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Functional Models in De Branges Spaces of One Class Commutative Operators

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For a commutative system of the linear bounded operators T_1, T_2, which operate in the Hilbert space H and none of the operators T_1, T_2 is a compression, the functional model is constructed. The model is built for a circle in de Branges space.

Key words: functional model, de Branges space, commutative systems of operators.

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The functional model of the compression operator T acting in the Hilbert space H was first obtained by B.S. Nagy and C. Foias [5]. The model allows to present the operator T as an operator of multiplication by the independent variable in a special space of functions [5, 2]. The study of the spectral characteristics of this model has led to a number of non-trivial problems on either functional analysis or theory of functions including the issues of interpolation, tasks of basis, completeness, etc. [2].

When the Nagy–Foias dilation technique [5] was used, there appeared significant difficulties in the constructing of similar functional models for the commutative systems of the operators $\{T_1, T_2\}$ defined in the Hilbert space H. Thus, the above problem could not be solved even for T_1 and T_2 being compressible. The solution was found in [7], which is based on a generalization of the concept node for commutative system operators, and in fact was proposed by Livshits.

In [8], a functional model of a pair of commutative operators is built when one of them is compressed. The construction is based on the Fourier transformation technique. If none of the operators $\{T_1, T_2\}$ is not a compression, then the given method is not applicable. In this paper, we construct the functional models for a commutative system of the operators $\{T_1, T_2\}$ where neither T_1 nor T_2 is compressed. For this case the functional model is constructed in de Branges space corresponding to the unit circumference obtained in [6].

1. Background Information

Let us consider the bounded linear operator T acting in the Hilbert space H. The collection

$$\triangle = (J; H \oplus E; V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix}; H \oplus \tilde{E}; \tilde{J}) \tag{1.1}$$

is called a unitary knot [1-4] if the linear operator

$$V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix} : H \oplus E \mapsto H \oplus \tilde{E} \tag{1.2}$$

satisfies the correlation

$$V^* \begin{bmatrix} I & 0 \\ 0 & \tilde{J} \end{bmatrix} V = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \qquad V \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} V^* = \begin{bmatrix} I & 0 \\ 0 & \tilde{J} \end{bmatrix}, \tag{1.3}$$

where J and $\tilde{J}$ are involutions in the Hilbert spaces E and $\tilde{E}$, respectively, $J = J^* = J^{-1}$, $\tilde{J} = \tilde{J}^* = \tilde{J}^{-1}$. Any bounded linear operator T in H can always be included into a unitary knot $\triangle$ (1.1) if we set [2], $-E = \overline{D_{T^*}H}; \tilde{E} = \overline{D_T H}; \Psi = \sqrt{|D_T|}; \Phi = \sqrt{|D_{T^*}|}; J = \mathrm{sign} D_{T^*}; \tilde{J} = \mathrm{sign} D_T; K = -\tilde{J}T^*$; where, as usually, $D_T = I - T^*T$ are defective operators of T, and $\sqrt{|A|}$, $\mathrm{sign} A$ of the self-adjoint operator A are understood in terms of the corresponding spectral decompositions.

The knot $\triangle$ (1.1) is called simple [2] if $H = H_1$, where

$$H_1 = \mathrm{span}\{T^n \Phi f + T^{*m} \Psi^* g; f \in E; g \in \tilde{E}; n, m \in \mathbb{Z}_+\}. \tag{1.4}$$

The subspaces H_1 and $H_0 = H_1^\perp = H \ominus H_1$ reduce the operator T, and the reducing of T to H_0 is a unitary operator [2].

The main invariant of the knot $\triangle$ (1), which describes simple knots, is a characteristic operator function introduced by Livshits in 1946, [1],

$$S_\triangle = K + \Psi(zI - T)^{-1}\Phi, \tag{1.5}$$

which plays the main role in the theory of triangular [2] and functional models [4, 5] for the operators close to the unitary ones (in terms of definition (1.1)).

Suppose that $\dim E = \dim \tilde{E} = r < \infty$ and $J = \tilde{J}$. Let us choose the orthonormalized bases $\{e_\alpha\}_1^r$ and $\{e'_\alpha\}_1^r$ in E and $\tilde{E}$. Then from the results of Potapov [2] it follows that the matrix-function $S_\triangle(z) = \| < S_\triangle(z) e_\alpha, e'_\beta > \|$, in the case when the spectrum $\sigma(T)$ of the operator T belongs to the unit circumference $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$, has the multiplicative structure

$$S_\triangle(z) = \overset{\curvearrowleft}{\int_0^l} \exp\left\{ \frac{e^{\imath\varphi_t} + z}{e^{\imath\varphi_t} - z} J\, dF_t \right\}, \tag{1.6}$$

where φ_t is a non-negative non-decreasing on $[0,\ell]$ function, and $0 \leq \varphi_t \leq 2\pi$; F_t is a non-decreasing hermitian $(r \times r)$ matrix-function on $[0,\ell]$ for which $trF_t \equiv t$.

Using Potapov's presentation (1.6), it is not difficult to build a triangular model of the operator for $S_\triangle(z)$ (5). By $L^2_{r,l}(F_x)$, denote the Hilbert space of the vector functions

$$L^2_{r,l}(F_x) = \left\{ f(x) = (f_1(x), \ldots, f_r(x)); \int_0^l f(x)\, dF_x f^*(x) < \infty \right\}. \tag{1.7}$$

In $L^2_{r,l}(F_x)$ (1.7), define the linear operator T,

$$Tf(x) = f(x) e^{\imath\varphi_x} - 2 \int_x^l f(t)\, dF_t \Phi_t^* \Phi_x^{*-1} J e^{\imath\varphi_x}, \tag{1.8}$$

where the matrix Φ_x is a solution of the integral equation

$$\Phi_x + \int_0^x \Phi_t\, dF_t J = I, \qquad x \in [0,l]. \tag{1.9}$$

Similarly, the matrix-function Ψ_x is a solution of

$$\Psi_x + \int_x^l \Psi_t\, dF_t J = J, \qquad x \in [0,l]. \tag{1.10}$$

Let us define the operators $\Phi : E \mapsto L^2_{r,l}(F_x)$ and $\Psi : L^2_{r,l}(F_x) \mapsto E$ (here $E = \mathbb{C}^n$) as follows:

$$\Phi f(x) = \sqrt{2} f \Psi_x e^{\imath\varphi_x}, \qquad \Psi f(x) = \sqrt{2} \int_0^l f(x)\, dF_x \Phi_x^*, \tag{1.11}$$

where $f \in E$ and $K = S_\triangle(\infty)$ (1.6). The collection

$$\triangle_c = (J; L^2_{r,l}(F_x) \oplus E; V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix}; L^2_{r,l}(F_x) \oplus E; J) \tag{1.12}$$

is a unitary knot (1.1)–(1.3) and is called a triangular model of the simple knot $\triangle$ (1.1), where $L^2_{r,l}(F_x)$, T, Φ, Ψ are from (1.7), (1.8)–(1.11). The latter means that simple components (1.4) of the knots $\triangle$ (1.1) and $\triangle_c$ (1.12), when the spectrum of the operator T is on the unit circumference $\sigma(T) \subseteq \mathbb{T}$, are unitarily equivalent

[2] under the condition that $J = \tilde{J}$ and $\dim E = \dim \tilde{E} = r < \infty$.
Let us suppose that $\dim E = 2$ and $J = J_N$, where

$$J_N = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.13}$$

According to [6], we introduce

$$L_x(z) = (1 - zT)^{-1}\Phi(1, 1), \tag{1.14}$$

$$\widetilde{L}_x(z) = (1 - zT^*)^{-1}\Psi^*(1, -1). \tag{1.15}$$

Definition 1. *The de Branges space $\mathcal{B}(E, G)$ is a Hilbert space formed by the vector functions $F(z) = [F_1(z), F_2(z)]$, where $F_k(z)$, $(k = 1, 2)$ are*

$$F_1(z) = \int\limits_0^l f(t)\, dF_t L_t^*(\bar{z}), \quad F_2(z) = \int\limits_0^l f(t)\, dF_t \tilde{L}_t^*(\bar{z}). \tag{1.16}$$

