,Y)\rangle =: Y^X$ is an object of $\bar{\mathcal{F}}$. With this definition, see, e.g., [18] or [2], it is easy to prove that $\bar{\mathcal{F}}$ is cartesian closed, i.e., that the $\bar{\mathcal{F}}$-isomorphism $(-)^\vee$ realizes

$$(Y^X)^Z \simeq Y^{Z\times X}.$$

8. Observables on $\boldsymbol{\mathcal{C}}^n$ Spaces

If our aim is to embed the category of $\mathcal{C}^n$ manifolds into a cartesian closed category, then the most natural way to apply the results of the previous Sect. 7 is to take as category $\mathcal{F}$ of types of figures $\mathcal{F} = \mathbf{Man}$, that is to consider directly the cartesian closure of the category of finite dimensional $\mathcal{C}^n$ manifolds⋆. We shall not follow this idea for several reasons. We will consider instead $\boldsymbol{\mathcal{C}}^n := \overline{\mathbf{O}\mathbb{R}^n}$, that is the cartesian closure of the category $\mathbf{O}\mathbb{R}^n$ of open sets and $\mathcal{C}^n$ maps. For $n=\infty$ this gives exactly diffeological spaces. Indeed, as we noted in the previous Sect. 7., $\overline{\mathbf{Man}}$ is in the second Grothendieck universe and, essentially for simplicity, from this point of view the choice $\mathcal{F} = \mathbf{O}\mathbb{R}^n$ is better. In spite of this choice, it is natural to expect, and in fact we will prove it, that the categories of both finite and infinite-dimensional manifolds are faithfully embedded in the previous $\boldsymbol{\mathcal{C}}^n = \overline{\mathbf{O}\mathbb{R}^n}$. Another reason to choose our definition of $\boldsymbol{\mathcal{C}}^n$ is that in this way the category $\boldsymbol{\mathcal{C}}^n$ is more natural to accept against $\overline{\mathbf{Man}}$; hence, ones again, we are opting for a reason of simplicity. We will see that manifolds modelled in convenient vector spaces are faithfully embedded in $\boldsymbol{\mathcal{C}}^n$, hence our choice to take finite dimensional objects in the definition of $\boldsymbol{\mathcal{C}}^n = \overline{\mathbf{O}\mathbb{R}^n}$ is not restrictive from this point of view.

In this section, we pay attention to another type of map which goes "in the opposite direction" with respect to figures $d : K \longrightarrow X$. They are important also

⋆We shall not formally assume any hypothesis on the topology of a manifold because we will not need it in what follows. Moreover, if not differently specified, the word "manifold" will always mean "finite dimensional manifold".

because we shall use them to introduce new infinitesimally closed points for any $X \in \boldsymbol{\mathcal{C}}^n$.

Definition 27. *Let X be an $\bar{\mathcal{F}}$ space, then we say that*

$$UK \text{ is a zone (in } X)$$

iff $U \in \mathcal{T}_X$, i.e., U is open in X, and $K \in \mathcal{F}$ is a type of figure. Moreover we say that

$$c \text{ is an observable on } UK \quad \textit{and we will write} \quad c \in^{UK} X$$

iff $c : (U \prec X) \longrightarrow \bar{K}$ is a map of the cartesian closure $\bar{\mathcal{F}}$.

So, the observables of a $\boldsymbol{\mathcal{C}}^n$ space X are simply the maps of class $\boldsymbol{\mathcal{C}}^n$ (i.e., are the arrows of this category) defined on an open set of X and with values in an open set $K \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$. Recall (see Sect. 7.1) that for any open set $K \in \mathbf{O}\mathbb{R}^n$, in the $\boldsymbol{\mathcal{C}}^n$ space $\bar{K}$ we take as figures of type $H \in \mathbf{O}\mathbb{R}^n$ all the ordinary $\mathcal{C}^n$-maps $\mathcal{C}^n(H, K)$, i.e., we have

$$\bar{K} = (\mathcal{C}^n(-, K), K).$$

Therefore, the composition of figures $d \in_H X$ with observables $c \in^{UK} X$ gives ordinary $\mathcal{C}^n$ maps:

$$d|_S \cdot c \in \mathcal{C}^n(S, K), \quad \text{where} \quad S := d^{-1}(U),$$

$$\boldsymbol{\mathcal{C}}^n \vDash \overline{(S \prec H)} \xrightarrow{d|_S} (U \prec X) \xrightarrow{c} \bar{K}.$$

From our previous theorems of Sect. 7., it follows that $\boldsymbol{\mathcal{C}}^n$ functions $f : X \longrightarrow Y$ take observables on the codomain to observables on the domain, i.e.:

$$c \in^{UK} Y \quad \Longrightarrow \quad f|_S \cdot c \in^{SK} X, \tag{15}$$

where $S := f^{-1}(U)$:

$$\begin{tikzcd} (S \prec X) \arrow[r, "f|_S"] \arrow[dr, "f|_S \cdot c"'] & (U \prec Y) \arrow[d, "c"] \\ & \bar{K} \end{tikzcd}$$

Therefore, isomorphic $\boldsymbol{\mathcal{C}}^n$ spaces have isomorphic sets of figures and observables, and the isomorphisms are given by suitable simple compositions.

The following definition is useful to understand when the points of a space are uniquely identified by all the observables. This condition is also connected with the faithfulness of the extension functor ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ which adds infinitesimally closed points to every diffeological space $X \in \boldsymbol{\mathcal{C}}^\infty$, obtaining ${}^\bullet X \supseteq X$ (see the following Sect. 12).

Definition 28. *If $X \in \mathcal{C}^n$ is a $\mathcal{C}^n$ space and x, $y \in |X|$ are two points, then we write*

$$x \asymp y$$

iff for every zone UK and every observable $c \in^{UK} X$ we have

1. $x \in U \iff y \in U$;
2. $x \in U \implies c(x) = c(y)$.

In this case we will read the relation $x \asymp y$ saying x and y are identified *in X. Moreover, we say that X* is separated *iff $x \asymp y$ implies $x = y$ for any x, $y \in |X|$.*

We point out that if two points are identified in X, then a generic open set $U \in \tau_X$ contains the first one if and only if it contains the second too (take a constant observable $c : U \longrightarrow \mathbb{R}$). Furthermore, from (15) it follows that $\mathcal{C}^n$ functions $f : X \longrightarrow Y$ preserve the relation $\asymp$:

$$x \asymp y \text{ in } X \quad \implies \quad f(x) \asymp f(y) \text{ in } Y \qquad \forall x, y \in |X|.$$

Trivial examples of separated spaces can be obtained considering the objects $\bar{U} \in \mathcal{C}^n$ with $U \in \mathbf{O}\mathbb{R}^n$ (here $\overline{(-)} : \mathbf{O}\mathbb{R}^n \longrightarrow \mathcal{C}^n$ is the embedding of the types of figures $\mathbf{O}\mathbb{R}^n$ into $\mathcal{C}^n$, see 7.1) or taking subobjects of separated spaces. But the full subcategory of separated $\mathcal{C}^n$ spaces has sufficiently good properties, as proved in the following

Theorem 29. *The category of separated $\mathcal{C}^n$ spaces is complete and admits co-products. Moreover, if X, Y are separated, then Y^X is separated too, and hence separated spaces form a cartesian closed category.*

Sketch of the proof. We only do some considerations about co-products, because from the definition of lifting (see Theorem 26) it can be directly proved that products and equalizers of separated spaces are separated too. Let us consider a family $(\mathcal{X}_i)_{i \in I}$ of separated spaces with support sets $X_i := |\mathcal{X}_i|$. Constructing their sum in **Set**

$$X := \sum_{i \in I} X_i,$$

$$j_i \ : \ x \in X_i \longmapsto (x, i) \in X,$$

from the completeness of $\mathcal{C}^n$ we can lift this co-product of sets into a co-product $(\mathcal{X}_i \xrightarrow{\mathrm{j}_i} \mathcal{X})_{i \in I}$ in $\mathcal{C}^n$. To prove that $\mathcal{X}$ is separated we take two points x, $y \in X = |\mathcal{X}|$ identified in $\mathcal{X}$. These points are of the form $x = (x_r, r)$ and $y = (y_s, s)$, with $x_r \in X_r, y_s \in X_s$ and $r, s \in I$. We want to prove that r and s are necessarily equal. In fact, from (14), for a generic $A \subseteq \mathcal{X}$ we have that

$$A \in \tau_{\mathcal{X}} \quad \iff \quad \forall\, i \in I : \quad j_i^{-1}(A) \in \tau_{\mathcal{X}_i},$$

and hence $X_r \times \{r\}$ is open in $\mathcal{X}$ and $x = (x_r, r) \asymp y = (y_s, s)$ implies

$$(x_r, r) \in X_r \times \{r\} \iff (y_s, s) \in X_r \times \{r\} \qquad \text{hence} \qquad r = s.$$

Thus $x = y$ iff x_r and $y_s = y_r$ are identified in $\mathcal{X}_r$ and this is a consequence of the following facts:

1. if U is open in $\mathcal{X}_r$ then $U \times \{r\}$ is open in $\mathcal{X}$;
2. if $c \in^{UK} \mathcal{X}_r$, then $\gamma(x, r) := c(x)\ \forall\, x \in U$ is an observable of $\mathcal{X}$ defined on $U \times \{r\}$.

