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Eigenvalue Distribution of a Large Weighted Bipartite Random Graph

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We study an eigenvalue distribution of the adjacency matrix $A^{(N,p,\alpha)}$ of the weighted random bipartite graph $\Gamma = \Gamma_{N,p}$. We assume that the graph has N vertices, the ratio of parts is $\dfrac{\alpha}{1-\alpha}$, and the average number of the edges attached to one vertex is αp or $(1-\alpha)p$. To every edge of the graph e_{ij}, we assign the weight given by a random variable a_{ij} with all moments finite.

We consider the moments of the normalized eigenvalue counting measure $\sigma_{N,p,\alpha}$ of $A^{(N,p,\alpha)}$. The weak convergence in probability of the normalized eigenvalue counting measures is proved.

Key words: random bipartite graph, eigenvalue distribution, counting measure.

Mathematics Subject Classification 2010: 15B52.

1. Introduction

The spectral theory of graphs is an actively developing field of mathematics involving a variety of methods and deep results (see the monographs [4, 5, 10]). Given a graph with N vertices, one can associate with it many different matrices, but the most studied are the adjacency matrix and the Laplacian matrix of the graph. Commonly, the set of N eigenvalues of the adjacency matrix is referred to as the spectrum of the graph. In these studies, the dimension of the matrix N is usually regarded as a fixed parameter. The spectra of infinite graphs are considered in certain particular cases of graphs having one or another regular structure (see, for example, [12]).

Another large class of graphs, where the limiting transition $N \to \infty$ provides a natural approximation, is represented by random graphs [2, 11]. In this branch, geometrical and topological properties of graphs (e.g., the number of connected components, the size of a maximal connected component) are studied for the immense number of random graph ensembles. One of the classes of the prime reference is the *binomial random graph* (see, e.g., [11]). Given a number $p_N \in (0,1)$, the family of graphs $\mathbf{G}(N, p_N)$ is defined by taking as Ω the set of all graphs on N vertices with the probability

$$P(G) = p_N^{e(G)}(1-p_N)^{\binom{N}{2}-e(G)}, \tag{1.1}$$

where $e(G)$ is the number of the edges of G. Most of the random graphs studies are devoted to the cases where $p_N \to 0$ as $N \to \infty$.

Intersection of these two branches of the theory of graphs includes the spectral theory of random graphs which is not properly studied. However, a number of powerful tools can be employed here, because the ensemble of random symmetric $N \times N$ adjacency matrices A_N is a particular representative of the random matrix theory, where the limiting transition $N \to \infty$ has been intensively studied since the pioneering work by E. Wigner [19] was published. Initiated by theoretical physics applications, the spectral theory of random matrices has revealed deep nontrivial links with many fields of mathematics.

The spectral properties of random matrices corresponding to (1.1) were studied in the limit $N \to \infty$ both in numerical and theoretical physics [6–8, 16–18]. There are two major asymptotic regimes: $p_N \gg 1/N$ and $p_N = O(1/N)$ and the corresponding models are called the *dilute random matrices* and *sparse random matrices*, respectively. The first studies of spectral properties of sparse and dilute random matrices in the physical literature are related with the works [16–18], where equations for the limiting density of states of sparse random matrices were derived. In [16] and [9], a number of important results on the universality of the correlation functions and the Anderson localization transition were obtained. But all these results were obtained by using non-rigorous replica and supersymmetry methods.

At the mathematical level of rigour, the eigenvalue distribution of dilute random matrices was studied in [14]. It was shown that the normalized eigenvalue counting function of

$$\frac{1}{\sqrt{Np_N}} A_{N,p_N} \tag{1.2}$$

converges in the limit $N, p_N \to \infty$ to the distribution of explicit form known as the semicircle, or Wigner law [19]. The moments of this distribution verify the well-known recurrent relation for the Catalan numbers and can be found explicitly. Therefore one can say that the dilute random matrices represent an explicitly solvable model (see also [17, 18]).

In the series of papers [1–3] and in [13], the adjacency matrix and the Laplace matrix of random graphs (1.1) with $p_N = pN$ were studied. It was shown that the sparse random matrix ensemble can also be viewed as the explicitly solvable model.

In the present paper we consider a bipartite analogue of the large sparse random graph. The paper is a modification of one part of [13] for this case.

2. Main Results

We consider the randomly weighted adjacency matrix of random bipartite graphs. Let $\Xi = \{a_{ij},\ i \le j,\ i,j \in \mathbb{N}\}$ be the set of jointly independent identically distributed (i.i.d.) random variables determined on the same probability space and possessing the moments

$$\mathbb{E} a_{ij}^k = X_k < \infty, \qquad \forall\, i,j,k \in \mathbb{N}, \tag{2.1}$$

where $\mathbb{E}$ denotes the mathematical expectation corresponding to Ξ. We set $a_{ji} = a_{ij}$ for $i \le j$.

Given $0 < p \le N$, let us define the family $D_N^{(p)} = \{d_{ij}^{(N,p)},\ i \le j,\ i,j \in \overline{1,N}\}$ of jointly independent random variables

$$d_{ij}^{(N,p)} = \begin{cases} 1, & \text{with probability } \ p/N, \\ 0, & \text{with probability } 1 - p/N. \end{cases} \tag{2.2}$$

We determine $d_{ji} = d_{ij}$ and assume that $D_N^{(p)}$ is independent from Ξ.

Let $\alpha \in (0,1)$, define $I_{\alpha,N} = \overline{1, [\alpha N]}$, where $[\cdot]$ is a floor function. Now one can consider the real symmetric $N \times N$ matrix $A^{(N,p,\alpha)}(\omega)$:

$$\left[A^{(N,p,\alpha)}\right]_{ij} = \begin{cases} a_{ij}\, d_{ij}^{(N,p)}, & \text{if } (i \in I_{\alpha,N} \wedge j \notin I_{\alpha,N}) \vee (i \notin I_{\alpha,N} \wedge j \in I_{\alpha,N}), \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

that has N real eigenvalues $\lambda_1^{(N,p,\alpha)} \le \lambda_2^{(N,p,\alpha)} \le \ldots \le \lambda_N^{(N,p,\alpha)}$.