Present the de Branges space as follows:

$$\mathcal{B}_\phi f = [F_1(z), F_2(z)]. \tag{1.17}$$

The scalar product in $\mathcal{B}(E, G)$ is induced by the prototype mapping $\mathcal{B}_\varphi$ (1.17),

$$< F(z), \hat{F}(z) >_{\mathcal{B}_\varphi(E,G)} = < f(t), \hat{f}(t) >_{L^2_{2,l}(F_t)}, \tag{1.18}$$

while $F(z) = \mathcal{B}_\varphi f(t)$, $\hat{F}(z) = \mathcal{B}_\varphi \hat{f}(t)$, where $f(t), \hat{f}(t) \in L^2_{2,l}(F_t)$.
The functions $E_x(z), \widetilde{E}_x(z), G_x(z), \widetilde{G}_x(z)$ are defined by the relations [6]

$$L_x(z) = (e^{-i\phi_x} - z)^{-1}[E_x(z), \widetilde{E}_x(z)], \tag{1.19}$$

$$\widetilde{L}_x(z) = (1 - ze^{-i\phi_x})^{-1}[G_x(z), \widetilde{G}_x(z)]. \tag{1.20}$$

Let T_1, T_2 be a commutative system of the linear bounded operators acting in the Hilbert space H. The collection of the Hilbert spaces $E, \tilde{E}$ and the operators $\Phi \in [E, H]; \Psi \in [H, \tilde{E}]; K \in [E, \tilde{E}]; \sigma_s, \tau_s, N_s, \Gamma_1 \in [E, E]; \tilde{\sigma}_s, \tilde{\tau}_s, \tilde{N}_s, \tilde{\Gamma}_1 \in [\tilde{E}, \tilde{E}] (s = 1, 2)$ is called a commutative unitary metric knot $\triangle$,

$$\triangle = (\Gamma_1, \sigma_s, \tau_s, N_s, H \oplus E, V_s, \overset{+}{V}_s, H \oplus \tilde{E}, \tilde{N}_s, \tilde{\tau}_s, \tilde{\sigma}_s, \tilde{\Gamma}_1), \tag{1.21}$$

if for the expansions

$$V_s = \begin{bmatrix} T_s & \Phi N_s \\ \Psi & K \end{bmatrix}, \qquad \overset{+}{V}_s = \begin{bmatrix} T_s^* & \Psi^* \tilde{N}_s^* \\ \Phi^* & K^* \end{bmatrix}$$

the following relations are true:

1) $V_s^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V_s = \begin{bmatrix} I & 0 \\ 0 & \tau_s \end{bmatrix}, \qquad \overset{+}{V}{}_s^* \begin{bmatrix} I & 0 \\ 0 & \sigma_s \end{bmatrix} \overset{+}{V}_s = \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau}_s \end{bmatrix},$

2) $T_2\Phi N_1 - T_1\Phi N_2 = \Phi\Gamma_1,\ \tilde{N}_1\Psi T_2 - \tilde{N}_2\Psi T_1 = \tilde{\Gamma}_1\Psi,$

3) $\tilde{N}_2\Psi\Phi N_1 - \tilde{N}_1\Psi\Phi N_2 = K\Gamma_1 - \tilde{\Gamma}_1 K,\ KN_s = \tilde{N}_s K (s = 1, 2),$

where $\sigma_s, \tau_s, (\tilde{\sigma}_s, \tilde{\tau}_s)$ are self-adjoint in $E(\tilde{E})$, $(s = 1, 2)$.
The operators acting in the spaces E and $\tilde{E}$ of the knot $\triangle$ (1.21) are dependent. An arbitrary commutative system of the linear bounded operators T_1, T_2 can always be included into the knot $\triangle$ (1.21) [1]. If the "defective" operators σ_1 and $\tilde{\sigma}_1$ in E and $\tilde{E}$ are reversible, we can always suppose that N_1 and $\tilde{N}_1$ are reversible. Let us introduce $N, \tilde{N}, \Gamma, \tilde{\Gamma}$ in the following form:

$$N = N_1^{-1}N_2,\ \ \Gamma = N_1^{-1}\Gamma_1,\ \ \tilde{N} = \tilde{N}_1^{-1}\tilde{N}_2,\ \ \tilde{\Gamma} = \tilde{N}_1^{-1}\tilde{\Gamma}_1. \tag{1.22}$$

Let us set the linear operators T_1 and T_2 in $L^2_{r,l}(F_x)$ (1.7):

$$T_1 f(x) = f(x)e^{\imath\varphi_x} - 2\int_x^l f(t)\, dF_t \Phi_t^* \Phi_x^{*-1} J e^{\imath\varphi_x}, \tag{1.23}$$

$$T_2 f(x) = f(x)\,(N(x)e^{\imath\varphi_x} + \Gamma(x)) - 2\int_x^l f(t)\, dF_t \Phi_t^* \Phi_x^{*-1} J N(x) e^{\imath\varphi_x}, \tag{1.24}$$

where $N(x)$ and $\Gamma(x)$ satisfy the differential Lax equations [7]:

$$N'(x) = [a_x J, N(x)],\ \ N(0) = \tilde{N}_2,\ \ \Gamma'(x) = [\Gamma(x), a_x J],\ \ \Gamma(0) = \tilde{\Gamma}_2,$$

$$[a_x J, \Gamma(x) + e^{\imath\varphi_x} N(x)] = 0,$$

where $dF_x = a_x dx$.

2. Effect of Operators T_1 and T_1^* on Vectors L_x and $\widetilde{L}_x$

Let the knot $\triangle$ (1.21) corresponds to the commutative system of the operators $\{T_1, T_2\}$. Suppose that $E = \widetilde{E}$, $\dim E = \dim \widetilde{E} = 2$ and $\sigma_1 = \tilde{\sigma}_1 = J_N$ (1.13), the spectrum of the operator T_1 consists of one point $\{1\}$, and therefore, $\varphi_x = 0$. By $L_x(z)$ and $\widetilde{L}_x(z)$, denote the vector functions (1.14),(1.15) which correspond to the operator $T_1 (T = T_1)$. We also denote the functions $E_x(z), \widetilde{E}_x(z), G_x(z), \widetilde{G}_x(z)$ by (1.19), (1.20).

Lemmas 1–4 were proved in [9]. They define the effect of the operators T_1 and T_1^* on the vectors L_x and $\widetilde{L}_x$.

Lemma 1. [9] *The operator T_1 affects the vector function $L_x(z)$ (1.14) in the following way:*

$$T_1 L_x(z) = \frac{L_x(z) - L_x(0)}{z}. \tag{2.1}$$

Lemma 2. [9] *The operator T_1 affects the vector function $\widetilde{L}_x(z)$ (1.15) in the following way:*

$$T_1 \widetilde{L}_x(z) = z\widetilde{L}_x(z) + \frac{\widetilde{G}_l(z) - G_l(z)}{2} L_x(0) - \frac{\widetilde{G}_l(z) + G_l(z)}{2}(1,-1)\Psi_x. \tag{2.2}$$

Lemma 3. [9] *The operator T_1^* affects the vector function $\widetilde{L}_x(z)$ in the following way:*

$$T_1^* \widetilde{L}_x(z) = \frac{\widetilde{L}_x(z) - \widetilde{L}_x(0)}{z}. \tag{2.4}$$

Lemma 4. [9] *The operator T_1^* affects the vector function $L_x(z)$ in the following way:*

$$T_1^* L_x(z) = zL_x(z) + \frac{E_0(z) - \widetilde{E}_0(z)}{2}\widetilde{L}_x(0) + \frac{E_0(z) + \widetilde{E}_0(z)}{2}(1,1)\Phi_x. \tag{2.5}$$

Let us prove the lemma below.