Now, let us consider exponential objects. If $f,\ g \in |Y^X|$ are identified, to prove that they are equal is equivalent to prove that $f(x)$ and $g(x)$ are identified in Y for any x. To obtain this conclusion, it is sufficient to consider that the evaluation in x, i.e., the application $\varepsilon_x : \varphi \in |Y^X| \longmapsto \varphi(x) \in |Y|$, is a $\boldsymbol{\mathcal{C}}^n$ map, and hence from any observable $c \in^{UK} Y$ we can always obtain the observable $\varepsilon_x|_{U'} \cdot c \in^{U'K} Y^X$ where $U' := \varepsilon_x^{-1}(U)$. ■

Finally let us consider two $\boldsymbol{\mathcal{C}}^n$ spaces such that the topology $\tau_{X\times Y}$ is equal to the product of the topologies τ_X and τ_Y (recall that in general we have $\tau_X \times \tau_Y \subseteq \tau_{X\times Y}$). Then, if $x,\ x' \in |X|$ and $y,\ y' \in |Y|$, directly from the definition of the relation $\asymp$, it is possible to prove that $x \asymp x'$ in X and $y \asymp y'$ in Y if and only if $(x, y) \asymp (x', y')$ in $X \times Y$.

9. Manifolds as Objects of $\boldsymbol{\mathcal{C}}^n$

We can associate in a very natural way a $\boldsymbol{\mathcal{C}}^n$ space $\bar{M}$ to any $\mathcal{C}^n$ manifold $M \in \mathbf{Man}$ with the following

Definition 30. *The underlying set of $\bar{M}$ is the underlying set of the manifold, i.e. $|\bar{M}| := |M|$, and for every $H \in \mathbf{O}\mathbb{R}^n$ the figures $d : H \longrightarrow M$ of type H are all the ordinary $\mathcal{C}^n$ maps from H to the manifold M, i.e.,*

$$d \in_H \bar{M} \quad :\iff \quad d \in \mathbf{Man}(H, M).$$

This definition is only the trivial generalization from the smooth case to $\mathcal{C}^n$ of the embedding of manifolds into the category of diffeological spaces (see, e.g., [6]).

The space $\bar{M}$ is a $\boldsymbol{\mathcal{C}}^n$ space with the same topology $\tau_{\bar{M}}$ of the starting manifold M. Moreover, the observables of $\bar{M}$ are the most natural ones we could expect. In fact, as a consequence of the Definition 30, it follows that

$$c \in^{UK} \bar{M} \quad \iff \quad c \in \mathbf{Man}(U, K). \tag{16}$$

Hence, it is clear that the space $\bar{M}$ is separated, because from (16) we get that charts are observables of the space. The following theorem says that the application $M \mapsto \bar{M}$ from **Man** to $\boldsymbol{\mathcal{C}}^n$ is a full embedding. Therefore, it also states that the notion of $\boldsymbol{\mathcal{C}}^n$-space is a nontrivial generalization of the notion of manifold which includes infinite-dimensional spaces too.

Theorem 31. *Let M and N be $\mathcal{C}^n$ manifolds, then*

1. $\bar{M} = \bar{N} \quad \Longrightarrow \quad M = N$;
2. $\boldsymbol{\mathcal{C}}^n \vDash \bar{M} \xrightarrow{f} \bar{N} \quad \Longleftrightarrow \quad \mathbf{Man} \vDash M \xrightarrow{f} N.$

Hence **Man** *is fully embedded in* $\boldsymbol{\mathcal{C}}^n$.

Proof. 1) If (U, φ) is a chart on M and $A := \varphi(U)$, then $\varphi^{-1}|_A : A \longrightarrow M$ is a figure of $\bar{M}$, that is $\varphi^{-1}|_A \in_A \bar{M} = \bar{N}$. But, if $\psi : U \longrightarrow \psi(U) \subseteq \mathbb{R}^k$ is a chart of N, then it is also an observable of $\bar{N}$. We have hence obtained a figure $\varphi^{-1}|_A \in_A \bar{N}$ and an observable $\psi \in^{U\psi(U)} \bar{N}$ of the space $\bar{N}$. But composition of figures and observables gives ordinary $\mathcal{C}^n$ maps, that is the atlases of M and N are compatible.

2) For the implication $\Rightarrow$ we use the same ideas as above and furthermore that $\varphi^{-1}|_A \in_A \bar{M}$ implies $\varphi^{-1}|_A \cdot f \in_A \bar{N}$. Finally we can compose this A-figure of $\bar{N}$ with a chart (observable) of N obtaining an ordinary $\mathcal{C}^n$ map. The implication $\Leftarrow$ follows directly from the Definition 30. ■

Directly from these definitions we can prove that for two manifolds we also have

$$\overline{M \times N} = \bar{M} \times \bar{N}.$$

This property is useful to prove the properties stated in the following examples.

10. Examples of $\boldsymbol{\mathcal{C}}^n$ Spaces and Functions

1. Let M be a $\mathcal{C}^\infty$ manifold modelled on convenient vector spaces (see Sect. 4). We can define $\bar{M}$ analogously as above, saying that $d \in_H \bar{M}$ iff $d : H \longrightarrow M$ is a smooth map from H (open in some $\mathbb{R}^h$) to the manifold M. In this way, smooth curves on M are exactly the figures $c \in_{\mathbb{R}} \bar{M}$ of type $\mathbb{R}$ in $\bar{M}$. On M we obviously think of the natural topology, that is the identification topology with respect to some smooth atlas, which is also the final topology with respect to all smooth curves and hence is also the final topology $\mathcal{T}_{\bar{M}}$ with respect to all figures of $\bar{M}$. More easily, with respect to the previous case of finite dimensional manifolds (due to the results available for manifolds modelled on convenient vector spaces, see Sect. 4), it is possible to

study observables, obtaining that $c \in^{UK} \bar{M}$ if and only if $c : U \longrightarrow K$ is smooth as a map between manifolds modelled on convenient vector spaces. Moreover if (U, φ) is a chart of M on the convenient vector space E, then $\varphi : (U \prec \bar{M}) \longrightarrow (\varphi(U) \prec \bar{E})$ is $\boldsymbol{\mathcal{C}}^\infty$. Using these results it is easy to prove the analogous of Theorem 31 for the category of manifolds modelled on convenient vector spaces. Hence also classical smooth manifolds modelled on Banach spaces are embedded in $\boldsymbol{\mathcal{C}}^\infty$.

2. It is not difficult to prove that the following applications, frequently used e.g., in calculus of variations, are smooth, that is they are arrows of $\boldsymbol{\mathcal{C}}^\infty$.

   (a) The operator of derivation:

   $$\partial_i : u \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k) \longmapsto \frac{\partial u}{\partial x_i} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k).$$

   To prove that this operator is smooth, i.e., it is an arrow of the category $\boldsymbol{\mathcal{C}}^\infty$, we have to show that it takes figures of type $H \in \mathbf{O}\mathbb{R}^infty$ on its domain to figures of the same type on the codomain. Figures of type H of the space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$ are maps of type $d : H \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$, so that we have to consider the composition $d \cdot \partial_i$. Using cartesian closedness we get that $d^\vee : H \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is an ordinary smooth map. But, always due to cartesian closedness, the composition $d \cdot \partial_i : H \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$ is a figure if and only if its adjoint $(d \cdot \partial_i)^\vee : H \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is an ordinary smooth map, and by a direct calculation we get that $(d \cdot \partial_i)^\vee = \partial_{u+i} d^\vee$, where $u \in \mathbb{N}$ is the dimension of $H \subseteq \mathbb{R}^u$. In fact

   $$\begin{aligned} (d \cdot \partial_i)^\vee (h, r) &= \partial_i(d(h))(r) = \frac{\partial d(h)}{\partial x_i}(r) \\ &= \lim_{\delta \to 0} \frac{d(h)(r + \delta \vec{e}_i) - d(h)(r)}{\delta} \\ &= \lim_{\delta \to 0} \frac{d^\vee(h, r + \delta \vec{e}_i) - d^\vee(h, r)}{\delta} \\ &= \partial_{u+i} d^\vee(h, r), \end{aligned}$$

   where $\vec{e}_i = (0, \overset{i-1}{\ldots}, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. This equality proves that $d \cdot \partial_i$ is a figure and hence that the operator ∂_i is smooth.