The normalized eigenvalue counting function (or integrated density of states (IDS)) of $A^{(N,p,\alpha)}$ is determined by the formula

$$\sigma\left(\lambda; A^{(N,p,\alpha)}\right) = \frac{\sharp\{j : \lambda_j^{(N,p,\alpha)} < \lambda\}}{N}.$$

Theorem 1. *Under the condition*

$$X_{2m} \le (Cm)^{2m}, \quad m \in \mathbb{N}, \tag{2.4}$$

the measure $\sigma\left(\lambda; A^{(N,p,\alpha)}\right)$ converges weakly in probability to the nonrandom measure $\sigma_{p,\alpha}$

$$\sigma\left(\cdot\ ; A^{(N,p,\alpha)}\right) \to \sigma_{p,\alpha}\,,\ \ N \to \infty, \tag{2.5}$$

which can be uniquely determined by its moments

$$\int \lambda^s d\sigma_{p,\alpha} = \begin{cases} m_k^{(p,\alpha)} = \sum\limits_{i=0}^{k} \left(S^{(1)}(k,i) + S^{(2)}(k,i)\right), & \text{if } s = 2k, \\ 0, & \text{if } s = 2k-1, \end{cases} \tag{2.6}$$

where the numbers $S^{(1)}(k,i)$, $S^{(2)}(k,i)$ *can be found from the system of recurrent relations*

$$S^{(1)}(l,r) = p \sum_{f=1}^{r} \binom{r-1}{f-1} X_{2f} \sum_{u=0}^{l-r} S^{(1)}(l-u-f, r-f) \sum_{v=0}^{u} \binom{f+v-1}{f-1} S^{(2)}(u,v), \tag{2.7}$$

$$S^{(2)}(l,r) = p \sum_{f=1}^{r} \binom{r-1}{f-1} X_{2f} \sum_{u=0}^{l-r} S^{(2)}(l-u-f, r-f) \sum_{v=0}^{u} \binom{f+v-1}{f-1} S^{(1)}(u,v) \tag{2.8}$$

with the initial conditions

$$S^{(1)}(l,0) = \alpha \delta_{l,0}, \;\; S^{(2)}(l,0) = (1-\alpha)\delta_{l,0}. \tag{2.9}$$

The following denotations are used:

$$\mathcal{M}_k^{(N,p,\alpha)} = \int \lambda^k d\sigma\left(\lambda; A^{(N,p,\alpha)}\right), \;\; M_k^{(N,p,\alpha)} = \mathbb{E}\mathcal{M}_k^{(N,p,\alpha)},$$

$$C_{k,m}^{(N,p,\alpha)} = \mathbb{E}\left\{\mathcal{M}_k^{(N,p,\alpha)}\mathcal{M}_m^{(N,p,\alpha)}\right\} - \mathbb{E}\left\{\mathcal{M}_k^{(N,p,\alpha)}\right\}\mathbb{E}\left\{\mathcal{M}_m^{(N,p,\alpha)}\right\}.$$

Theorem 1 is a corollary of Theorem 2

Theorem 2. *Assuming conditions* (2.4),

(*i*) *The correlators* $C_{k,m}^{(N,p,\alpha)}$ *vanish in the limit* $N \to \infty$*:*

$$C_{k,m}^{(N,p,\alpha)} \le \frac{C(k,m,p,\alpha)}{N}, \;\; \forall\; k,m \in \mathbb{N}. \tag{2.10}$$

(*ii*) *The limit of the s-th moment exists for all* $s \in \mathbb{N}$*:*

$$\lim_{N\to\infty} M_s^{(N,p,\alpha)} = \begin{cases} \sum\limits_{i=0}^{k} \left(S^{(1)}(k,i) + S^{(2)}(k,i)\right), & \text{if } s = 2k, \\ 0, & \text{if } s = 2k-1, \end{cases} \tag{2.11}$$

where the numbers $S^{(1)}(k,i)$, $S^{(2)}(k,i)$ *are determined by* (2.7)–(2.9).

(*iii*) *The limiting moments* $\left\{m_k^{(p,\alpha)}\right\}_{k=1}^{\infty}$ *satisfy Carleman's condition*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{m_k^{(p,\alpha)}}} = \infty. \tag{2.12}$$

3. Proof of Theorem 1

3.1. Walks and contributions

Using the independence of families Ξ and $D_N^{(p)}$, we have

$$\begin{aligned}
M_k^{(N,p)} &= \int \mathbb{E}\{\lambda^k d\sigma_{A^{(N,p,\alpha)}}\} = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}[\lambda_i^{(N,p,\alpha)}]^k\right) = \frac{1}{N}\mathbb{E}\left(\mathrm{Tr}[A^{(N,p,\alpha)}]^k\right) \\
&= \frac{1}{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_k=1}^{N}\mathbb{E}\left(A_{j_1,j_2}^{(N,p,\alpha)}A_{j_2,j_3}^{(N,p,\alpha)}\dots A_{j_k,j_1}^{(N,p,\alpha)}\right) \\
&= \frac{1}{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_k=1}^{N}\mathbb{E}\left(a_{j_1,j_2}a_{j_2,j_3}\dots a_{j_k,j_1}\right) \\
&\quad \times\mathbb{E}\left(d_{j_1,j_2}^{(N,p)}d_{j_2,j_3}^{(N,p)}\dots d_{j_k,j_1}^{(N,p)}\right)\xi_{j_1,j_2}^{(N,\alpha)}\xi_{j_2,j_3}^{(N,\alpha)}\dots\xi_{j_k,j_1}^{(N,\alpha)}, \qquad (3.1)
\end{aligned}$$

where

$$\xi_{ij}^{(N,\alpha)} = \begin{cases} 1, \text{if } (i\in I_{\alpha,N}\wedge j\notin I_{\alpha,N})\vee(i\notin I_{\alpha,N}\wedge j\in I_{\alpha,N}), \\ 0, \text{otherwise.}\end{cases}$$