Lemma 5. *If the vector functions $L_x(z)$ and $\widetilde{L}_x(z)$ are set by* (1.14) *and* (1.15)*, and Φ_x, Ψ_x are the solutions of integral equations* (1.9) *and* (1.10)*, then*

$$\int_0^l (1,-1)\Psi_t dF_t L_t^*(\overline{z}) = -1 - \frac{1}{2}R_1 J \begin{pmatrix} \overline{E_0(\overline{z})} \\ \overline{\widetilde{E}_0(\overline{z})} \end{pmatrix}, \tag{2.6}$$

$$\int_0^l (1,-1)\Psi_t dF_t \widetilde{L}_t^*(\overline{z}) = \frac{1}{2}R_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\overline{G_l(\overline{z})} - \overline{\widetilde{G}_l(\overline{z})}}{2}, \tag{2.7}$$

$$\int_0^l (1,1)\Phi_t dF_t L_t^*(\overline{z}) = \frac{1}{2z}R_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2z}, \tag{2.8}$$

$$\int_0^l (1,1)\Phi_x dF_t \widetilde{L}_t^*(\overline{z}) = \frac{1}{2z}R_2 J \begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix}, \tag{2.9}$$

where R_1 and R_2 have the forms

$$R_1 = \left(\frac{\overline{G_l(0)}E_0(0) - \overline{\widetilde{G}_l(0)}E_0(0) \pm 1}{\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}}, \frac{(\overline{G_l(0)} - \overline{\widetilde{G}_l(0)})(\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}) \pm 1}{\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}} \right), \tag{2.10}$$

$$R_2 = \left(\frac{G_l(\infty)\overline{E_0(\overline{\infty})} - G_l^2(\infty) - G_l(\infty)\widetilde{G}_l(\infty) \pm 1}{\overline{G_l(\infty)} + \overline{\widetilde{G}_l(\infty)}}, \right.$$

$$\left. \frac{-3G_l(\infty)\overline{E_0(\infty)} + \widetilde{G}_l^2(\infty) + G_l(\infty)\widetilde{G}_l(\infty) - 2\widetilde{G}_l(\infty)\overline{E_0(\infty)} \pm 1}{\overline{G_l(\infty)} + \overline{\widetilde{G}_l(\infty)}} \right). \tag{2.11}$$

P r o o f. Let us consider the equation for the vector function $L_x(z)$

$$(1-z)L_x(z) + 2z\int_x^l L_t(z)dF_t\Phi_t^*\Phi_x^{*-1}J = (1,1)\Psi_x$$

and differentiate it by x to get

$$(1-z)L'_x(z) - 2zL_x(z)a_x\Phi_x^*\Phi_x^{*-1}J + 2z\int_x^l L_t(z)dF_t\Phi_t^*\Phi_x^{*-1}(\Phi_x^{*-1})'J = (1,1)\Psi'_x.$$

Since $\Psi'_x = \Psi_x a_x J$ and $\Phi'_x = -\Phi_x a_x J$, then $\Phi_x^{*\prime} = -Ja_x\Phi_x^*$ and $(\Phi_x^{*-1})' = \Phi_x^{*-1}Ja_x$. By using these statements, we obtain

$$(1-z)L'_x(z) - 2zL_x(z)a_xJ + ((1,1)\Psi_x - (1-z)L_x(x))a_xJ = (1,1)\Psi_x a_x J,$$

$$(1-z)L'_x(z) = (1+z)L_x(z)a_xJ,$$

i.e., $L'_x(z) = \frac{1+z}{1-z}L_x(z)a_xJ$ and $L_x^{*\prime}(z) = \frac{1+z}{1-z}Ja_xL_x^*(z)$. Let us consider the following statements:

$$(\Psi_xJL_x^*(z))' = \Psi_x a_x JJL_x(z) + \Psi_xJ\frac{1+\overline{z}}{1-\overline{z}}Ja_xL_x(z) = \frac{2}{1-\overline{z}}\Psi_x a_x L_x^*(z),$$

$$((1,-1)\Psi_xJL_x^*(z))' = (1,-1)\frac{2}{1-\overline{z}}\Psi_x a_x L_x^*(z).$$

Since $\Psi_l = J$, $\Phi_0 = I$, and $L_l(z) = (1,1)J\frac{1}{1-z}$, $L_l^*(\overline{z}) = J\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{1-z}$, $L_0(z) = \frac{1}{1-z}(E_0(z), \widetilde{E}_0(z))$ and $L_0^*(\overline{z}) = \frac{1}{1-z}(\overline{E_0(\overline{z})}, \overline{\widetilde{E}_0(\overline{z})})$, then after integrating the statement

$$(1,-1)\Psi_x a_x L_x^*(\overline{z}) = \frac{1-z}{2}((1,-1)\Psi_xJL_x^*(\overline{z})),$$

we obtain

$$\int\limits_0^l (1,-1)\Psi_x dF_x L_x^*(\overline{z}) = \frac{1-z}{2}(1,-1)\left(JJJ\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{1-z} - \Psi_0 J\frac{1}{1-z}\begin{pmatrix}\overline{E_0(\overline{z})}\\ \overline{\widetilde{E}_0(\overline{z})}\end{pmatrix}\right)$$

$$= -1 - \frac{1}{2}(1,-1)\Psi_0 J\begin{pmatrix}\overline{E_0(\overline{z})}\\ \overline{\widetilde{E}_0(\overline{z})}\end{pmatrix}. \tag{2.12}$$

Similarly, using $(\Phi_x J L_x^*(\overline{z}))' = \frac{2z}{1-z}\Phi_x a_x L_x^*(\overline{z})$, we integrate the following statement:

$$\int\limits_0^l (1,1)\Phi_t dF_t L_t^*(\overline{z}) = \frac{1-z}{2z}(1,1)\left(\Phi_l JJ\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{1-z} - IJ\frac{1}{1-z}(\overline{E_0(\overline{z})},\overline{\widetilde{E}_0(\overline{z})})\right)$$

$$= \frac{1}{2z}(1,1)\left(\Phi_l\begin{pmatrix}1\\1\end{pmatrix} - J(\overline{E_0(\overline{z})},\overline{\widetilde{E}_0(\overline{z})})\right) = \frac{1}{2z}(1,1)\Phi_l\begin{pmatrix}1\\1\end{pmatrix} + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2z}. \tag{2.13}$$

Now we take the equation for the vector function $\widetilde{L}_x(z)$,

$$(1-z)\widetilde{L}_x(z) + 2z\int\limits_0^l \widetilde{L}_t(z)dF_t J\Phi_t^{-1}\Phi_x = (1,-1)\Phi_x,$$

and differentiate it by x to get

$$(1-z)\widetilde{L}'_x(z) + 2z\widetilde{L}_x(z)a_x J\Phi_x^{-1}\Phi_x - 2z\int\limits_0^l \widetilde{L}_t(z)dF_t J\Phi_t^{-1}\Phi_x a_x J = -(1,-1)\Phi_x a_x J,$$

$$(1-z)\widetilde{L}'_x(z) + 2z\widetilde{L}_x(z)a_x J + ((1-z)\widetilde{L}_x(z) - (1,-1)\Phi_x)a_x J = -(1,-1)\Phi_x a_x J,$$
$$(1-z)\widetilde{L}'_x(z) + (1+z)\widetilde{L}_x(z)a_x J = 0.$$