   (b) We can proceed in an analogous way (but here we have to use the derivation under the integral sign) to prove that the integral operator

   $$\begin{aligned} i :\ & \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \\ & u \longmapsto \int_a^b u(-, s)\, \mathrm{d}s \end{aligned}$$

is smooth.

3. Because of cartesian closedness, the set-theoretical operations like the following ones are always $\mathcal{C}^n$ arrows (see, e.g., [26]):

   - composition:
$$(f, g) \in B^A \times C^B \quad \mapsto \quad g \circ f \in C^A;$$
   - evaluation:
$$(f, x) \in Y^X \times X \quad \mapsto \quad f(x) \in Y;$$
   - insertion:
$$x \in X \quad \mapsto \quad (x, -) \in (X \times Y)^Y.$$

4. Using the smoothness of the previous set-theoretical operations and the smoothness of the derivation and integral operators, we can easily prove that the classical operator of the calculus of variations is smooth
$$\mathcal{I}(u)(t) := \int_a^b F[u(t,s), \partial_2 u(t,s), s]\, \mathrm{d}s,$$
$$\mathcal{I} : \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^k) \longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}),$$
where the function $F : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \longrightarrow \mathbb{R}$ is smooth.

5. Inversion between smooth manifolds modelled on Banach spaces
$$(-)^{-1} : f \in \mathrm{Diff}(N, M) \quad \mapsto \quad f^{-1} \in \mathrm{Diff}(M, N)$$
is a smooth mapping, where $\mathrm{Diff}(M, N)$ is the subspace of $\mathcal{C}^\infty(\bar{M}, \bar{N})$ given by the diffeomorphisms between M and N.
So $(\mathrm{Diff}(M, M), \circ)$ is a (generalized) Lie group. To prove that $(-)^{-1}$ is smooth let us consider a figure $d \in_U \mathrm{Diff}(N, M)$, then, using cartesian closedness, the map $f := (d \cdot i)^\vee : U \times N \longrightarrow M$, where $i : \mathrm{Diff}(N, M) \hookrightarrow M^N$ is the inclusion, is an ordinary smooth function between Banach manifolds. We have to prove that $g := [d \cdot (-)^{-1} \cdot j]^\vee : U \times M \longrightarrow N$ is smooth, where $j : \mathrm{Diff}(M, N) \hookrightarrow N^M$ is the inclusion. But $f[u, g(u, m)] = m$ and $\mathbf{D}_2 f(u, n) = \mathbf{D}[d(u)](n)$, hence the conclusion follows from the implicit function theorem because $d(u) \in \mathrm{Diff}(N, M)$.

6. Since the category $\mathcal{C}^n$ is complete, we can also have $\mathcal{C}^n$ spaces with singular points like, e.g., the equalizer $\{x \in X \mid f(x) = g(x)\}$. In this way, any algebraic curve is a $\mathcal{C}^\infty$ separated space too.

7. Another type of space with singular points is the following. Let $\varphi \in \mathcal{C}^n(\mathbb{R}^k, \mathbb{R}^m)$ and consider the subspace $([0,1]^k \prec \mathbb{R}^k)$, then $(\varphi([0,1]^k) \prec \mathbb{R}^m) \in \boldsymbol{\mathcal{C}}^n$ is a deformation in $\mathbb{R}^m$ of the hypercube $[0,1]^k$.

8. Let C be a continuum body, I the interval for time, and $\mathcal{E}$ the 3-dimensional Euclidean space. We can define on C a natural structure of smooth diffeological space. In fact, for any point $p \in C$ let $p_r(t) \in \mathcal{E}$ be the position of p at time t in the frame of reference r; we define figures of type U on C ($U \in \mathbf{O}R^n$) the functions $d : U \longrightarrow C$ for which the following application

$$\begin{aligned} \tilde{d} : U \times I &\longrightarrow \mathcal{E}, \\ (u,t) &\longmapsto d(u)_r(t) \end{aligned}$$

is smooth. For example, if $U = \mathbb{R}$, then we can think of $d : \mathbb{R} \longrightarrow C$ as a curve traced on the body and parametrized by $u \in \mathbb{R}$. Hence we are requiring that the position $d(u)_r(t)$ of the particle $d(u) \in C$ in the frame of reference r varies smoothly with the parameter u and the time t. This is a generalization of the continuity of motion of any point of the body (take d constant). This smooth (that is diffeological) space will be separated, as an object of $\boldsymbol{\mathcal{C}}^\infty$, if different points of the body cannot have the same motion:

$$p_r(-) = q_r(-) \quad \Longrightarrow \quad p = q \qquad \forall p, q \in C.$$

The configuration space of C can be viewed (see [85]) as a space of type

$$M := \sum_{t \in I} M_t \qquad \text{where} \qquad M_t \subseteq \mathcal{E}^C$$

and so, for the categorical properties of $\boldsymbol{\mathcal{C}}^\infty$ the spaces $\mathcal{E}^C$, M_t (no matter how we choose these subspaces M_t) and M are always objects of $\mathcal{C}^\infty$ as well. With this structure the motion of C in the frame r:

$$\begin{aligned} \mu_r : C \times I &\longrightarrow \mathcal{E}, \\ (p,t) &\longmapsto p_r(t) \end{aligned}$$

is a smooth map. Note that to obtain these results we need neither M_t nor C to be manifolds, but only the possibility to associate to any point p of C a motion $p_r(-) : I \longrightarrow \mathcal{E}$. If we had the possibility to develop a differential geometry for these spaces too, we would have the possibility to obtain many results of continuum mechanics for bodies which cannot be naturally represented using a manifold or having an infinite-dimensional configuration space. Moreover, in the next section we will see how to extend any diffeological space with infinitesimal points, so that we can also consider infinitesimal sub-bodies of C.

11. The Ring of Fermat Reals

Surprisingly, it is quite simple to define an extension of the real field $\mathbb{R}$ containing nilpotent infinitesimals and having properties similar to those of SDG. Due to its simplicity, this construction does not need any background in mathematical logic. For the proof of these sections, see [34, 52, 86].

We need firstly the following class of functions

Definition 32. *We say that* x is a little-oh polynomial, *and we write* $x \in \mathbb{R}_o[t]$ *iff*

1. $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$;
2. *We can write*

$$x(t) = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \text{as} \quad t \to 0^+$$

for suitable

$$k \in \mathbb{N}$$

$$r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

$$a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}.$$

Hence, a little-oh polynomial* $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \geq 0$, with generic positive powers of t, and up to a little-oh function as $t \to 0^+$.

Remark 33. *Sometimes, but not always, we will use a notation like* $h_t := h(t)$ *for real functions of the real variable* t*. This permits to decrease the number of parenthesis used in formulas and to leave the classical notation* $f(x)$ *for functions of the form* $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$*. Moreover, we will use a slight modification of Landau's little-oh notation: writing* $x_t = y_t + o(t)$ *as* $t \to 0^+$ *we will always mean*

$$\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0 \quad \text{and} \quad x_0 = y_0.$$

In other words, every little-oh function we will consider is continuous as $t \to 0^+$*.*

We can now define:

Definition 34. *Let* $x,\ y \in \mathbb{R}_o[t]$*, then we say that* $x \sim y$ *or that* $x = y$ *in* ${}^\bullet\mathbb{R}$ *iff* $x(t) = y(t) + o(t)$ *as* $t \to 0^+$*. Because it is easy to prove that* $\sim$ *is an equivalence relation, we can define the quotient set* ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$*.*

---

*Actually in the following notation the variable t is mute.

The equivalence relation $\sim$ is a congruence with respect to pointwise operations, hence $^\bullet\mathbb{R}$ is a commutative ring, called *ring of Fermat reals*. Where it will be useful to simplify notations, we will write "$x = y$ in $^\bullet\mathbb{R}$" instead of $x \sim y$, and we will talk directly about the elements of $\mathbb{R}_o[t]$ instead of their equivalence classes. For example, we can say that $x = y$ in $^\bullet\mathbb{R}$ and $z = w$ in $^\bullet\mathbb{R}$ imply $x + z = y + w$ in $^\bullet\mathbb{R}$. The immersion of $\mathbb{R}$ in $^\bullet\mathbb{R}$ is $r \longmapsto \hat{r}$ defined by $\hat{r}(t) := r$, and in the sequel we will always identify $\hat{\mathbb{R}}$ with $\mathbb{R}$, which is hence a subring of $^\bullet\mathbb{R}$. Conversely, the map $^\circ(-) : x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$, which evaluates each Fermat real in 0, is well-defined. We will call $^\circ(-)$ the *standard part map*. In the following, we will also use the notation $\mathrm{d}t_a := [t \in \mathbb{R}_{\geq 0} \mapsto t^{1/a} \in \mathbb{R}]_\sim \in {}^\bullet\mathbb{R}$ so that e.g. $\mathrm{d}t_2 = \left[t^{1/2}\right]_\sim$ is a second order infinitesimal. In general, as we will see from the definition of order of a generic infinitesimal, $\mathrm{d}t_a$ is an infinitesimal of order a. Let us note that $\mathrm{d}t_a \cdot \mathrm{d}t_b = \mathrm{d}t_{\frac{ab}{a+b}}$, moreover $\mathrm{d}t_a^\alpha := (\mathrm{d}t_a)^\alpha = \mathrm{d}t_{\frac{a}{\alpha}}$ for every $\alpha \geq 1$, and finally $\mathrm{d}t_a = 0$ for every $a < 1$. For example, $\mathrm{d}t_a^{[a]+1} = 0$ for every $a \in \mathbb{R}_{>0}$, where $[a] \in \mathbb{N}$ is the integer part of a, i.e. $[a] \leq a < [a] + 1$.