Let $W_k^{(N)}$ be a set of closed walks of k steps over the set $\overline{1,N}$:

$$W_k^{(N)} = \{w=(w_1,w_2,\cdots,w_k,w_{k+1}=w_1): \forall i\in\overline{1,k+1}\ \ w_i\in\overline{1,N}\}.$$

For $w\in W_k^{(N)}$, let us denote $a(w)=\prod_{i=1}^k a_{w_i,w_{i+1}}$, $d^{(N,p)}(w)=\prod_{i=1}^k d_{w_i,w_{i+1}}^{(N,p)}$ and $\xi^{(N,\alpha)}(w)=\prod_{i=1}^k \xi_{w_i,w_{i+1}}^{(N,\alpha)}$. Then we have

$$M_k^{(N,p)} = \frac{1}{N}\sum_{w\in W_k^{(N)}}\mathbb{E}a(w)\,\mathbb{E}d^{(N,p)}(w)\,\xi^{(N,\alpha)}(w). \qquad (3.2)$$

Let $w\in W_k^{(N)}$ and $f,g\in\overline{1,N}$. Denote by $n_w(f,g)$ the number of steps $f\to g$ and $g\to f$,

$$n_w(f,g) = \#\{i\in\overline{1,k}:\ (w_i=f\ \wedge\ w_{i+1}=g)\vee(w_i=g\ \wedge\ w_{i+1}=f)\}.$$

Then

$$\mathbb{E}a(w) = \prod_{f=1}^{N}\prod_{g=f}^{N}X_{n_w(f,g)}.$$

Given $w\in W_k^{(N)}$, let us define the sets $V_w=\cup_{i=1}^k\{w_i\}$ and $E_w=\cup_{i=1}^k\{(w_i,w_{i+1})\}$, where (w_i,w_{i+1}) is a non-ordered pair. It is easy to see that $G_w=(V_w,E_w)$ is a simple non-oriented graph and the walk w covers the graph G_w. Let us call G_w

the skeleton of walk w. We denote by $n_w(e)$ the number of passages of the edge e by the walk w in direct and inverse directions. For $(w_j, w_{j+1}) = e_j \in E_w$, we denote $a_{e_j} = a_{w_j,w_{j+1}} = a_{w_{j+1},w_j}$. Then we obtain

$$\mathbb{E}a(w) = \prod_{e\in E_w} \mathbb{E}a_e^{n_w(e)} = \prod_{e\in E_w} X_{n_w(e)}.$$

Similarly, we can write

$$\mathbb{E}d^{(N,p)}(w) = \prod_{e\in E_w} \mathbb{E}\left([d_e^{(N,p)}]^{n_w(e)}\right) = \prod_{e\in E_w} \frac{p}{N}.$$

Then we can write (3.2) in the form

$$M_k^{(N,p)} = \frac{1}{N} \sum_{w\in W_k^{(N)}} \xi^{(N,\alpha)}(w) \prod_{e\in E_w} \frac{p\, X_{n_w(e)}}{N}$$

$$= \sum_{w\in W_k^{(N)}} \xi^{(N,\alpha)}(w) \left(\frac{p^{|E_w|}}{N^{|E_w|+1}} \prod_{e\in E_w} X_{n_w(e)} \right) = \sum_{w\in W_k^{(N)}} \theta(w), \qquad (3.3)$$

where $\theta(w)$ is the contribution of the walk w to the mathematical expectation of the corresponding moment. To perform the limiting transition $N \to \infty$, it is natural to divide $W_k^{(N)}$ into the classes of equivalence. The walks $w^{(1)}$ and $w^{(2)}$ are equivalent $w^{(1)} \sim w^{(2)}$ if and only if there exists a bijection ϕ between the sets of vertices $V_{w^{(1)}}$ and $V_{w^{(2)}}$ such that for $i = 1, 2, \ldots, k$, $w_i^{(2)} = \phi(w_i^{(1)})$,

$$w^{(1)} \sim w^{(2)} \iff \exists \phi :\ V_{w^{(1)}} \overset{bij}{\to} V_{w^{(2)}} :\ \forall\, i \in \overline{1, k+1}\,,$$
$$w_i^{(2)} = \phi(w_i^{(1)}) \wedge \mathbf{1}_{I_{\alpha,N}}(w_i^{(2)}) = \mathbf{1}_{I_{\alpha,N}}(\phi(w_i^{(1)})).$$

The last condition requires that every vertex and its image be in the same component. It is essential for further computations because the contribution of a walk equals zero when the origin and the end of the step are in the same component. Let us denote by $[w]$ the class of equivalence of walk w, and by $C_k^{(N)}$ the set of these classes. It is obvious that if two walks $w^{(1)}$ and $w^{(2)}$ are equivalent, then their contributions are equal:

$$w^{(1)} \sim w^{(2)} \implies \theta(w^{(1)}) = \theta(w^{(2)}).$$

The cardinality of the class of equivalence $[w]$ is equal to the number of all mappings $\phi : V_w \to \overline{1, N}$ such that $\phi(V_{1,w}) \subset I_{\alpha,N}$ and $\phi(V_{2,w}) \subset \overline{1, N} \setminus I_{\alpha,N}$ (where $V_{1,w} = V_w \cap I_{\alpha,N}$ and $V_{2,w} = V_w \setminus I_{\alpha,N}$) is equal to the number $[\alpha N]([\alpha N] - 1)$