Thus, $\widetilde{L}'_x(z) = -\frac{1+z}{1-z}\widetilde{L}_x(z)a_x J$ and $(\widetilde{L}_x^*(z))' = -\frac{1+z}{1-z}Ja_x\widetilde{L}_x^*(z)$. Let us consider the following statements:

$$(\Psi_x J\widetilde{L}_x^*(\overline{z}))' = \Psi_x a_x JJ\widetilde{L}_x^*(\overline{z}) - \Psi_x J\frac{1+z}{1-z}Ja_x\widetilde{L}_x^*(\overline{z}) = \frac{-2z}{1-z}\Psi_x a_x\widetilde{L}_x^*(\overline{z}).$$

And after integration we obtain

$$\int\limits_0^l (1,-1)\Psi_x a_x\widetilde{L}_x^*(\overline{z}) = \frac{1-z}{-2z}(1,-1)\left(JJ\frac{1}{1-z}\begin{pmatrix}\overline{G_l(\overline{z})}\\ \overline{\widetilde{G}_l(\overline{z})}\end{pmatrix} - \Psi_x J\frac{1}{1-z}I\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= \frac{1}{-2z}(1,-1)\begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} - \frac{1}{-2z}(1,-1)\Psi_0 \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence,

$$\int_0^l (1,-1)\Psi_x a_x \widetilde{L}_x^*(\overline{z}) = -\frac{1}{2z}(1,-1)\Psi_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\overline{G_l(\overline{z})} - \overline{\widetilde{G}_l(\overline{z})}}{2z}. \tag{2.14}$$

Similarly, we can get

$$(\Phi_x J \widetilde{L}_x^*(\overline{z}))' = \frac{2z}{1-z}\Phi_x a_x \widetilde{L}_x^*(\overline{z}),$$

then

$$\int_0^l (1,1)\Phi_x a_x \widetilde{L}_x^*(\overline{z}) = \frac{1-z}{2z}(1,1)\left(\Phi_l J \frac{1}{1-z}\begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} - IJ\frac{1}{1-z} I \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \tag{2.15}$$

and

$$\int_0^l (1,1)\Phi_x a_x \widetilde{L}_x^*(\overline{z}) = \frac{1}{2z}\Phi_l J \begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} + \frac{1}{z}.$$

Write down a characteristic matrix-function $S_\triangle(z)$ element-wisely, $S_\triangle(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$, and find its coefficients. Since $N_0(z) = -S_\triangle(z)$, $\widetilde{N}_l^*(\overline{z}) = S_\triangle(z)$, $(1,1)N_x(z)J = (E_0(z), \widetilde{E}_0(z))$ and $(1,-1)\widetilde{N}_l(z) = (G_l(z), \widetilde{G}_l(z))$, then $\widetilde{N}_l^*(z)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix}$.

For $S_\triangle(z)$, we get the equations

$$-(1,1)\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} J = (E_0(z), \widetilde{E}_0(z)),$$

$$\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix}.$$

By solving this system, we obtain the coefficients of the matrix-function $S_\triangle(z)$:

$$c(z) = E_0(z) - a(z), \quad b(z) = a(z) - \overline{G_l(\overline{z})}, \quad d(z) = E_0(z) - \overline{\widetilde{G}_l(\overline{z})} - a(z).$$

Now we will use the condition $|\det S_{\triangle}(z)|^2 = 1$, i.e., $|\det S_{\triangle}(z)| = \pm 1$, or $a(z)d(z) - b(z)c(z) = \pm 1$, to get the expression for $a(z)$

$$a(z) = \frac{\overline{G_l(\overline{z})}E_0(z)1}{\overline{G_l(\overline{z})} + \overline{\widetilde{G}_l(\overline{z})}} .$$

Now we can find the expression of $(1,-1)\Psi_x$:

$$(1,-1)\Psi_x = N_0(0)J = -(1,-1)S_{\triangle}(0)J$$

$$= \left(\frac{\overline{G_l(0)}E_0(0) - \overline{\widetilde{G}_l(0)}E_0(0) \pm 1}{\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}}, \frac{(\overline{G_l(0)} - \overline{\widetilde{G}_l(0)})(\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}) \pm 1}{\overline{G_l(0)} + \overline{\widetilde{G}_l(0)}} \right)$$

and the expression of $(1,1)\Phi_l$:

$$(1,1)\Phi_l = (1,1)\widetilde{N}_l(\infty) = \left(\frac{G_l(\infty)\overline{E_0(\overline{\infty})} - G_l^2(\infty) - G_l(\infty)\widetilde{G}_l(\infty) \pm 1}{\overline{G_l(\infty)} + \overline{\widetilde{G}_l(\infty)}}, \right.$$

$$\left. \frac{-3G_l(\infty)\overline{E_0(\infty)} + \widetilde{G}_l^2(\infty) + G_l(\infty)\widetilde{G}_l(\infty) - 2\widetilde{G}_l(\infty)\overline{E_0(\infty)} \pm 1}{\overline{G_l(\infty)} + \overline{\widetilde{G}_l(\infty)}} \right).$$

Having defined these expressions as R_1 and R_2, respectively, and using integrals (2.12)–(2.15), we obtain the expressions stated in the lemma definition. ■

Lemma 6. *The operator T_1^* affects the vector function $L_x(z)$ (1.14) in the following way:*

$$T_1^* L_x(z) = (z + \mu(z))L_x(z) + \nu(z)\widetilde{L}_x(z) + \frac{E_0(z) - \widetilde{E}_0(z)}{2}\widetilde{L}_x(0), \tag{2.16}$$

where

$$\nu(z) = \frac{c_2(z)c_3(z) - c_1(z)c_4(z)}{c_2(z) - c_4(z)}, \tag{2.17}$$

$$\mu(z) = \frac{c_1(z) - c_3(z)}{c_2(z) - c_4(z)}, \tag{2.18}$$

$$c_1(z) = \frac{(E_0(z) + \widetilde{E}_0(z))(1 - z^2)}{2(E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})} \left(\frac{1}{2z} R_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2z} \right), \tag{2.19}$$

$$c_2(z) = \frac{(G_l'(z) + \widetilde{G}_l'(z))(1 - z^2)}{2(E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})} , \tag{2.20}$$

$$c_3(z) = \frac{E_0(z) + \widetilde{E}_0(z)}{E_0'(z) - \widetilde{E}_0'(z)} \left(\frac{1}{2z} R_2 J \begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} \right), \tag{2.21}$$

$$c_4(z) = \frac{2(G_l(z)\overline{G_l(\overline{z})} - \widetilde{G}_l(z)\overline{\widetilde{G}_l(\overline{z})})}{(E_0'(z) - \widetilde{E}_0'(z))(1 - z^2)}\,, \tag{2.22}$$

and R_1*,* R_2 *have the forms of* (2.10) *and* (2.11)*, respectively.*

P r o o f. According to Lemma 4,

$$T_1^* L_x(z) = zL_x(z) + \frac{E_0(z) - \widetilde{E}_0(z)}{2}\widetilde{L}_x(0) + \frac{E_0(z) + \widetilde{E}_0(z)}{2}(1,1)\Phi_x.$$

Now we will show that the vectors $L_x(z)$ and $\widetilde{L}_x(z)$ are linearly independent with each fixed $x \in [0, l]$ and any $z \in C$. Assuming the opposite, $\delta(z)L_x(z) = \widetilde{L}_x(z)$, let us suppose that

$$\delta(z)(1 - zT)^{-1}\Phi(1,1) = (1 - zT^*)^{-1}\Psi^*(1,-1).$$

Apply the operator T_1 to both parts of the equation

$$\delta(z)\frac{(1 - zT)^{-1}\Phi(1,1) - \Phi(1,1)}{z} = z(1 - zT^*)^{-1}\Psi^*(1,-1) - \Phi JS^*\left(\frac{1}{\overline{z}}\right)(1,-1),$$

$$(1 - zT)^{-1}\Phi(1,1)(\delta(z) - z^2) = \Phi(1,1)\delta(z) + z\Phi JS^*\left(\frac{1}{\overline{z}}\right)(1,-1).$$