With the following theorem, we will introduce the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$, that is a unique notation for its standard part and all its infinitesimal parts.

Theorem 35. *If $x \in {}^\bullet\mathbb{R}$, then there exists one and only one sequence*

$$(k, r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k)$$

such that

$$k \in \mathbb{N}$$

$$r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k \in \mathbb{R}$$

and

1. $x = r + \sum_{i=1}^{k} \alpha_i \cdot \mathrm{d}t_{a_i}$ *in* $^\bullet\mathbb{R}$*;*
2. $0 > a_1 > a_2 > \dots > a_k \geq 1$*;*
3. $\alpha_i \neq 0 \quad \forall i = 1, \dots, k$.

On the basis of this theorem, we introduce the following notation

Definition 36. *If $x \in {}^\bullet\mathbb{R}$, on the basis of Theorem 35, we will use the notations $^\circ x_i = \alpha_i$ and we will say that*

$$x = {}^\circ x + \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{a_i} \quad \textit{is the decomposition (of } x). \tag{17}$$

Finally, if $k \ge 1$, *that is if* $x \in {}^\bullet\mathbb{R}\setminus\mathbb{R}$, *we set* $\omega(x) := a_1$ *and* $\omega_i(x) := a_i$. *The real number* $\omega(x) = a_1$ *is the greatest order in the decomposition* (17) *and is called the* order *of the Fermat real* $x \in {}^\bullet\mathbb{R}$. *The number* $\omega_i(x) = a_i$ *is called the ith order of* x. *If* $x \in \mathbb{R}$, *we set* $\omega(x) := 0$.

11.1. The ideals D_k

In this section, we will introduce the sets of nilpotent infinitesimals corresponding to a kth order neighborhood of 0. Every smooth function restricted to this neighborhood becomes a polynomial of order k, obviously given by its kth order Taylor's formula (without remainder). We start with a theorem characterizing infinitesimals of order less than k.

Theorem 37. *If* $x \in {}^\bullet\mathbb{R}$ *and* $k \in \mathbb{N}_{>1}$, *then* $x^k = 0$ *in* ${}^\bullet\mathbb{R}$ *if and only if* ${}^\circ x = 0$ *and* $\omega(x) < k$.

Definition 38. *If* $a \in \mathbb{R}_{\ge 0} \cup \{\infty\}$, *then*

$$D_a := \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0,\ \omega(x) < a + 1\}.$$

Moreover, we will simply denote D_1 *by* D.

1. If $x = \mathrm{d}t_3$, then $\omega(x) = 3$ and $x \in D_3$. More in general, $\mathrm{d}t_k \in D_a$ if and only if $\omega(\mathrm{d}t_k) = k < a + 1$. For example, $\mathrm{d}t_k \in D$ if and only if $1 \le k < 2$.
2. $D_\infty = \bigcup_a D_a = \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0\}$ is the set of all the infinitesimals of ${}^\bullet\mathbb{R}$.
3. $D_0 = \{0\}$ because the only infinitesimal having order strictly less than 1 is, by definition of order, $x = 0$.

The following theorem gathers several expected properties of the sets D_a and of the order of an infinitesimal $\omega(x)$. In this statement, if $r \in \mathbb{R}$, then $\lceil r \rceil$ is the *ceiling* of the real r, i.e., the unique integer $\lceil r \rceil \in \mathbb{Z}$ such that $\lceil r \rceil - 1 < r \le \lceil r \rceil$.

Theorem 39. *Let* $a, b \in \mathbb{R}_{>0}$ *and* $x, y \in D_\infty$, *then*

1. $a \le b \implies D_a \subseteq D_b$,
2. $x \in D_{\omega(x)}$,
3. $a \in \mathbb{N} \implies D_a = \{x \in {}^\bullet\mathbb{R} \mid x^{a+1} = 0\}$,
4. $x \in D_a \implies x^{\lceil a \rceil + 1} = 0$,
5. $x \in D_\infty \setminus \{0\}$ *and* $k = [\omega(x)] \implies x \in D_k \setminus D_{k-1}$,

6. $x \cdot y \neq 0 \quad \Longrightarrow \quad \dfrac{1}{\omega(x \cdot y)} = \dfrac{1}{\omega(x)} + \dfrac{1}{\omega(y)}$,

7. $x + y \neq 0 \quad \Longrightarrow \quad \omega(x+y) = \max(\omega(x), \omega(y))$,

8. D_a *is an ideal.*

Because of properties 6 and 7 of the previous theorem, we have that $v(x) := \frac{1}{\omega(x)}$ if $x \in {}^\bullet\mathbb{R}_{\neq 0}$ and $v(0) := +\infty$ is a valuation on the ring ${}^\bullet\mathbb{R}$, i.e., it is a function $v : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that $v(0) = +\infty$, $v(x) \in \mathbb{R}$ for $x \neq 0$, and such that $v(x \cdot y) = v(x) + v(y)$ and $v(x+y) \geq \min(v(x), v(y))$ (in our case the equality holds). This permits to mention here some analogies between the A. Robinson's valuation field ${}^\rho\mathbb{R}$ (also called the field of asymptotic numbers, see [87, 88]) and our ring of Fermat reals.

11.2. Invertible Fermat reals

We can see more formally that to prove an equality of the form

$$f(x+h) = f(x) + h \cdot f'(x) \quad \forall h \in D \tag{18}$$

(analogous of the Kock–Lawvere axiom (10)), we cannot embed the reals $\mathbb{R}$ into a field but only into a ring, necessarily containing nilpotent element. In fact, applying (18) to the function $f(h) = h^2$ for $h \in D$, where $D \subseteq {}^\bullet\mathbb{R}$ is a given subset of ${}^\bullet\mathbb{R}$, we have

$$f(h) = h^2 = f(0) + h \cdot f'(0) = 0 \quad \forall h \in D,$$

where we have supposed the preservation of the equality $f'(0) = 0$ from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. In other words, if D and $f(h) = h^2$ verify (18), then necessarily each element $h \in D$ must be a new type of number whose square is zero. Of course, in a field the only subset D verifying this property is $D = \{0\}$.

Because we cannot have property (18) and a field at the same time, we need a sufficiently good family of cancellation laws as substitutes. The simplest one of them is the following:

Theorem 40. *If $x \in {}^\bullet\mathbb{R}$ is a Fermat real and r, $s \in \mathbb{R}$ are standard real numbers, then*

$$(x \cdot r = x \cdot s \text{ in } {}^\bullet\mathbb{R} \quad \text{and} \quad x \neq 0) \quad \Longrightarrow \quad r = s.$$

As a consequence of this result, we can always cancel a nonzero Fermat real in an equality of the form $x \cdot r = x \cdot s$ where r, s are *standard* reals. This is obviously tied with the univocal identification of the first derivative in (18) and implies that formula (18) uniquely identifies the first derivative in case it is a standard real number. For a partial reduction of this limitation using the notion of equality $=_k$ up to kth order infinitesimals, i.e.,

$$x =_k y \quad \iff \quad {}^\circ x = {}^\circ y \ \text{ and } \ \omega(x-y) \le k,$$

see [34].

The last result of this section takes its ideas from the similar situations of formal power series and gives also a formula to compute the inverse of an invertible Fermat real.

Theorem 41. *Let $x = {}^\circ x + \sum_{i=1}^n {}^\circ x_i \cdot \mathrm{d}t_{a_i}$ be the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$. Then x is invertible if and only if ${}^\circ x \ne 0$, and in this case*

$$\frac{1}{x} = \frac{1}{{}^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_{i=1}^n \frac{{}^\circ x_i}{{}^\circ x} \cdot \mathrm{d}t_{a_i} \right)^j . \tag{19}$$

In the formula (19) we have to note that the series is actually a finite sum because any $\mathrm{d}t_{a_i}$ is nilpotent. For example, $(1+\mathrm{d}t_2)^{-1} = 1 - \mathrm{d}t_2 + \mathrm{d}t_2^2 - \mathrm{d}t_2^3 + \cdots = 1 - \mathrm{d}t_2 + \mathrm{d}t$ because $\mathrm{d}t_2^3 = 0$.