$\dots([\alpha N]-|V_{1,w}|+1)(N-[\alpha N])(N-[\alpha N]-1)\dots(N-[\alpha N]-|V_{2,w}|+1)$. Then (3.3) can be written in the form

$$M_k^{(N,p)}=\sum_{w\in W_k^{(N)}}\xi^{(N,\alpha)}(w)\left(\frac{p^{|E_w|}}{N^{|E_w|+1}}\prod_{e\in E_w}X_{n_w(e)}\right)$$

$$=\sum_{[w]\in\mathcal{C}_k^{(N)}}\xi^{(N,\alpha)}(w)\prod_{e\in E_w}X_{n_w(e)}\Bigg(\frac{[\alpha N]([\alpha N]-1)\dots([\alpha N]-|V_{1,w}|+1)}{N^{|E_w|+1}p^{-|E_w|}}$$

$$\times(N-[\alpha N])(N-[\alpha N]-1)\dots(N-[\alpha N]-|V_{2,w}|+1)\Bigg)=\sum_{[w]\in\mathcal{C}_k^{(N)}}\hat{\theta}([w]). \quad (3.4)$$

In the second line of (3.4), for every class $[w]$ we choose an arbitrary walk w corresponding to this class of equivalence.

3.2. Minimal and Essential Walks

The class of walks $[w]$ of $\mathcal{C}_k^{(N)}$ has at most k vertices. Hence, $\mathcal{C}_k^{(1)}\subset\mathcal{C}_k^{(2)}\subset\dots\subset\mathcal{C}_k^{(i)}\subset\dots\mathcal{C}_k^{(k\ \min(\alpha,1-\alpha)^{-1})}=\mathcal{C}_k^{(k\ \min(\alpha,1-\alpha)^{-1}+1)}=\dots$. It is natural to denote $\mathcal{C}_k=\mathcal{C}_k^{(\min(\alpha,1-\alpha)^{-1})}$. Then (3.4) can be written as

$$m_k^{(p)}=\lim_{N\to\infty}\sum_{[w]\in\mathcal{C}_k}\xi^{(N,\alpha)}(w)\alpha^{|V_{1,w}|}(1-\alpha)^{|V_w|-|V_{1,w}|}\left(N^{|V_w|-|E_w|-1}\prod_{e\in E_w}\frac{X_{n_w(e)}}{p^{-1}}\right). \quad (3.5)$$

The set $\mathcal{C}_k$ is finite. Regarding this and (3.5), we conclude that the class $[w]$ has non-vanishing contribution if $|V_w|-|E_w|-1\geq 0$ and w is a bipartite walk on the complete bipartite graph $K_{I_{\alpha,N},\overline{1,N}\setminus I_{\alpha,N}}$. But for each simple connected graph $G=(V,E)$ $|V_w|\leq|E_w|+1$, and the equality takes place if and only if the graph G is a tree.

It is convenient to use the notion of a minimal walk.

Definition 1. *The walk w is a minimal walk if w_1 (the root of walk) has the number* 1 *and the number of every new vertex is equal to the number of all already passed vertices plus* 1.

Let us denote the set of all minimal walks of $W_k^{(N)}$ by $\mathcal{I}_k^{(N)}$.

E x a m p l e 1. The sequences (1,2,1,2,3,1,4,2,1,4,3,1) and (1,2,3,2,4,2,3,2,1,2,4,1,5,1) represent the minimal walks.

Definition 2. *The minimal walk w that has a tree as a skeleton is an essential walk.*

Let us denote the set of all essential walks of $W_k^{(N)}$ by $\mathcal{E}_k^{(N)}$. Therefore we can rewrite (3.5) in the form

$$m_k^{(p)} = \sum_{w \in \mathcal{E}_k} \left(\theta_1(w) + \theta_2(w)\right), \tag{3.6}$$

where

$$\theta_1(w) = \alpha^{\beta(w)}(1-\alpha)^{|V_w|-\beta(w)} \left(\prod_{e \in E_w} \left(p\, X_{n_w(e)}\right) \right),$$

$$\theta_2(w) = (1-\alpha)^{\beta(w)}\alpha^{|V_w|-\beta(w)} \left(\prod_{e \in E_w} \left(p\, X_{n_w(e)}\right) \right),$$

where $\beta(w)$ is the number of vertices v such that the distance between v and the first vertex w_1 is even. The number of passages of each edge e belonging to the essential walk w is even. Hence, the limiting mathematical expectation $m_k^{(p)}$ depends only on the even moments of the random variable a. It is clear that the limiting mathematical expectation $\lim\limits_{N\to\infty} M_{2s+1}^{(N,p,\alpha)}$ is equal to zero.