Let us consider the case where $\delta(z) = z^2$. From the previous equation we obtain that $(1,-1)S^*(\frac{1}{\overline{z}}) = -z(1,-1)$, which is impossible because $S^*(\frac{1}{\overline{z}}) = K^* + z\Psi^*(1 - zT_1^*)^{-1}\Phi^*$ and $S^*(\frac{1}{\overline{z}}) \neq 0$ where $z = 0$.
If $\delta(z) \neq z^2$, then

$$(1 - zT)^{-1}\Phi(1,1) = \Phi\left(\frac{(1,1)\delta(z) + zJS^*(\frac{1}{\overline{z}})(1,-1)}{\delta(z) - z^2}\right)$$

and $(1 - zT)^{-1}\Phi(1,1) \in \Phi E$ for $\forall z$, but $L_x(z) \notin \Phi E$ for $\forall z$.
Thus the functions $L_x(z)$ and $\widetilde{L}_x(z)$ are linearly independent and form basis in E^2 for each fixed x for $\forall z$. Therefore we present the last term in the form

$$\frac{E_0(z) + \widetilde{E}_0(z)}{2}(1,1)\Phi_x = \mu(z)L_x(z) + \nu(z)\widetilde{L}_x(z) \tag{2.23}$$

subsequently multiplying (2.23) by $\widetilde{L}_x^*(z)$,

$$\frac{E_0(z) + \widetilde{E}_0(z)}{2}\int_0^l (1,1)\Phi_x dF_t \widetilde{L}_t^*(\overline{z}) = \mu(z)\int_0^l L_t(z) dF_t \widetilde{L}_t^*(\overline{z}) + \nu(z)\int_0^l \widetilde{L}_t(z) dF_t \widetilde{L}_t^*(\overline{z}).$$

Let us calculate the integrals in the above statement. First we get

$$N_0(z) - \widetilde{N}_l^*(\omega) = 2(\overline{\omega} - z)\int_0^l M_t(z)dF_t\widetilde{M}_t^*(\omega), \tag{2.24}$$

$$\widetilde{N}_l(z) - N_0^*(\omega) = 2(z - \overline{\omega})\int_0^l \widetilde{M}_t(z)dF_tM_t^*(\omega). \tag{2.25}$$

We multiply (2.25) on the left by $(-1,1)$ and on the right by $(1,1)^T$. Since $(-1,1)\widetilde{N}_l(z) = (G_l(z), \widetilde{G}_l(z))$ and $(1,1)N_0(\omega) = (E_0(\omega), \widetilde{E}_0(\omega))$, then

$$G_l(z) + \widetilde{G}_l(z) + \overline{E_0(\omega)} - \overline{\widetilde{E}_0(\omega)} = 2(z - \overline{\omega})\int_0^l \widetilde{L}_t(z)dF_tL_t^*(\omega).$$

Write the expression in the form

$$(z-\omega)\int_0^l \widetilde{L}_t(z)dF_tL_t^*(\overline{\omega}) = \frac{G_l(z) + \widetilde{G}_l(z)}{2} + \frac{\overline{\widetilde{E}_0(\overline{\omega})} - \overline{E_0(\overline{\omega})}}{2}.$$

Let us define $f(z) = \frac{G_l(z)+\widetilde{G}_l(z)}{2}$ and $g(\omega) = \frac{\overline{\widetilde{E}_0(\overline{\omega})}-\overline{E_0(\overline{\omega})}}{2}$. Since $f(z) = -g(z)$, then

$$\frac{f(z) - g(\omega)}{z-\omega} \to f'(z), \quad \omega \mapsto z,$$

$$\int_0^l \widetilde{L}_t(z)dF_tL_t^*(\overline{z}) = \frac{1}{2}\left(\frac{dG_l(z)}{dz} + \frac{d\widetilde{G}_l(z)}{dz}\right). \tag{2.26}$$

Now, if we multiply (2.24) on the left by $(1,1)$ and on the right by $(-1,1)^T$, then

$$(E_0(z), \widetilde{E}_0(z))\begin{pmatrix} -1 \\ 1 \end{pmatrix} - (1,1)\begin{pmatrix} \overline{G_l(\omega)} \\ \overline{\widetilde{G}_l(\omega)} \end{pmatrix} = 2(\overline{\omega} - z)\int_0^l L_t(z)dF_t\widetilde{L}_t^*(\omega),$$

$$E_0(z) - \widetilde{E}_0(z) + \overline{G_l(\omega)} + \overline{\widetilde{G}_l(\omega)} = 2(z - \overline{\omega})\int_0^l L_t(z)dF_t\widetilde{L}_t^*(\omega)$$

and, similarly,

$$\int_0^l L_t(z)dF_t\widetilde{L}_t^*(\overline{z}) = \frac{1}{2}\left(\frac{dE_l(z)}{dz} - \frac{d\widetilde{E}_l(z)}{dz}\right). \tag{2.27}$$

We also have the expressions for two integrals:

$$\int_0^l \widetilde{L}_t(z)dF_t\widetilde{L}_t^*(\overline{\omega}) = \frac{G_x(z)\overline{G_x(\overline{\omega})} - \widetilde{G}_x(z)\overline{\widetilde{G}_x(\overline{\omega})}}{1 - z\overline{\omega}}, \tag{2.28}$$

$$\int_0^l L_t(z)dF_tL_t^*(\overline{\omega}) = \frac{E_0(z)\overline{E_0(\overline{\omega})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{\omega})}}{1 - z\overline{\omega}}. \tag{2.29}$$

By using (2.26)–(2.29), we obtain

$$\frac{E_0(z) + \widetilde{E}_0(z)}{2}\int_0^l (1,1)\Phi_x dF_t\widetilde{L}_t^*(\overline{z})$$

$$= \nu(z)\left(\frac{E_l'(z) - \widetilde{E}_l'(z)}{2}\right) + \mu(z)\left(\frac{G_x(z)\overline{G_x(\overline{z})} - \widetilde{G}_x(z)\overline{\widetilde{G}_x(\overline{z})}}{1 - z^2}\right).$$

Now we multiply statement (2.23) on the right by $L_x^*(\overline{z})$,

$$\frac{E_0(z) + \widetilde{E}_0(z)}{2}\int_0^l (1,1)\Phi_x dF_tL_t^*(\overline{z}) = \nu(z)\int_0^l L_t(z)dF_tL_t^*(\overline{z}) + \mu(z)\int_0^l \widetilde{L}_t(z)dF_tL_t^*(\overline{z}).$$

By using expressions (2.26)–(2.29), in a similar way, we obtain

$$\frac{E_0(z) + \widetilde{E}_0(z)}{2}\int_0^l (1,1)\Phi_x dF_tL_t^*(\overline{z})$$

$$= \nu(z)\left(\frac{E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})}}{1 - z^2}\right) + \mu(z)\left(\frac{G_l'(z) + \widetilde{G}_l'(z)}{2}\right).$$

Now let us calculate $\nu(z)$ and $\mu(z)$. Taking into account (2.10) and (2.11), we will define the coefficients:

$$c_1(z) = \frac{(E_0(z) + \widetilde{E}_0(z))(1 - z^2)}{2(E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})}\left(\frac{1}{2z}R_2\begin{pmatrix}1\\1\end{pmatrix} + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2z}\right),$$

$$c_2(z) = \frac{(G_l'(z) + \widetilde{G}_l'(z))(1 - z^2)}{2(E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})},$$

$$c_3(z) = \frac{E_0(z) + \widetilde{E}_0(z)}{E_0'(z) - \widetilde{E}_0'(z)}\left(\frac{1}{2z}R_2J\begin{pmatrix}\overline{G_l(\overline{z})}\\ \overline{\widetilde{G}_l(\overline{z})}\end{pmatrix}\right),$$

$$c_4(z) = \frac{2(G_l(z)\overline{G_l(\overline{z})} - \widetilde{G}_l(z)\overline{\widetilde{G}_l(\overline{z})})}{(E_0'(z) - \widetilde{E}_0'(z))(1-z^2)}.$$