11.3. The derivation formula

In this section we want to give a proof of (18), called *derivation formula* in the context of Fermat reals. Anyhow, before considering the proof of the derivation formula, we have to extend a given smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to a certain function ${}^\bullet f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$.

Definition 42. *If U is an open subset of $\mathbb{R}^n$, then ${}^\bullet U := \{x \in {}^\bullet\mathbb{R}^n \mid {}^\circ x \in U\}$. Here, with the symbol ${}^\bullet\mathbb{R}^n$ we mean ${}^\bullet\mathbb{R}^n := {}^\bullet\mathbb{R} \times \overset{n}{\ldots} \times {}^\bullet\mathbb{R}$.*

Definition 43. *Let A be an open subset of $\mathbb{R}^n$, $f : A \longrightarrow \mathbb{R}$ a smooth function and $x \in {}^\bullet A$, then we define*

$${}^\bullet f(x) := f \circ x \ \text{ in } {}^\bullet\mathbb{R}.$$

In other words, using the notation $[x]_\sim \in {}^\bullet\mathbb{R}$ for the equivalence class generated by $x \in \mathbb{R}_o[t]$ modulo the relation $\sim$ defined in Definition 34, we can write the previous definition as ${}^\bullet f([x]_\sim) := [f \circ x]_\sim$.

This definition is correct because it is easy to prove that little-oh polynomials are preserved by smooth functions, and because the function f is locally Lipschitz, so

$$\left|\frac{f(x_t)-f(y_t)}{t}\right| \le K \cdot \left|\frac{x_t-y_t}{t}\right| \qquad \forall t \in (-\delta,\delta)$$

for a sufficiently small δ and some constant K, and hence if $x=y$ in ${}^\bullet\mathbb{R}$, then also ${}^\bullet f(x) = {}^\bullet f(y)$ in ${}^\bullet\mathbb{R}$.

The function ${}^\bullet f$ is an extension of f, that is

$${}^\bullet f(r) = f(r) \quad \text{in} \quad {}^\bullet\mathbb{R} \quad \forall \mathrm{r} \in \mathbb{R},$$

as it follows directly from the definition of equality in ${}^\bullet\mathbb{R}$ (i.e. Definition 34), thus we can still use the symbol $f(x)$ both for $x \in {}^\bullet\mathbb{R}$ and $x \in \mathbb{R}$ without confusion.

Theorem 44. *Let A be an open set in $\mathbb{R}$, $x \in A$ and $f : A \longrightarrow \mathbb{R}$ a smooth function, then*

$$\exists!\, m \in \mathbb{R}\ \forall h \in D : \quad f(x+h) = f(x) + h \cdot m. \tag{20}$$

In this case we have $m = f'(x)$, where $f'(x)$ is the usual derivative of f at x.

P r o o f. Uniqueness follows from the previous cancellation law Theorem 40, indeed, if $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$ both verify (20), then $h \cdot m_1 = h \cdot m_2$ for every $h \in D$. But there exists a nonzero first order infinitesimal, e.g., $\mathrm{d}t \in D$, so it follows from Theorem 40 that $m_1 = m_2$.

To prove the existence part, take $h \in D$, so that $h^2 = 0$ in ${}^\bullet\mathbb{R}$, i.e., $h_t^2 = o(t)$ for $t \to 0^+$. But f is smooth, hence from its second order Taylor's formula we have

$$f(x+h_t) = f(x) + h_t \cdot f'(x) + \frac{h_t^2}{2} \cdot f''(x) + o(h_t^2).$$

But

$$\frac{o(h_t^2)}{t} = \frac{o(h_t^2)}{h_t^2} \cdot \frac{h_t^2}{t} \to 0 \quad \text{for } t \to 0^+,$$

so

$$\frac{h_t^2}{2} \cdot f''(x) + o(h_t^2) = o_1(t) \quad \text{for } t \to 0^+,$$

and we can write

$$f(x+h_t) = f(x) + h_t \cdot f'(x) + o_1(t) \quad \text{for } t \to 0^+$$

that is

$$f(x+h) = f(x) + h \cdot f'(x) \quad \text{in } {}^\bullet\mathbb{R}$$

and this proves the existence part because $f'(x) \in \mathbb{R}$. ■

For example, $e^h = 1 + h$, $\sin(h) = h$ and $\cos(h) = 1$ for every $h \in D$.
Analogously, we can prove the following infinitesimal Taylor's formula.

Lemma 45. *Let A be an open set in $\mathbb{R}^d$, $x \in A$, $n \in \mathbb{N}_{>0}$ and $f : A \longrightarrow \mathbb{R}$ a smooth function, then*

$$\forall h \in D^d_n : \quad f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x).$$

For example, $\sin(h) = h - \frac{h^3}{6}$ if $h \in D_3$ so that $h^4 = 0$. Note that $m = f'(x) \in \mathbb{R}$, i.e., the slope is a standard real number, and that we can use the previous formula with standard real numbers x only, and not with a generic $x \in {}^\bullet\mathbb{R}$, but it is possible to remove these limitations (see, e.g., [34]).

In other words, we can say that the derivation formula (20) allows us to differentiate the usual differentiable functions using a language with infinitesimal numbers and to obtain from this an ordinary function.

If we apply this theorem to the smooth function $p(r) := \int_x^{x+r} f(t)\,\mathrm{d}t$ for f smooth, then we immediately obtain the following

Corollary 46. *Let A be open in $\mathbb{R}$, $x \in A$ and $f : A \longrightarrow \mathbb{R}$ smooth. Then*

$$\forall h \in D : \quad \int_x^{x+h} f(t)\,\mathrm{d}t = h \cdot f(x).$$

Moreover, $f(x) \in \mathbb{R}$ is uniquely determined by this equality.

11.4. Order relation

From the previous sections one can draw the conclusion that the ring of Fermat reals ${}^\bullet\mathbb{R}$ is essentially "the little-oh" calculus. But, on the other hand the Fermat reals give us more flexibility than this calculus: working with ${}^\bullet\mathbb{R}$ we do not have to bother ourselves with remainders made of "little-oh", but we can neglect them and use the useful algebraic calculus with nilpotent infinitesimals. Anyway, thinking the elements of ${}^\bullet\mathbb{R}$ as new numbers, and not simply as "little-oh functions", permits to treat them in a different and new way, for example, to define on them an order relation with a clear geometrical interpretation.

First of all, let us introduce the useful notation

$$\forall^0 t \ge 0 : \quad \mathcal{P}(t),$$

and we will read the quantifier $\forall^0 t \geq 0$ saying *"for every $t \geq 0$ (sufficiently) small"* to indicate that the property $\mathcal{P}(t)$ is true for all t in some right* neighborhood of $t = 0$, i.e.

$$\exists\, \delta > 0\; \forall t \in [0, \delta) : \; \mathcal{P}(t).$$

The first heuristic idea to define an order relation is the following:

$$x \leq y \iff x - y \leq 0 \iff \exists z : \quad z = 0 \quad \text{in} \quad {}^\bullet\mathbb{R} \quad \text{and} \quad x - y \leq z.$$

More precisely, if x, $y \in {}^\bullet\mathbb{R}$ are two little-oh polynomials, we want to ask locally that** x_t is less than or equal to y_t, but up to a $o(t)$ for $t \to 0^+$, where the little-oh function $o(t)$ depends on x and y. Formally:

Definition 47. *Let x, $y \in {}^\bullet\mathbb{R}$, then we say*

$$x \leq y$$

iff we can find $z \in {}^\bullet\mathbb{R}$ such that $z = 0$ in ${}^\bullet\mathbb{R}$ and

$$\forall^0 t \geq 0 : \quad x_t \leq y_t + z_t.$$

Recall that $z = 0$ in ${}^\bullet\mathbb{R}$ is equivalent to $z_t = o(t)$ for $t \to 0^+$. It is immediate to see that we can equivalently define $x \leq y$ if and only if we can find $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ such that $x_t \leq y_t$ for every t sufficiently small. From this it also follows that the relation $\leq$ is well-defined on ${}^\bullet\mathbb{R}$, i.e. if $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ and $x \leq y$, then $x' \leq y'$. As usual, we will use the notation $x < y$ for $x \leq y$ and $x \neq y$.

Theorem 48. *The relation $\leq$ is an order, i.e., is reflexive, transitive and antisymmetric; it extends the order relation of $\mathbb{R}$ and with it $({}^\bullet\mathbb{R}, \leq)$ is an ordered ring. Finally the following sentences are equivalent:*

1. *$h \in D_\infty$, i.e., h is an infinitesimal;*
2. *$\forall r \in \mathbb{R}_{>0} : \quad -r < h < r$.*

*We recall that by Definition 32 our little-oh polynomials are always defined on $\mathbb{R}_{\geq 0}$.