3.3. First Edge Decomposition of Essential Walks

Let us start with necessary definitions. The first vertex $w_1 = 1$ of the essential walk w is called the root of the walk. We denote it by ρ. Let us denote the second vertex $w_2 = 2$ of the essential walk w by ν. We denote by l the half of walk's length, and by r the number of steps of w starting from the root ρ. In this subsection, we derive the recurrent relations by splitting the walk (or the tree) into two parts. To describe this procedure, it is convenient to consider the set of essential walks of length $2l$ such that they have r steps starting from the root ρ. We denote this set by $\Lambda(l,r)$. One can see that this description is exact in the sense that it is minimal and gives a complete description of the walks we need. Denote by $S^{(1)}(l,r)$, $S^{(2)}(l,r)$ the sum of contributions of the walk of $\Lambda(l,r)$ with the weights θ_1 and θ_2, respectively. Let us remove the edge $(\rho,\nu) = (1,2)$ from G_w and denote the obtained graph by $\hat{G}_w$. The graph $\hat{G}_w$ has two components. Denote the component that contains the vertex ν by G_2, and the component containing the root ρ by G_1. Add the edge (ρ,ν) to the edge set of the tree G_2. Denote the result by $\hat{G}_2$. Denote by u the half of the walk's length over the tree G_2, and by f the number of steps (ρ,ν) in the walk w. It is clear that the following inequalities hold for all essential walks (except the walk of length zero): $1 \le f \le r$, $r + u \le l$. Let us denote by $\Lambda_1(l,r,u,f)$ the set of essential walks with the fixed parameters l, r, u, f, and by $S_1^{(1)}(l,r,u,f)$ ($S_1^{(2)}(l,r,u,f)$), the sum of contributions of the walks of $\Lambda_1(l,r,u,f)$ with the weight θ_1 (θ_2). Denote by $\Lambda_2(l,r)$ the set of essential walks of $\Lambda(l,r)$ such that their skeleton has only one edge attached to the root ρ. We also denote by $S_2^{(1)}(l,r)$ and $S_2^{(2)}(l,r)$ the sum of the weights θ_1 and θ_2 of the walk of $\Lambda_2(l,r)$, respectively. Now we can formulate the first lemma of decomposition. It allows us to express $S^{(1)}$, $S^{(2)}$ as the functions of $S^{(1)}$, $S^{(2)}$, $S_2^{(1)}$, $S_2^{(2)}$.

Lemma 1 (First decomposition lemma)**.** *The following relations hold:*

$$S^{(1)}(l,r)=\sum_{f=1}^{r}\sum_{u=0}^{l-r} S_1^{(1)}(l,r,u,f), \tag{3.7}$$

$$S^{(2)}(l,r)=\sum_{f=1}^{r}\sum_{u=0}^{l-r} S_1^{(2)}(l,r,u,f), \tag{3.8}$$

where

$$S_1^{(1)}(l,r,u,f)=\alpha^{-1}\binom{r-1}{f-1}S_2^{(1)}(f+u,f)\,S^{(1)}(l-u-f,r-f), \tag{3.9}$$

$$S_1^{(2)}(l,r,u,f)=(1-\alpha)^{-1}\binom{r-1}{f-1}S_2^{(2)}(f+u,f)\,S^{(2)}(l-u-f,r-f). \tag{3.10}$$

P r o o f. The first two equalities are obvious. The last two equalities follow from the bijection F,

$$\Lambda_1(l,r,u,f) \overset{bij}{\to} \Lambda_2(f+u,f)\times\Lambda(l-u-f,r-f)\times\Theta_1(r,f), \tag{3.11}$$

where $\Theta_1(r,f)$ is the set of sequences of 0 and 1 of the length r such that there are exactly f symbols 1 in the sequence, and the first symbol is 1.

Let us construct the mapping F. Regarding one particular essential walk w of $\Lambda_1(l,r,u,f)$, we consider the first edge e_1 of the graph G_w and divide w into two parts, the left and the right ones with respect to the edge e_1. Then we add a special code that determines the transitions from the left part to the right one and back through the root ρ. Obviously these two parts are walks, but not necessary minimal walks. Then we minimize these walks. This decomposition is constructed by the following algorithm. We run over w and simultaneously draw the left part, the right part, and the code. If the current step belongs to G_l, we add it to the first part, otherwise we add this step to the second part. The code is constructed as follows. Each time the walk leaves the root, the sequence is enlarged by one symbol. If the current step is $\rho\to\nu$, then this symbol is "0", otherwise this symbol is "1". It is clear that the first element of the sequence is "1", the number of signs "1" is equal to f, and the full length of the sequence is r. Now we minimize the left and the right parts. Thus, we have constructed the decomposition of the essential walk w and the mapping F. The weight $\theta_1(w)(\theta_2(w))$ of the essential walk is multiplicative with respect to the edges and vertices. In the factors $S_2^{(1)}(f+u,f)$, $S^{(1)}(l-u-f,r-f)$, we twice count the multiplier corresponding to the root, so we need to add the factor α^{-1} to (3.9).

E x a m p l e 2. For $w=(1,2,1,2,3,2,1,4,1,2,5,2,1,4,6,4,1,2,5,2,3,2,3,2,1,4,1)$ the left part, the right one, and the code are $(1,2,1,2,3,2,1,2,4,2,1,2,4,2,3,2,3,2,1)$, $(1,2,1,2,3,2,1,2,1)$, $(1,1,0,1,0,1,0)$, respectively.

Let us denote the left part by $(w^{(f)})$ and the right part by $(w^{(s)})$. These parts are walks with the root ρ. For each edge e in the tree $\hat{G}_2$ the number of passages of e of the essential walk w is equal to the corresponding number of passages of e of the left part $(w^{(f)})$. Also, for each edge e belonging to the tree G_1 the number of passages of e of the essential walk w is equal to the corresponding number of passages of e of the right part $(w^{(s)})$. The weight of the essential walk is multiplicative with respect to the edges. Then the weight of the essential walk w is equal to the product of the weights of left and right parts. The walk of zero length has a unit weight. Combining this with (3.11), we obtain

$$S_1^{(1)}(l,r,u,f)=\alpha^{-1}\left|\Theta_1(r,f)\right|S_2^{(1)}(f+u,f)S^{(1)}(l-u-f,r-f), \tag{3.12}$$

$$S_1^{(2)}(l,r,u,f)=(1-\alpha)^{-1}\left|\Theta_1(r,f)\right|S_2^{(2)}(f+u,f)S^{(2)}(l-u-f,r-f). \tag{3.13}$$

Taking into account that $|\Theta_1(r,f)|=\binom{r-1}{f-1}$, we derive (3.9), (3.10) from (3.12), (3.13).