Hence,

$$\nu(z) = \frac{c_2(z)c_3(z) - c_1(z)c_4(z)}{c_2(z) - c_4(z)}, \qquad \mu(z) = \frac{c_1(z) - c_3(z)}{c_2(z) - c_4(z)},$$

and thus the expression $c_2(z) - c_4(z)$ is not identically equal to zero. Finally we get

$$T^* L_x(z) = zL_x(z) + \frac{E_0(z) - \widetilde{E}_0(z)}{2}\widetilde{L}_x(0) + \mu(z)L_x(z) + \nu(z)\widetilde{L}_x(z),$$

which proves the lemma. ■

Lemma 7. *The operator* T_1 *affects the vector function* $\widetilde{L}_x(z)$ (1.15) *in the following way:*

$$T_1\widetilde{L}_x(z) = (z - \widetilde{\nu}(z))\widetilde{L}_x(z) + \frac{\widetilde{G}_l(z) - G_l(z)}{2}L_x(0) - \widetilde{\mu}(z)L_x(z), \tag{2.30}$$

where

$$\widetilde{\nu}(z) = \frac{c_2(z)\widetilde{c}_3(z) - \widetilde{c}_1(z)c_4(z)}{c_2(z) - c_4(z)}, \tag{2.31}$$

$$\widetilde{\mu}(z) = \frac{\widetilde{c}_1(z) - \widetilde{c}_3(z)}{c_2(z) - c_4(z)}, \tag{2.32}$$

$$\widetilde{c}_1(z) = \frac{(E_0(z) + \widetilde{E}_0(z))(1-z^2)}{2(E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})}\left(-1 - \frac{1}{2}R_1 J \begin{pmatrix} \overline{E_0(\overline{z})} \\ \overline{\widetilde{E}_0(\overline{z})} \end{pmatrix}\right), \tag{2.33}$$

$$\widetilde{c}_3(z) = \frac{E_0(z) + \widetilde{E}_0(z)}{E_0'(z) - \widetilde{E}_0'(z)}\left(\frac{1}{2}R_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\overline{G_l(\overline{z})} - \overline{\widetilde{G}_l(\overline{z})}}{2}\right), \tag{2.34}$$

and $c_2(z)$, $c_4(z)$ *are* (2.20) *and* (2.22), *and* R_1, R_2 *are* (2.10) *and* (2.11), *respectively.*

P r o o f. According to Lemma 2,

$$T_1\widetilde{L}_x(z) = z\widetilde{L}_x(z) + \frac{\widetilde{G}_l(z) - G_l(z)}{2}L_x(0) - \frac{\widetilde{G}_l(z) + G_l(z)}{2}(1,-1)\Psi_x.$$

We can perform the calculations that are similar to those made in Lemma 5. Since the functions $L_x(z)$ and $\widetilde{L}_x(z)$ are linearly independent and form the basis in L_2, we can present the latter term of the above statement in the form

$$\frac{\widetilde{G}_l(z) + G_l(z)}{2}(1,-1)\Psi_x = \widetilde{\mu}(z)L_x(z) + \widetilde{\nu}(z)\widetilde{L}_x(z).$$

Similarly, we multiply it by $L_x(z)$ and $\widetilde{L}_x(z)$ and using the expressions for (2.26)–(2.29), we obtain

$$\frac{\widetilde{G}_l(z)+G_l(z)}{2}\int_0^l(1,-1)\Psi_t dF_t \widetilde{L}_t^*(\overline{z})$$

$$=\widetilde{\nu}(z)\left(\frac{E_l'(z)-\widetilde{E}_l'(z)}{2}\right)+\widetilde{\mu}(z)l\left(\frac{G_x(z)\overline{G_x(\overline{z})}-\widetilde{G}_x(z)\overline{\widetilde{G}_x(\overline{z})}}{1-z^2}\right),$$

$$\frac{\widetilde{G}_l(z)+G_l(z)}{2}\int_0^l(1,-1)\Psi_t dF_t L_t^*(\overline{z})$$

$$=\widetilde{\nu}(z)\left(\frac{E_0(z)\overline{E_0(\overline{z})}-\widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})}}{1-z^2}\right)+\widetilde{\mu}(z)\left(\frac{G_l'(z)+\widetilde{G}_l'(z)}{2}\right).$$

By using (2.10) and (2.11) and introducing similar coefficients, we obtain

$$\widetilde{c}_1(z)=\frac{(E_0(z)+\widetilde{E}_0(z))(1-z^2)}{2(E_0(z)\overline{E_0(\overline{z})}-\widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})}\left(-1-\frac{1}{2}R_1J\left(\begin{array}{c}\overline{E_0(\overline{z})}\\ \overline{\widetilde{E}_0(\overline{z})}\end{array}\right)\right), \tag{2.35}$$

$$\widetilde{c}_2(z)=\frac{(G_l'(z)+\widetilde{G}_l'(z))(1-z^2)}{2(E_0(z)\overline{E_0(\overline{z})}-\widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})})}=c_2(z),$$

$$\widetilde{c}_3(z)=\frac{E_0(z)+\widetilde{E}_0(z)}{E_0'(z)-\widetilde{E}_0'(z)}\left(\frac{1}{2}R_1\left(\begin{array}{c}1\\1\end{array}\right)+\frac{\overline{G_l(\overline{z})}-\overline{\widetilde{G}_l(\overline{z})}}{2}\right), \tag{2.36}$$

$$\widetilde{c}_4(z)=\frac{2(G_l(z)\overline{G_l(\overline{z})}-\widetilde{G}_l(z)\overline{\widetilde{G}_l(\overline{z})})}{(E_0'(z)-\widetilde{E}_0'(z))(1-z^2)}=c_4(z).$$

Thus, for $\widetilde{\nu}(z)$ and $\widetilde{\mu}(z)$ we get

$$\widetilde{\nu}(z)=\frac{c_2(z)\widetilde{c}_3(z)-\widetilde{c}_1(z)c_4(z)}{c_2(z)-c_4(z)}, \tag{2.37}$$

$$\widetilde{\mu}(z)=\frac{\widetilde{c}_1(z)-\widetilde{c}_3(z)}{c_2(z)-c_4(z)}. \tag{2.38}$$

Finally we obtain the expression

$$T_1\widetilde{L}_x(z)=z\widetilde{L}_x(z)+\frac{\widetilde{G}_l(z)-G_l(z)}{2}L_x(0)-\widetilde{\mu}(z)L_x(z)-\widetilde{\nu}(z)\widetilde{L}_x(z),$$

which proves the lemma. ∎

3. De Branges Transformation

In [9], the following results were obtained, namely Lemmas 8–10.

Lemma 8. [9] *De Branges transformation B_L (Definition 1) affects $T_1 f$ in the following way:*

$$B_L(T_1 f) = (z + \overline{\mu(\overline{z})})F_1(z) + \nu(\overline{z})F_2(z) + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2} F_2(0), \tag{3.1}$$

where F_1 and F_2 have the same form as in (1.16).

Lemma 9. [9] *De Branges transformation $B_{\widetilde{L}}$ affects $T_1 f$ in the following way:*

$$B_{\widetilde{L}}(T_1 f) = \frac{F_2(z) - F_2(0)}{z}, \tag{3.2}$$

where F_1 and F_2 have the same form as in (1.16).

Lemma 10. [9] *If the vector $(1, -1)$ is latent for $\widetilde{N}^* + z\widetilde{\Gamma}^*$ with each z, then de Branges transformation $B_{\widetilde{L}}$ affects $T_2 f$, where $T_2 f$ is from the knot $\triangle$ (1.21), in the following way:*

$$B_{\widetilde{L}}(T_2 f(z)) = \frac{F_2(z)n(z) - F(0)n(0)}{z}, \tag{3.3}$$

where F_1 and F_2 have the form of (1.16)*, and the function $n(z)$ satisfies the statement*

$$(\widetilde{N}^* + z\widetilde{\Gamma}^*)(1, -1) = n(z)(1, -1). \tag{3.4}$$

Let us prove the lemma below.