**We recall that, to simplify the notations, we do not use equivalence classes as elements of ${}^\bullet\mathbb{R}$ but directly little-oh functions. The only notion of equality between little-oh functions is, of course, the equivalence relation defined in Definition 34 and, as usual, we must always prove that our relations between little-oh polynomials are well-defined.

11.5. Geometrical representation of Fermat reals

One of the conducting idea in the construction of Fermat reals is to maintain always a clear intuitive meaning. More precisely, we always tried to keep a good dialectic between provable formal properties and their intuitive meaning. In this direction we can see the possibility to find a geometrical representation of Fermat reals.

The idea is that to any Fermat real $x \in {}^\bullet\mathbb{R}$ we can associate the function

$$t \in \mathbb{R}_{\geq 0} \mapsto {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} \in \mathbb{R}, \tag{21}$$

where N is, of course, the number of addends in the decomposition of x. Therefore, a geometric representation of this function is also a geometric representation of the number x, because different Fermat reals have different decompositions, see Theorem 35. Finally, we can guess that, because the notion of equality in ${}^\bullet\mathbb{R}$ depends only on the germ generated by each little-oh polynomial (see Definition 34), we can represent each $x \in {}^\bullet\mathbb{R}$ with only the first small part of the function (21).

Definition 49. *If $x \in {}^\bullet\mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$, then*

$$\mathrm{graph}_\delta(x) := \left\{ ({}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)}, t) \mid 0 \leq t < \delta \right\},$$

where N is the number of addends in the decomposition of x.

Note that the values of the function are placed in the abscissa position. This inversion of abscissa and ordinate in the $\mathrm{graph}_\delta(x)$ permits to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture (see, e.g., the following figure). Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N = 0$ and the $\mathrm{graph}_\delta(x)$ is a vertical line passing through ${}^\circ x = x$.

The following theorem permits to represent geometrically the Fermat reals

Theorem 50. *If $\delta \in \mathbb{R}_{>0}$, then the function*

$$x \in {}^\bullet\mathbb{R} \mapsto \mathrm{graph}_\delta(x) \subset \mathbb{R}^2$$

is injective. Moreover, if x, $y \in {}^\bullet\mathbb{R}$, then we can find $\delta \in \mathbb{R}_{>0}$ (depending on x and y) such that $x < y$ if and only if

$$\forall p, q, t : \quad (p, t) \in \mathrm{graph}_\delta(x), \quad (q, t) \in \mathrm{graph}_\delta(y) \quad \Longrightarrow \quad p < q. \tag{22}$$

We have seen how a simple extension ${}^\bullet\mathbb{R}$ of the real field, having the properties similar to those of SDG, is possible. In the next section, we will see how to extend this construction $\mathbb{R} \mapsto {}^\bullet\mathbb{R}$ to every diffeological space. Finally, using our cartesian closure, we will insert all our extended spaces ${}^\bullet X$, for $X \in \boldsymbol{\mathcal{C}}^\infty$, in a cartesian closed category.

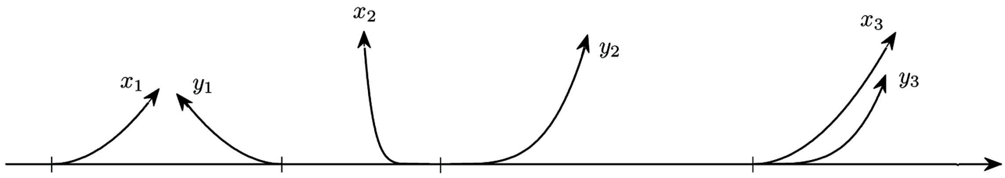

Fig. 1. Different cases in which $x_i < y_i$.

12. Extending Smooth Spaces with Infinitesimals

The main aim of this section is to extend any $\boldsymbol{\mathcal{C}}^\infty$ space (i.e., any diffeological space) and any $\boldsymbol{\mathcal{C}}^\infty$ function by means of our "infinitesimally close points". First of all, we will extend to a generic space $X \in \boldsymbol{\mathcal{C}}^\infty$ the notion of little-oh polynomial. The set of these paths will be denoted by $X_o[t]$. Afterward, we shall use the observables φ of the space X to generalize the equivalence relation $\sim$ (i.e., the equality in ${}^\bullet\mathbb{R}$, see Definition 34) using the following idea:

$$\varphi(x_t) = \varphi(y_t) + o(t) \qquad \text{with} \qquad \varphi \in^{UK} X.$$

Using this equivalence relation, we will define ${}^\bullet X := X_o[t]/\sim$, which will be the generalization of the Definition ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$. Following this idea, the main problem is to understand how to relate the little-oh polynomials x, y with the domain U of φ. The second problem is that with this definition, ${}^\bullet X$ is a set only, without any kind of structure. Indeed, using the cartesian closure, we will tackle the problem to define a meaningful category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ and a suitable structure on ${}^\bullet X$ so that ${}^\bullet X \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$.

12.1. Little-oh polynomials in $\boldsymbol{\mathcal{C}}^\infty$

At first, we will define a little-oh polynomial in the space $\mathbb{R}^d$, and secondly, we will generalize this notion to a generic space $X \in \boldsymbol{\mathcal{C}}^\infty$ using observables.

Definition 51. *We say that* x is a little-oh polynomial in $\mathbb{R}^d$*, and we write* $x \in \mathbb{R}^d_o[t]$ *iff*

1. $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^d$.

2. *We can write*

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad as \quad t \to 0^+$$

for suitable

$$k \in \mathbb{N}$$

$$r, \alpha_1, \dots, \alpha_k \in \mathbb{R}^d$$

$$a_1, \dots, a_k \in \mathbb{R}_{\geq 0}.$$

Now let $X \in \boldsymbol{\mathcal{C}}^\infty$ and let $\mathcal{C}_0(X)$ be the set of all the maps $x : \mathbb{R}_{\geq 0} \longrightarrow X$ which are continuous at the origin $t = 0$ (recall that any diffeological space is a topological space, see Definition 21), then we say that $x \in \mathcal{C}_0(X)$ is a little-oh polynomial (of X) *iff for every zone UK of X, with $K \subseteq \mathbb{R}^{\mathsf{k}}$, and every observable $\varphi \in^{UK} X$ we have*

$$x_0 \in U \quad \Longrightarrow \quad \varphi \circ x \in \mathbb{R}_o^{\mathsf{k}}[t].$$

Moreover,

$$X_o[t] := X_o := \{x \in \mathcal{C}_0(X) \,|\, x \text{ is a little-oh polynomial of } X\}.$$

Let us note that for $d = 1$ we have exactly the old Definition 32. A direct verification proves that being a little-oh polynomial is a local property. Moreover, we will prove later that the two parts of this definition (i.e. that of $X_o[t]$ and that of $\mathbb{R}_o^d[t]$) are equivalent if $X = \mathbb{R}^d$.

Because every $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ preserves the observables, we have that $\boldsymbol{\mathcal{C}}^\infty$ functions preserve little-oh polynomials too,

$$x \in X_o[t] \quad \Longrightarrow \quad f \circ x \in Y_o[t].$$

Theorem 52. *If M is a $\mathcal{C}^\infty$ manifold and $x : \mathbb{R}_{\geq 0} \longrightarrow |M|$ is a map, then we have that $x \in \bar{M}_o[t]$ if and only if there exists a chart (U, φ) of M such that:*

1. $x(0) \in U$
2. $\varphi \circ x \in \mathbb{R}_o^d[t]$, *where* $d := \dim(M)$.

12.2. The Fermat extension of spaces and functions

Considering the previous definition of little-oh paths and the Definition 28, it is now clear how to generalize the definition of equality in ${}^\bullet\mathbb{R}$ (see Definition 34) to a generic $X \in \boldsymbol{\mathcal{C}}^\infty$:

Definition 53. *Let X be a $\mathcal{C}^\infty$ space and let x, $y \in X_o[t]$ be two little-oh polynomials, then we say that*

$$x \sim y \quad in \quad X \quad or\ simply \quad x = y \quad in \quad {}^\bullet X$$

iff for every zone UK of X and every observable $\varphi \in^{UK} X$ we have

1. $x_0 \in U \iff y_0 \in U$;
2. $x_0 \in U \quad \Longrightarrow \quad \varphi(x_t) = \varphi(y_t) + o(t)$.

Obviously we will write ${}^\bullet X := X_o[t]/\sim$ and ${}^\bullet f(x) := f \circ x$ if $f \in \mathcal{C}^\infty(X, Y)$ and $x \in {}^\bullet X$ and we will call them the *Fermat extension of X and of f*, respectively. As usual, we will also define the *standard part of* $x \in {}^\bullet X$ as ${}^\circ x := x(0) \in X$.