Now let us prove that for any given elements $w^{(f)}$ of $\Lambda_2(f+u,f)$, $w^{(s)}$ of $\Lambda(l-u-f,r-f)$, and the sequence $\theta\in\Theta_1(r,f)$, one can construct one and only one element w of $\Lambda_1(l,r,u,f)$. To do this, we use the following gathering algorithm. We go along either $w^{(f)}$ or $w^{(s)}$ and simultaneously draw the walk w. The switch from $w^{(f)}$ to $w^{(s)}$ and back is governed by the code sequence θ. In fact, this procedure is inverse to the decomposition procedure described above up to the fact that $w^{(s)}$ is minimal. This difficulty can be easily resolved, for example, by coloring the vertices of $w^{(f)}$ and $w^{(s)}$ in red and blue colors, respectively. Certainly, the common root of $w^{(f)}$ and $w^{(s)}$ has only one color. To illustrate the gathering procedures we give the following example.

E x a m p l e 3. For $w^{(f)}=(1,2,1,2,3,2,1,2,4,2,1,2,4,2,3,2,3,2,1)$, $w^{(s)}=(1,2,1,2,3,2,1,2,1)$, $\theta=(1,1,0,1,0,1,0)$ the gathering procedure gives $w=(1,2,1,2,3,2,1,4,1,2,5,2,1,4,6,4,1,2,5,2,3,2,3,2,1,4,1)$.

It is clear that the decomposition and gathering are injective mappings. Their domains are finite sets, and therefore the corresponding mapping (3.11) is bijective. This completes the proof of Lemma 1. ■

To formulate Lemma 2, let us give necessary definitions. We denote by v the number of steps starting from the vertex ν excepting the steps $\nu \to \rho$, and by $\Lambda_3(u+f,f,v)$ the set of essential walks of $\Lambda_2(u+f,f)$ with the fixed parameter v. We also denote by $S_3^{(1)}(u+f,f,v)$ ($S_3^{(2)}(u+f,f,v)$) the sum of the weights θ_1 (θ_2) of the walks of $\Lambda_3(u+f,f,v)$. Let us denote by $G_{1,2}$ the graph consisting of only one edge (ρ,ν), and by $\Lambda_4(f)$ the set of essential walks of length $2f$ such that their skeleton coincides with the graph $G_{1,2}$. It is clear that $\Lambda_4(f)$ consists of only one walk (1,2,1,2,...,2,1) of the weight $\frac{X_{2f}}{p^{-1}}$. Lemma 1 allows us to express $S^{(1)}$, $S^{(2)}$ as the functions of $\mathrm{S}_2^{(1)}$, $S_2^{(2)}$, $S^{(1)}$, $\mathrm{S}^{(2)}$. By the next lemma, $S_2^{(1)}$, $S_2^{(2)}$ can be expressed as the functions of $S^{(1)}$, $S^{(2)}$. Thus, two lemmas allow us to express $S^{(1)}$, $S^{(2)}$ as the functions of $\mathrm{S}^{(1)}$, $S^{(2)}$.

Lemma 2 (Second decomposition lemma). *We have:*

$$S_2^{(1)}(f+u,f) = \sum_{v=0}^{u} S_3^{(1)}(f+u,f,v), \tag{3.14}$$

$$S_2^{(2)}(f+u,f) = \sum_{v=0}^{u} S_3^{(2)}(f+u,f,v), \tag{3.15}$$

$$S_3^{(1)}(f+u,f,v) = \alpha \binom{f+v-1}{f-1} \frac{X_{2f}}{p^{-1}} S^{(2)}(u,v), \tag{3.16}$$

$$S_3^{(2)}(f+u,f,v) = (1-\alpha) \binom{f+v-1}{f-1} \frac{X_{2f}}{p^{-1}} S^{(1)}(u,v). \tag{3.17}$$

The first two equalities are trivial, the second two follow from the bijection

$$\Lambda_3(f+u,f,v) \overset{bij}{\to} \Lambda(u,v) \times \Lambda_4(f) \times \Theta_2(f+v,f), \tag{3.18}$$

where $\Theta_2(f+v,f)$ is the set of sequences of 0 and 1 of the length $f+v$ such that there are exactly f symbols 1 in the sequence and its last symbol is 1. The proof is analogous to that of the first decomposition lemma. The factor α in (3.16) is a contribution of the root in the weight.

Combining these two decomposition lemmas and changing the order of summation, we get recurrent relations (2.7), (2.8) with initial conditions (2.9).

Let us denote the set of double closed walks of k and m steps over $\overline{1,N}$ by $\mathcal{D}_{k,m}^{(N)} \overset{\text{def}}{\equiv} W_k^{(N)} \times W_m^{(N)}$. For $dw = (w^{(1)}, w^{(2)}) \in \mathcal{D}_{k,m}^{(N)}$, we denote $a(dw) =$

$a(w^{(1)})a(w^{(2)})$, $d^{(N,p)}(dw) = d^{(N,p)}(w^{(1)})d^{(N,p)}(w^{(2)})$, $\xi^{(N,\alpha)}(dw) = \xi^{(N,\alpha)}(w^{(1)}) \times \xi^{(N,\alpha)}(w^{(2)})$.