Lemma 11. *If the vector $(1, 1)$ is latent for $(N + z\Gamma)$, then de Branges transformation B_L affects $T_2 f$, where $T_2 f$ is from the knot $\triangle$ (1.21), in the following way:*

$$B_L(T_2 f(z)) = \frac{F_1(z)}{m(z)} + \frac{\widetilde{\mu}(z)}{m(z)}F_1(z) + \frac{\widetilde{\nu}(z)}{m(z)}F_2(z), \tag{3.5}$$

where F_1 and F_2 have the form of (1.16)*, and the function $m(z)$ satisfies the statement*

$$(N + z\Gamma)^{-1}(1, 1) = \frac{1}{m(z)}(1, 1). \tag{3.6}$$

Therefore the coefficients $\widetilde{\mu}(z)$ and $\widetilde{\nu}(z)$ have the forms

$$\widetilde{\mu}(z) = \frac{I_1(z)d_3(z) - I_2(z)d_1(z)}{d_2(z)d_3(z) - d_1(z)d_4(z)}, \tag{3.7}$$

$$\widetilde{\nu}(z) = \frac{I_1(z)d_4(z) - I_2(z)d_2(z)}{d_1(z)d_4(z) - d_2(z)d_3(z)}\,, \tag{3.8}$$

where

$$I_1(z) = \frac{1}{2z}(1,1)\sqrt{2}S\left(\frac{1}{z}\right)\widetilde{\sigma}_2\left(\Phi_l\begin{pmatrix}1\\1\end{pmatrix} - J(\overline{E_0(\overline{z})}, \overline{\widetilde{E}_0(\overline{z})})\right), \tag{3.9}$$

$$I_2(z) = \frac{1}{2z}(1,1)\sqrt{2}S\left(\frac{1}{z}\right)\widetilde{\sigma}_2\left(\Phi_l J\begin{pmatrix}\overline{G_l(\overline{z})}\\ \overline{\widetilde{G}_l(\overline{z})}\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\right), \tag{3.10}$$

$$d_1(z) = \frac{E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})}}{1-|z|^2}\,, \tag{3.11}$$

$$d_2(z) = \frac{G_l'(z) + \widetilde{G}_l'(z)}{2}\,, \tag{3.12}$$

$$d_3(z) = \frac{E_0'(z) - \widetilde{E}_0'(z)}{2}\,, \tag{3.13}$$

$$d_4(z) = \frac{G_l(z)\overline{G_l(\overline{z})} - \widetilde{G}_l(z)\overline{\widetilde{G}_l(\overline{z})}}{1-|z|^2}\,. \tag{3.14}$$

P r o o f. Using the expressions for T_1 (1.23) and T_2 (1.24), we obtain

$$T_2\Phi = T_1\Phi N + \Phi\Gamma,$$

$$zT_2\Phi = zT_1\Phi N + z\Phi\Gamma = (zT_1 - 1)\Phi N + \Phi(\Gamma z + N),$$
$$z(zT_1 - 1)^{-1}T_2\Phi = \Phi N + (zT_1 - 1)^{-1}\Phi(\Gamma z + N),$$
$$T_2^*T_2 z(zT_1 - 1)^{-1}\Phi = T_2^*\Phi N + T_2^*(zT_1 - 1)^{-1}\Phi(\Gamma z + N).$$

Due to the knots relations, we get the statement $T_2^*T_2 + \Psi^*\widetilde{\sigma}_2\Psi = I$. Then

$$(I - \Psi^*\widetilde{\sigma}_2\Psi)z(zT_1 - 1)^{-1}\Phi = T_2^*\Phi N + T_2^*(zT_1 - 1)^{-1}\Phi(\Gamma z + N),$$

$$(\Psi^*\widetilde{\sigma}_2\Psi - I)z(1 - zT_1)^{-1}\Phi = T_2^*\Phi N + T_2^*(zT_1 - 1)^{-1}\Phi(\Gamma z + N),$$

$$z\Psi^*\widetilde{\sigma}_2\Psi z(1 - zT_1)^{-1}\Phi - z(1 - zT_1)^{-1}\Phi = T_2^*\Phi N - T_2^*(1 - zT_1)^{-1}\Phi(\Gamma z + N).$$

Since the characteristic function has the form $S(z) = K + \Psi(z - T_1)^{-1}\Phi$ (1.5), then after writing the expressions

$$S(\frac{1}{z}) - K = \Psi(\frac{1}{z} - T_1)^{-1}\Phi = z\Psi(1 - zT_1)^{-1}\Phi,$$

$$T_2\Phi N + \Psi\widetilde{\sigma}_2 K = 0,$$

we obtain the equality

$$\Psi^*\widetilde{\sigma}_2(S(\frac{1}{z}) - K) - z(1 - zT_1)^{-1}\Phi = T_2^*\Phi N - T_2^*(1 - zT_1)^{-1}\Phi(N + \Gamma z),$$

$$\Psi^*\widetilde{\sigma}_2 S(\frac{1}{z}) - z(1 - zT_1)^{-1}\Phi = -T_2^*(1 - zT_1)^{-1}\Phi(N + \Gamma z),$$

$$T_2^*(1 - zT_1)^{-1}\Phi = z(1 - zT_1)^{-1}\Phi(N + z\Gamma)^{-1} - \Psi^*\widetilde{\sigma}_2 S(\frac{1}{z})(N + z\Gamma)^{-1},$$

$$T_2^*(1-zT_1)^{-1}\Phi(1,1) = z(1-zT_1)^{-1}\Phi(N+z\Gamma)^{-1}(1,1)-\Psi^*\widetilde{\sigma}_2 S(\frac{1}{z})(N+z\Gamma)^{-1}(1,1).$$

Let us introduce the function $m(z)$ satisfying the equation

$$(N + z\Gamma)^{-1}(1,1) = \frac{1}{m(z)}(1,1),$$

i.e., suppose that (1,1) is a latent vector of $(N + z\Gamma)$.
Then the statement

$$T_2^* L_x(z) = \frac{L_x(z)}{m(z)} + \frac{\Psi^*\widetilde{\sigma}_2 S(\frac{1}{z})(1,1)}{m(z)}$$

can be presented in the form

$$\Psi^*\widetilde{\sigma}_2 S(\frac{1}{z})(1,1) = \widetilde{\mu} L_x(z) + \widetilde{\nu}\widetilde{L}_x(z),$$

or by using the operator Ψ^*,

$$(1,1)\sqrt{2}S\left(\frac{1}{z}\right)\widetilde{\sigma}_2\Phi_x = \widetilde{\mu} L_x(z) + \widetilde{\nu}\widetilde{L}_x(z). \tag{3.15}$$

Multiplying (3.15) by $L_x(z)$ and $\widetilde{L}_x(z)$, we obtain two statements

$$\int_0^l (1,1)\sqrt{2}S\left(\frac{1}{z}\right)\widetilde{\sigma}_2\Phi_t dF_t L_t^*(\overline{z}) = \widetilde{\mu}(z)\int_0^l L_t(z)dF_t L_t^*(\overline{z}) + \widetilde{\nu}\int_0^l \widetilde{L}_t(z)dF_t L_t^*(\overline{z}),$$

$$\int_0^l (1,1)\sqrt{2}S\left(\frac{1}{z}\right)\widetilde{\sigma}_2\Phi_t dF_t \widetilde{L}_t^*(\overline{z}) = \widetilde{\mu}(z)\int_0^l L_t(z)dF_t \widetilde{L}_t^*(\overline{z}) + \widetilde{\nu}\int_0^l \widetilde{L}_t(z)dF_t \widetilde{L}_t^*(\overline{z}).$$