The correctness of the definition of ${}^\bullet f$ is stated in the following:

Theorem 54. *If $f \in \mathcal{C}^\infty(X, Y)$ and $x = y$ in ${}^\bullet X$ then ${}^\bullet f(x) = {}^\bullet f(y)$ in ${}^\bullet Y$.*

Using the continuity of $\varphi \circ x$ we can note that $x = y$ in ${}^\bullet X$ implies that x_0 and y_0 are identified in X (see Definition 28) and thus using constant maps $\hat{x}(t) := x$, for $x \in X$, we obtain an injection $\hat{(-)} : |X| \longrightarrow {}^\bullet X$ if the space X is separated. Therefore, if Y is separated too, ${}^\bullet f$ is really an extension of f. Finally, note that the application ${}^\bullet(-)$ preserves compositions and identities.

Moreover, it is not hard to prove that if $X = M$ is a $\mathcal{C}^\infty$ manifold then we have that $x = y$ in ${}^\bullet M$ iff there exists a chart (U, φ) of M such that

1. x_0, $y_0 \in U$
2. $\varphi(x_t) = \varphi(y_t) + o(t)$.

Moreover, the previous conditions do not depend on the chart (U, φ). In particular, if $X = U$ is an open set in $\mathbb{R}^k$, then $x = y$ in ${}^\bullet U$ is simply equivalent to the limit relation $x(t) = y(t) + o(t)$ as $t \to 0^+$; hence, if $i : U \hookrightarrow \mathbb{R}^k$ is the inclusion map, it is easy to prove that its Fermat extension ${}^\bullet i : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}^k$ is injective. We will always identify ${}^\bullet U$ with ${}^\bullet i({}^\bullet U)$, so we simply write ${}^\bullet U \subseteq {}^\bullet \mathbb{R}^k$. According to this identification, if U is open in $\mathbb{R}^k$, we can also prove that

$${}^\bullet U = \{x \in {}^\bullet \mathbb{R}^k \mid {}^\circ x \in U\}. \tag{23}$$

This property says that the preliminary definition of ${}^\bullet U$ given in Definition 42 is equivalent to the previous, more general, Definition 53 of extension. Using the previous equivalent way to express the relation $\sim$ on manifolds, we see that $(x, y) = (x', y')$ in ${}^\bullet(M \times N)$ iff $x = x'$ in ${}^\bullet M$ and $y = y'$ in ${}^\bullet N$. From this conclusion we can prove that the following applications:

$$\alpha_{MN} := \alpha \ : \ ([x]_\sim, [y]_\sim) \in {}^\bullet M \times {}^\bullet N \longmapsto [(x, y)]_\sim \in {}^\bullet(M \times N), \tag{24}$$

$$\beta_{MN} := \beta \ : \ [z]_\sim \in {}^\bullet(M \times N) \longmapsto ([z \cdot p_M]_\sim, [z \cdot p_N]_\sim) \in {}^\bullet M \times {}^\bullet N \tag{25}$$

(for clarity we have used the notation with the equivalence classes) are well-defined bijections with $\alpha^{-1} = \beta$ (obviously p_M, p_N are the projections). We will use the first one of them in the following section with the temporary notation $\langle p, x\rangle := \alpha(p, x)$, hence $f\langle p, x\rangle = f(\alpha(p, x))$ for $f : {}^\bullet(M \times N) \longrightarrow Y$. This simplifies our notations but permits to avoid the identification of ${}^\bullet M \times {}^\bullet N$ with ${}^\bullet(M \times N)$ until we will have proved that α and β are arrows of the category ${}^\bullet\mathcal{C}^\infty$.

12.3. The category of Fermat spaces

Up to now, every ${}^\bullet X$ is a simple set only. Now we want to use the general passage from a category of types of figures $\mathcal{F}$ to its cartesian closure $\bar{\mathcal{F}}$ so as to put on any ${}^\bullet X$ a useful structure of $\bar{\mathcal{F}}$ space. Our aim is to obtain a new cartesian closed category $\bar{\mathcal{F}} =: {}^\bullet\mathcal{C}^\infty$, called the *category of Fermat spaces*, and a functor ${}^\bullet(-) : \mathcal{C}^\infty \longrightarrow {}^\bullet\mathcal{C}^\infty$, called the *Fermat functor*. Therefore, we have to choose $\mathcal{F}$, that is, we have to understand what types of figures of ${}^\bullet X$ we need. It may seem very natural to take ${}^\bullet g : {}^\bullet U \longrightarrow {}^\bullet V$ as arrow in $\mathcal{F}$ if $g : U \longrightarrow V$ is in $\mathbf{O}\mathbb{R}^\infty$ (in [89] we followed this way). The first problem in this idea is that, e.g.,

$${}^\bullet\mathbb{R} \xrightarrow{{}^\bullet f} {}^\bullet\mathbb{R} \quad \Longrightarrow \quad {}^\bullet f(0) = f(0) \in \mathbb{R},$$

hence there cannot exist a constant function of the type ${}^\bullet f$ to a nonstandard value, and so we cannot satisfy the closure of $\mathcal{F}$ with respect to generic constant functions (see the hypotheses about the types of figures $\mathcal{F}$ in Sect. 7.1). But we can make further considerations about this problem so as to motivate better the choice of $\mathcal{F}$. The first one is that we surely want to have the possibility to lift maps* as simple as the sum between Fermat reals:

$$s : (p, q) \in {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \longrightarrow p + q \in {}^\bullet\mathbb{R}.$$

Therefore, we have to choose $\mathcal{F}$ so that the map $s^\wedge(p) : q \in {}^\bullet\mathbb{R} \longrightarrow p + q \in {}^\bullet\mathbb{R}$ is an arrow of ${}^\bullet\mathcal{C}^\infty$. Note that this map is neither constant nor of the type ${}^\bullet f$ because $s^\wedge(p)(0) = p$ and p could be a non standard Fermat real.

The second consideration is about the map α defined in (24): if we want α to be an arrow of ${}^\bullet\mathcal{C}^\infty$, then in the following situation we have to obtain a ${}^\bullet\mathcal{C}^\infty$ arrow

$$\begin{array}{c} {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \xrightarrow{p \times 1_{{}^\bullet\mathbb{R}}} {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \xrightarrow{\alpha} {}^\bullet(\mathbb{R} \times \mathbb{R}) \xrightarrow{{}^\bullet g} {}^\bullet\mathbb{R} \\ (r, s) \longmapsto (p, s) \longmapsto \langle p, s\rangle \longmapsto {}^\bullet g\langle p, s\rangle, \end{array}$$

*I.e. to consider their adjoint function using cartesian closedness.

where $p \in {}^\bullet\mathbb{R}$ and $g \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$. The idea we shall follow is exactly to take as arrows of $\mathcal{F}$ all the maps that locally are of the form $\delta(s) = {}^\bullet g\langle p, s\rangle$, where $p \in {}^\bullet(\mathbb{R}^{\mathsf{p}})$ works as a parameter of ${}^\bullet g\langle -, -\rangle$. Obviously, in this way δ could also be a constant map to a nonstandard value (take as g a projection). Frequently one can find maps of the form ${}^\bullet g\langle p, -\rangle$ in informal calculations in physics or geometry. Actually, they simply are $\mathcal{C}^\infty$ maps with some fixed parameter p, which could be an infinitesimal distance (e.g., in the potential of the electric dipole), an infinitesimal coefficient associated to a metric, or a side $l := s(a, -)$ of an infinitesimal surface $s : [a, b] \times [c, d] \longrightarrow {}^\bullet\mathbb{R}$, where $[a, b], [c, d] \subseteq D_k$ (see [52] for several examples).