Then we obtain

$$C_{k,m}^{(N,p,\alpha)} = \frac{1}{N^2} \sum_{dw=(w^{(1)},w^{(2)})\in W_{k,m}^{(N)}} \xi^{(N,\alpha)}(dw) \Big\{ \mathbb{E}a(dw)\ \mathbb{E}d^{(N,p)}(dw)$$

$$-\mathbb{E}a(w^{(1)})\ \mathbb{E}d^{(N,p)}(w^{(1)})\ \mathbb{E}a(w^{(2)})\ \mathbb{E}d^{(N,p)}(w^{(2)}) \Big\}. \qquad (3.19)$$

For the closed double walks $dw = (w^{(1)}, w^{(2)}) \in \mathcal{D}_{k,m}^{(N)}$, we denote $n_{dw}(f,g) = n_{w^{(1)}}(f,g) + n_{w^{(2)}}(f,g)$. Introduce a simple non-oriented graph $G_{dw} = G_{w^{(1)}} \cup G_{w^{(2)}}$ for the double walk $dw = (w^{(1)}, w^{(2)}) \in \mathcal{D}_{k,m}^{(N)}$, i.e., $V_{dw} = V_{w^{(1)}} \cup V_{w^{(2)}}$ and $E_{dw} = E_{w^{(1)}} \cup E_{w^{(2)}}$. Then, we can rewrite 3.19 in the following form:

$$C_{k,m}^{(N,p,\alpha)} = \frac{1}{N^2} \sum_{dw=(w^{(1)},w^{(2)})\in W_{k,m}^{(N)}} \xi^{(N,\alpha)}(dw) \left\{ \prod_{e\in E_{dw}} \mathbb{E}a_e^{n_{dw}(e)}\ \mathbb{E}\left[d_e^{(N,p)}\right]^{n_{dw}(e)} \right.$$

$$\left. - \prod_{e\in E_{w(1)}} \mathbb{E}a_e^{n_{w^{(1)}}(e)}\ \mathbb{E}\left[d_e^{(N,p)}\right]^{n_{w^{(1)}}(e)} \prod_{e\in E_{w(1)}} \mathbb{E}a_e^{n_{w^{(2)}}(e)}\ \mathbb{E}\left[d_e^{(N,p)}\right]^{n_{w^{(2)}}(e)} \right\}$$

$$= \frac{1}{N^2} \sum_{dw=(w^{(1)},w^{(2)})\in W_{k,m}^{(N)}} \xi^{(N,\alpha)}(dw) \left\{ \left(\frac{p}{N}\right)^{|E_{dw}|} \prod_{e\in E_{dw}} X_{n_{dw}(e)} \right.$$

$$\left. - \left(\frac{p}{N}\right)^{|E_{w^{(1)}}|+|E_{w^{(2)}}|} \prod_{e\in E_{w^{(1)}}} X_{n_{w^{(1)}}(e)} \prod_{e\in E_{w^{(2)}}} X_{n_{w^{(2)}}(e)} \right\}. \qquad (3.20)$$

To perform the limiting transition, $N \to \infty$, it is natural to divide $\mathcal{D}_k^{(N)}$ into classes of equivalence. The double walks $dw = (w^{(1)}, w^{(2)})$ and $du = (u^{(1)}, u^{(2)})$ from $\mathcal{D}_{k,m}^{(N)}$ are equivalent if and only if their first walks are equivalent and their second walks are equivalent:

$$dw \sim du \Leftrightarrow \left(w^{(1)} \sim u^{(1)} \wedge w^{(2)} \sim u^{(2)} \right).$$

Let us denote by $[dw]$ the class of equivalence of the double walk dw, and by $\mathcal{F}_k^{(N)}$ the set of these classes. Then (3.20) can be written in the following form:

$$C_{k,m}^{(N,p,\alpha)} = \frac{1}{N^2} \sum_{[dw]\in \mathcal{F}_{k,m}^{(N)}} \xi^{(N,\alpha)}(dw) \left\{ \frac{p^{|E_{dw}|}[\alpha N]([\alpha N]-1)\dots([\alpha N]-|V_{1,w^{(1)}}|+1)}{N^{|E_{dw}|}} \right.$$

$$\times[\alpha N]([\alpha N]-1)\dots([\alpha N]-|V_{1,w^{(2)}}|+1)$$
$$\times(N-[\alpha N])(N-[\alpha N]-1)\dots(N-[\alpha N]-|V_{2,w^{(1)}}|+1)$$
$$\times(N-[\alpha N])(N-[\alpha N]-1)\dots(N-[\alpha N]-|V_{2,w^{(2)}}|+1)$$
$$\times\left(\prod_{e\in E_{dw}}X_{n_{dw}(e)}-\frac{p^{|E_{w^{(1)}}|+|E_{w^{(2)}}|-|E_{dw}|}}{N^{|E_{w^{(1)}}|+|E_{w^{(2)}}|-|E_{dw}|}}\prod_{e\in E_{w^{(1)}}}X_{n_{w^{(1)}}(e)}\prod_{e\in E_{w^{(2)}}}X_{n_{w^{(2)}}(e)}\right)\Bigg\}. \quad (3.21)$$

Let us define a formal order of the passes for the double walk $dw=(w^{(1)},w^{(2)})\in\mathcal{D}_{k,m}^{(N)}$:

$$dw_i=\begin{cases}w_i^{(1)}, \text{ if } 1\le i\le k,\\ w_{i-k}^{(2)}, \text{if } k+1\le i\le k+m.\end{cases}$$