By using previously obtained expressions for integrals (2.26)–(2.29), we introduce the following coefficients:

$$d_1(z) = \int_0^l L_t(z)dF_t L_t^*(\overline{z}) = \frac{E_0(z)\overline{E_0(\overline{z})} - \widetilde{E}_0(z)\overline{\widetilde{E}_0(\overline{z})}}{1 - |z|^2},$$

$$d_2(z) = \int_0^l \widetilde{L}_t(z) dF_t L_t^*(\overline{z}) = \frac{G_l'(z) + \widetilde{G}_l'(z)}{2},$$

$$d_3(z) = \int_0^l L_t(z) dF_t \widetilde{L}_t^*(\overline{z}) = \frac{E_0'(z) - \widetilde{E}_0'(z)}{2},$$

$$d_4(z) = \int_0^l \widetilde{L}_t(z) dF_t \widetilde{L}_t^*(\overline{z}) = \frac{G_l(z)\overline{G_l(\overline{z})} - \widetilde{G}_l(z)\overline{\widetilde{G}_l(\overline{z})}}{1-|z|^2}.$$

Now, using the calculations from Lemma 5, we obtain

$$\int_0^l \Phi_t dF_t L_t^*(\overline{z}) = \frac{1-z}{2z}\left(\Phi_l JJ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{1-z} - IJ\frac{1}{1-z}(\overline{E_0(\overline{z})}, \overline{\widetilde{E}_0(\overline{z})})\right),$$

$$\int_0^l \Phi_x a_x \widetilde{L}_x^*(\overline{z}) = \frac{1-z}{2z}\left(\Phi_l J \frac{1}{1-z}\begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} - IJ\frac{1}{1-z} I \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right).$$

By $I_1(z)$ and $I_2(z)$, we denote the following expressions:

$$I_1(z) = \frac{1}{2z}(1,1)\sqrt{2}S\begin{pmatrix} \frac{1}{z} \end{pmatrix}\widetilde{\sigma}_2\left(\Phi_l\begin{pmatrix} 1 \\ 1 \end{pmatrix} - J(\overline{E_0(\overline{z})}, \overline{\widetilde{E}_0(\overline{z})})\right),$$

$$I_2(z) = \frac{1}{2z}(1,1)\sqrt{2}S\begin{pmatrix} \frac{1}{z} \end{pmatrix}\widetilde{\sigma}_2\left(\Phi_l J\begin{pmatrix} \overline{G_l(\overline{z})} \\ \overline{\widetilde{G}_l(\overline{z})} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right),$$

then

$$\frac{I_1(z)}{d_1(z)} = \widetilde{\mu}(z) + \widetilde{\nu}(z)\frac{d_2(z)}{d_1(z)},$$

$$\frac{I_2(z)}{d_3(z)} = \widetilde{\mu}(z) + \widetilde{\nu}(z)\frac{d_4(z)}{d_3(z)},$$

$$\frac{I_1(z)d_3(z) - I_2(z)d_1(z)}{d_1(z)d_3(z)} = \nu(z)\frac{d_2(z)d_3(z) - d_1(z)d_4(z)}{d_1(z)d_3(z)}.$$

Hence we have

$$\nu(z) = \frac{I_1(z)d_3(z) - I_2(z)d_1(z)}{d_2(z)d_3(z) - d_1(z)d_4(z)}, \qquad \widetilde{\nu}(z) = \frac{I_1(z)d_4(z) - I_2(z)d_2(z)}{d_1(z)d_4(z) - d_2(z)d_3(z)}$$

and obtain the expression

$$B_L(T_2 f(z)) = \frac{F_1(z)}{m(z)} + \frac{\widetilde{\mu}(z)}{m(z)}F_1(z) + \frac{\widetilde{\nu}(z)}{m(z)}F_2(z),$$

which proves the lemma. ∎

From Lemmas 6–11 we have the following theorem.

Theorem. *Let a commutative knot* $\triangle$ (1.21) *be such that* $E = \tilde{E}, dim E = 2$, $\sigma_1 = \tilde{\sigma}_1 = J_N$ (1.13), *the spectrum of the operator* T_1 *be located at the point* $\{1\}$ *and the vector* $(1, 1)$ *be latent for* $(N + z\Gamma)$, *i.e., let the function* $m(z)$ *be such that* $(N + z\Gamma)(1, 1)^T = m(z)(1, 1)^T$, *and the vector* $(1, -1)$ *be latent for* $\widetilde{N}^* + z\widetilde{\Gamma}^*$, *i.e., let the function* $n(z)$ *be such that* $(\widetilde{N}^* + z\widetilde{\Gamma}^*)(1, -1)^T = n(z)(1, -1)^T$. *Then the main system of the commutative operators* $\{T_1, T_2\}$ *of the knot* $\triangle$ (1.21) *is unitarily equivalent to the system of operators that operates in the de Branges space* $\mathcal{B}(E, G)$ *in the following way:*

$$(T_1F)_1(z) = (z + \overline{\mu(\overline{z})})F_1(z) + \nu(\overline{z})F_2(z) + \frac{\overline{E_0(\overline{z})} - \overline{\widetilde{E}_0(\overline{z})}}{2}F_2(0),$$

$$(T_1F)_2(z) = \frac{F_2(z) - F_2(0)}{z},$$

$$(T_2F)_1(z) = \frac{F_1(z)}{m(z)} + \frac{\widetilde{\mu}(z)}{m(z)}F_1(z) + \frac{\widetilde{\nu}(z)}{m(z)}F_2(z),$$

$$(T_2F)_2(z) = \frac{F_2(z)n(z) - F(0)n(0)}{z},$$

where $(F_1(z), F_2(z)) \in \mathcal{B}(E, G)$. *The coefficients* $\mu(z)$, $\nu(z)$ *and* $\widetilde{\mu}(z)$, $\widetilde{\nu}(z)$ *have the forms of* (2.17), (2.18) *and* (3.7), (3.8), *respectively,* $N, \widetilde{N}, \Gamma, \widetilde{\Gamma}$ *are defined by* (1.22). *The correctness of this definition follows from the reversibility of* σ *and* $\widetilde{\sigma}$.

Note. *Let us consider the conditions* $(N + z\Gamma)(1, 1)^T = m(z)(1, 1)^T$ *and* $(\widetilde{N}^* + z\widetilde{\Gamma}^*)(1, -1)^T = n(z)(1, -1)^T$ *from the theorem. If we use* (1.22), *then the condition of intertwining* [7]

$$S(z)N_1{}^{-1}(N_2 + z\Gamma_1) = \widetilde{N}_1^{-1}(\widetilde{N}_2 + z\widetilde{\Gamma}_1)S(z)$$

will have the form

$$S(z)(N + z\Gamma) = (\widetilde{N} + z\widetilde{\Gamma})S(z).$$

After multiplying this equation from left by $(1, -1)$ *and from right by* $(1, 1)^T$ *and using* $m(z)(1, 1)^T = (N+z\Gamma)(1, 1)^T$ *and* $(1, -1)(\widetilde{N}+z\widetilde{\Gamma}) = (1, -1)\overline{n(\overline{z})}$, *we obtain*

$$m(z)(1, -1)S(z)(1, 1)^T = \overline{n(\overline{z})}(1, -1)S(z)(1, 1)^T.$$

Hence the conditions imply that either $m(z) = \overline{n(\overline{z})}$ *or* $(1, 1)S(z)(1, 1)^T = 0$ *for* $\forall z \in C$.

Thus the functional model is built for the commutative system of the operators T_1, T_2, which is the main for the commutative knot $\triangle$(1.21) satisfying the conditions of the theorem. However, T_1 and T_2 affect one of the components $[F_1(z), F_2(z)]$ as a shift and the other one as a multiplication by special holomorphic functions.

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