Note the importance of the map α to perform passages like the following:

$$M \times N \xrightarrow{f} Y \quad \text{in} \quad \boldsymbol{\mathcal{C}}^\infty,$$

$${}^\bullet(M \times N) \xrightarrow{{}^\bullet f} {}^\bullet Y \quad \text{in} \quad {}^\bullet\boldsymbol{\mathcal{C}}^\infty,$$

$${}^\bullet M \times {}^\bullet N \xrightarrow{{}^\bullet f} {}^\bullet Y \quad \text{in} \quad {}^\bullet\boldsymbol{\mathcal{C}}^\infty \text{ (identification via } \alpha),$$

$${}^\bullet N \xrightarrow{{}^\bullet f^\wedge} {}^\bullet Y^{{}^\bullet M} \quad \text{using cartesian closedness.}$$

This motivates the choice of arrows in $\mathcal{F}$, but there is a second problem about the choice of the objects of the category $\mathcal{F}$. Take a manifold M and an arrow $t : D \longrightarrow {}^\bullet M$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Even if we have not still defined formally the meaning of this "arrow", we want to think t as a tangent vector applied either to a standard point $t(0) \in M$ or to a nonstandard one, $t(0) \in {}^\bullet M \setminus M$. Roughly speaking, this is the case when we can write $t(h) = {}^\bullet g\langle p, h\rangle$ for every $h \in D$ and for some g, p. If we want to obtain this equality, it is useful to have two properties: the first one is that the identity map over D, i.e. 1_D, is a figure of D, i.e. $1_D \in_D D$. In this way, the property $t : D \longrightarrow {}^\bullet M$, being an arrow of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$, implies that t is a figure of ${}^\bullet M$ of the type D, i.e. $t \in_D {}^\bullet M$. The second property we would like to obtain is to have maps of the form ${}^\bullet g\langle p, -\rangle : D \longrightarrow {}^\bullet M$ as figures of ${}^\bullet M$. Of course, we can thus say that necessarily $t = {}^\bullet g\langle p, -\rangle$ for some $g \in \mathcal{C}^\infty(\mathbb{R}^{\mathsf{p}}, \mathbb{R})$ and $p \in \mathbb{R}^{\mathsf{p}}$. Therefore, to obtain these properties, it would be useful to have D as an object of $\mathcal{F}$. But D is not the extension of a standard subset of $\mathbb{R}$, thus what will be the objects of $\mathcal{F}$? We will take generic subsets S of ${}^\bullet(\mathbb{R}^{\mathsf{s}})$ with the topology $\mathcal{T}_S$ generated by $\mathcal{U} = {}^\bullet U \cap S$, for U open in $\mathbb{R}^{\mathsf{s}}$ (in this case we will say that the open set $\mathcal{U}$ *is defined by* U *in* S). In other words, $A \in \mathcal{T}_S$ if and only if

$$A = \bigcup \{{}^\bullet U \cap S \subseteq A \,|\, U \text{ is open in } \mathbb{R}^{\mathsf{s}}\}\,. \tag{26}$$

These are the motivations to introduce the category of the types of figures $\mathcal{F}$ by means of the following

Definition 55. *We call* $\mathbf{S}^\bullet\mathbb{R}^\infty$ *the category whose objects are topological spaces* $(S, \mathcal{T}_S)$, *with* $S \subseteq {}^\bullet(\mathbb{R}^\mathsf{s})$ *for some* $\mathsf{s} \in \mathbb{N}$ *which depends on* S, *and with the previous topology* $\mathcal{T}_S$. *In the following we will frequently use the simplified notation* S *instead of the complete* $(S, \mathcal{T}_S)$.

If $S \subseteq {}^\bullet(\mathbb{R}^\mathsf{s})$ *and* $T \subseteq {}^\bullet(\mathbb{R}^\mathsf{t})$ *then we say that*

$$S \xrightarrow{f} T \quad \text{in} \quad \mathbf{S}^\bullet\mathbb{R}^\infty$$

iff f *maps* S *in* T *and for every* $s \in S$ *we can write*

$$f(x) = {}^\bullet g\langle p, x\rangle \quad \forall x \in {}^\bullet V \cap S \tag{27}$$

for some

$$\begin{aligned} &V \text{ open in } \mathbb{R}^\mathsf{s} \text{ such that } s \in {}^\bullet V \\ &p \in {}^\bullet U, \text{ where } U \text{ is open in } \mathbb{R}^\mathsf{p} \\ &g \in \mathcal{C}^\infty(U \times V, \mathbb{R}^\mathsf{t}). \end{aligned}$$

Moreover we will consider on $\mathbf{S}^\bullet\mathbb{R}^\infty$ *the forgetful functor given by the inclusion* $|-| : \mathbf{S}^\bullet\mathbb{R}^\infty \hookrightarrow \mathbf{Set}$, *i.e.* $|(S, \mathcal{T}_S)| := S$. *The category* $\mathbf{S}^\bullet\mathbb{R}^\infty$ *will be called* the category of subsets of ${}^\bullet\mathbb{R}^\infty$ *(but note that here* ∞ *indicates the class of regularity of the functions we are considering).*

In other words, *locally* a $\boldsymbol{\mathcal{C}}^\infty$ function $f : S \longrightarrow T$ between two types of figures $S \subseteq {}^\bullet(\mathbb{R}^\mathsf{s})$ and $T \subseteq {}^\bullet(\mathbb{R}^\mathsf{t})$ is constructed in the following way:

1. Start with an ordinary standard function $g \in \mathcal{C}^\infty(U \times V, \mathbb{R}^\mathsf{t})$, with U open in $\mathbb{R}^\mathsf{p}$ and V open in $\mathbb{R}^\mathsf{s}$. The space $\mathbb{R}^\mathsf{p}$ has to be thought as a space of parameters for the function g;
2. Consider its Fermat extension obtaining ${}^\bullet g : {}^\bullet(U \times V) \longrightarrow {}^\bullet(\mathbb{R}^\mathsf{t})$;
3. Consider the composition ${}^\bullet g \circ \langle -, -\rangle : {}^\bullet U \times {}^\bullet V \longrightarrow {}^\bullet(\mathbb{R}^\mathsf{t})$, where $\langle -, -\rangle$ is the map α given by (24);
4. Fix a parameter $p \in {}^\bullet U$ as a first variable of the previous composition, i.e. consider ${}^\bullet g\langle p, -\rangle : {}^\bullet V \longrightarrow {}^\bullet(\mathbb{R}^\mathsf{t})$. Locally, the map f is of this form: $f = {}^\bullet g\langle p, -\rangle$;
5. Because in the Definition 55 we ask $s \in {}^\bullet V$ we have that $\mathcal{V} := {}^\bullet V \cap S$ is a neighborhood of s defined by V in S (see (26)). Analogously, ${}^\bullet U$ is a neighborhood of the parameter p.

Theorem 56. $\mathbf{S}^\bullet\mathbb{R}^\infty$ *is a category of types of figures.*

On the basis of this theorem, we can define

$${}^\bullet\boldsymbol{\mathcal{C}}^\infty := \overline{\mathbf{S}^\bullet\mathbb{R}^\infty}.$$

Each object of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ will be called a *Fermat space.*

12.4. The Fermat functor

Now the problem is: what Fermat space could we associate to sets like ${}^\bullet X$ or D?

Definition 57. *Let $X \in \boldsymbol{\mathcal{C}}^\infty$, then for any subset $Z \subseteq {}^\bullet X$ we call ${}^\bullet(ZX)$ the extended space generated on Z (see Sect. 7.1.) by the following set of figures $d : T \longrightarrow Z$ (where $T \subseteq {}^\bullet(\mathbb{R}^{\mathrm{t}})$ is a type of figure in $\mathbf{S}{}^\bullet\mathbb{R}^\infty$)*

$$d \in \mathcal{D}^0_T(Z) \quad :\Longleftrightarrow \quad \begin{array}{l} d \text{ is constant or we can write} \\ d = {}^\bullet h|_T \text{ for some } h \in_V X \text{ such that } T \subseteq {}^\bullet V. \end{array} \tag{28}$$

Thus, in the non-trivial case, we start from a standard figure $h \in_V X$ of type $V \in \mathbf{O}\mathbb{R}^\infty$ such that ${}^\bullet V \supseteq T$; we extend this figure obtaining ${}^\bullet h : {}^\bullet V \longrightarrow {}^\bullet X$, and finally the restriction ${}^\bullet h|_T$ is a generating figure if it maps T in Z. This choice is very natural, and the adding of the alternative "d is constant" in the previous disjunction is due to the need to have all constant figures in a family of generating figures.

Using this definition of ${}^\bullet(ZX)$, we set (with some abuses of language)

$$\begin{aligned} {}^\bullet X &:= {}^\bullet({}^\bullet X X), \\ D &:= {}^\bullet(D\mathbb{R}), \\ {}^\bullet\mathbb{R} &:= {}^\bullet({}^\bullet\mathbb{R}\mathbb{R}), \\ {}^\bullet\mathbb{R}^k &:= {}^\bullet({}^\bullet(\mathbb{R}^k)\mathbb{R}^k), \\ D_k &:= {}^\bullet(D_k\mathbb{R}^k). \end{aligned}$$

We will call ${}^\bullet(ZX)$ *the Fermat space induced on Z by $X \in \boldsymbol{\mathcal{C}}^\infty$*. We can now study the extension functor:

Theorem 58. *Let $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ and Z a subset of ${}^\bullet X$ with ${}^\bullet f(Z) \subseteq W \subseteq {}^\bullet Y$, then in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ we have that*

$${}^\bullet(ZX) \xrightarrow{{}^\bullet f|_Z} {}^\bullet(WY).$$

Therefore ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ is a functor called the Fermat functor.

It is possible to prove that this functor has very good properties like the preservation of product of manifolds ${}^\bullet(M \times N) \simeq {}^\bullet M \times {}^\bullet N$ and the preservation of intersections, unions, inclusions, counter-images of open sets, intuitionistic negations and quantifiers. Finally, this functor is also an embedding of the category of smooth manifolds into the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of Fermat spaces (see [34]).

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