Let us denote the set of all minimal double walks of $\mathcal{D}_{k,m}^{(N)}$ by $\mathcal{G}_{k,m}^{(N)}$. Then we obtain

$$N\,C_{k,m}^{(N,p,\alpha)}=\sum_{w\in\mathcal{G}_{k,m}}\gamma(dw)\lim_{N\to\infty}\left[\frac{N^{|V_{dw}|-|E_{dw}|-1}}{p^{-|E_{dw}|}}\left(\prod_{e\in E_{dw}}X_{n_{dw}(e)}\right.\right.$$
$$\left.\left.-\frac{p^{c(dw)}}{N^{c(dw)}}\prod_{e\in E_{w^{(1)}}}X_{n_{w^{(1)}}(e)}\prod_{e\in E_{w^{(2)}}}X_{n_{w^{(2)}}(e)}\right)\right], \quad (3.22)$$

where $c(dw)$ is the number of common edges of $G_{w(1)}$ and $G_{w(2)}$, i.e., $c(dw)=|E_{w^{(1)}}|+|E_{w^{(2)}}|-|E_{dw}|$,

$$\gamma(dw)=\left(\alpha^{\beta(w^{(1)})}(1-\alpha)^{|V_{w^{(1)}}|-\beta(w^{(1)})}+(1-\alpha)^{\beta(w^{(1)})}\alpha^{|V_{w^{(1)}}|-\beta(w^{(1)})}\right)$$
$$\times\left(\alpha^{\beta(w^{(2)})}(1-\alpha)^{|V_{w^{(2)}}|-\beta(w^{(2)})}+(1-\alpha)^{\beta(w^{(2)})}\alpha^{|V_{w^{(2)}}|-\beta(w^{(2)})}\right).$$

$\mathcal{G}_{k,m}$ is a finite set. G_{dw} has at most 2 connected components. But if G_{dw} has exactly 2 connected components, then $V_{w^{(1)}}\cap V_{w^{(2)}}=\varnothing\Rightarrow E_{w^{(1)}}\cap E_{w^{(2)}}=\varnothing\Rightarrow c(dw)=0\Rightarrow\left(\prod_{e\in E_{dw}}V_{n_{dw}(e)}-\frac{p^{c(dw)}}{N^{c(dw)}}\prod_{e\in E_{w^{(1)}}}V_{n_{w^{(1)}}(e)}\prod_{e\in E_{w^{(2)}}}V_{n_{w^{(2)}}(e)}\right)=0$. Hence the contribution of such minimal double walks to $N\,C_{k,m}^{(N,p,\alpha)}$ is equal to 0. Otherwise, G_{dw} is a connected graph and $|V_w|-|E_w|-1\le 0$. Thus (i) of Theorem 2 is proved.

Let us define a cluster of an essential walk w as a set of non-oriented edges incident to a given vertex v of G_w (v is a center of the cluster). Therefore the cluster is a subgraph of the skeleton G_w. Also, let us define an ordered cluster of the essential walk w as a cluster of the essential walk w with the sequence of numbers $n_w(e_0)$, $n_w(e_1), \dots, n_w(e_l)$, where $\{e_r\}_{r=0}^{l}$ is a sequence of edges of the cluster ordered by time of their first passing. We can derive from two decomposition lemmas that the number of passes covering the ordered cluster with numbers $2j$, $2i_1$, $2i_2$, $\dots 2i_l$ ($s = \sum_{j=1}^{l} i_j$) is equal to

$$n(j; i_1, i_2, \dots, i_l) = \binom{j+s-1}{j-1} \frac{s! \prod_{r=1}^{l} i_r}{s(s-i_1)(s-i_1-i_2)\dots i_l \prod_{r=1}^{l} i_r!}. \tag{3.23}$$

Indeed, the steps to the center of ordered cluster v are uniquely determined by the choice of steps from the vertex v. We can choose the steps from v along the edge e_0 in $\binom{j+s-1}{j-1}$ ways. After that we can choose the steps from v along the edge e_1 in $\binom{s-1}{i_1-1}$ ways. Then we can choose the steps from v along the edge e_2 in $\binom{s-i_1-1}{i_2-1}$ ways, and so on. Thus, the number of the passes covering the ordered cluster with numbers $2j$, $2i_1$, $2i_2$, $\dots 2i_l$ ($s = \sum_{j=1}^{l} i_j$) is equal to

$$\binom{j+s-1}{j-1}\binom{s-1}{i_1-1}\binom{s-i_1-1}{i_2-1}\dots\binom{s-i_1-i_2-\dots-i_{l-1}-1}{i_l-1} = \binom{j+s-1}{j-1}$$
$$\times \frac{s!\, i_1}{s(s-i_1)!\, i_1!} \frac{(s-i_1)!\, i_2}{(s-i_1)(s-i_1-i_2)!\, i_2!} \dots \frac{(s-i_1-i_2-\dots-i_{l-1})!\, i_l}{(s-i_1-i_2-\dots-i_{l-1})\, 0!\, i_l!}. \tag{3.24}$$

But it is evident that the numbers in (3.23) and (3.24) coincide.

Let us define the weight of the ordered cluster by

$$\theta(j; i_1, i_2, \dots, i_l) = n(j; i_1, i_2, \dots, i_l) \prod_{r=1}^{l} (p\, X_{2i_r}). \tag{3.25}$$

Combining (3.25) with (2.4) and estimating $C_1 k^k e^{-k} \le k! \le C_2 k^k e^{-k}$, we obtain

$$\theta(j; i_1, i_2, \dots, i_l) \le 2^j C_3^s s^{2s} (1+p)^s. \tag{3.26}$$

Let us define the ordered skeleton as a skeleton with all ordered clusters and a chosen root. First we define the weight of the ordered skeleton as the number of passes covering it multiplied by $\prod_{e \in E_w} (p\, X_{n_w(e)})$. From (3.26), we get that the weight of the ordered skeleton of essential walk w from $\mathcal{E}_k$ is not greater than $C_4^k k^{2k} (1+p)^k$. We can regard the ordered skeleton of essential walk as a half-plane rooted tree in which for each edge there is a natural number assigned to it. To calculate the number of all ordered skeletons corresponding to $2k$-th moment, we consider all appropriate half-plane rooted trees with i edges and all distributions of $k-i$ undistinguishable balls into i distinguishable boxes. It is seen that the number of all ordered skeletons is not greater than $\sum_{i=0}^{k}\left(\frac{2i!}{i!(i+1)!}\binom{k-1}{i-1}\right)\leq 2^{3k}$. Thus (iii) of Theorem 2 is proved.

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