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Asymptotic Behavior of Fractional Derivatives of Bounded Analytic Functions

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We find sharp sufficient conditions for the boundedness of fractional derivatives of a bounded analytic function in a Stolz angle. If $F \neq 0$ in the unit disc, the necessary and sufficient conditions for the boundedness of fractional derivatives of its argument in a Stolz angle are established.

Key words: bounded analytic function, Stolz angle, Blaschke product, fractional derivative.

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1. Introduction and Main Results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $D(\zeta, \rho) = \{z \in \mathbb{C} : |z - \zeta| < \rho\}$. The symbol $C(\cdot)$ stands for some positive constant depending on the values in the parentheses not necessarily the same in each occurrence. Let H^∞ be the Hardy class of bounded analytic functions in $\mathbb{D}$. Let B be a Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n(a_n - z)}{|a_n|(1 - \overline{a}_n z)}, \qquad 0 < |a_n| < 1, n \in \mathbb{N}. \tag{1}$$

For a fixed $\theta_0 \in \mathbb{R}$ the following theorem of O. Frostman ([7, 12]) gives the necessary and sufficient conditions for the existence of the radial limits of B and its derivative.

Theorem A. (i) *Necessary and sufficient that*

$$\lim_{r \to 1-0} f(re^{i\theta_0}) = L,$$

exist and $|L| = 1$ *for* $f = B$*, and every subproduct of* B*, is that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta_0} - a_n|} < \infty. \tag{2}$$

(ii) *Necessary and sufficient that*

$$\lim_{r\to 1-0} B(re^{i\theta_0}) = L, \ \lim_{r\to 1-0} B'(re^{i\theta_0}) = M$$

exist and $|L| = 1$ *is that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta_0} - a_n|^2} < \infty. \tag{3}$$

Note that condition (2) is often called Frostman's condition.

Theorem A was generalized and complemented by many authors (e.g., G. Cargo ([5]), P. Ahern, D. Clark ([1, 2]), K.-K. Leung, C.N. Linden ([13]) and others). Since the proof of the necessity of Theorem A is based on the estimates of the argument, one may expect to describe the local behavior of $\arg B(z)$ in terms of Frostman's type conditions. In [6], one can find necessary and sufficient conditions for the local growth $\arg F$, $F \in H^\infty$, in terms of the generalized Frostman's condition. The relations between conditions on the zeros of the Blaschke product B and the membership of $\arg B(e^{i\theta})$ in $L^p, 0 < p \le \infty$ were studied in [14].

It is known that every function $F \in H^\infty$, $F(0) \ne 0$, $|F(z)| < 1$, $z \in \mathbb{D}$, can be represented in the form ([7, 9])

$$F(z) = B(z) \exp\left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right), \tag{4}$$

where μ is a non-decreasing function on $[-\pi, \pi]$. We use the same letters to denote the non-decreasing functions on $[-\pi, \pi]$ and the Stieltjes measures on $\partial\mathbb{D}$ associated with them.

Let μ, μ_* be finite Borel measures on $\partial\mathbb{D}$. We write that $\mu_* \prec \mu$ if $\mu_*(M) \le \mu(M)$ for an arbitrary Borel set $M \subset \partial\mathbb{D}$. We say that F_* is a divisor of $F \in H^\infty$ if $F_* \in H^\infty$ and if there exists a function $G \in H^\infty$ such that $F = GF_*$. Note that F_* is a divisor of F if and only if $\mu_* \prec \mu$ and the zero set of F_* is a subset of that for F.

P. Ahern and D. Clark proved the following theorem ([1]).

Theorem B. *Let* $f \in H^\infty$ *be of the form* (4)*, and* $N \in \mathbb{N}$.

(*i*) *Suppose that N is even, and $\mu(\{x\}) = 0$. Necessary and sufficient that $F_*^{(N)}(re^{ix})$ be bounded as $r \to 1-0$ for every divisor F_* of F is that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|e^{ix}-a_n|^{N+1}} + \int_0^{2\pi} \frac{d\mu(t)}{|e^{it}-e^{ix}|^{N+1}} < \infty \tag{5}$$

hold.

(*ii*) *Suppose that N is odd. Necessary and sufficient that*

$$\lim_{r\to 1-0} F^{(j)}(re^{ix}) = L_j$$

exist for $j = 0, ..., N-1$, that $F^{(N)}(re^{ix})$ be bounded as $r \to 1-0$ and that

$$\lim_{R\to 1+0} F^{(j)}(Re^{ix}) = L_j$$

for $0 \le j \le N-1$ is that (5) hold.

Note that the set of points e^{ix} such that (5) is satisfied with $N = 1$ is often called the Ahern-Clark set. This notion has many applications, see, e.g., [3], [8, Chap. IX]. In particular, a function F of the form (4) is said to have an angular derivative $F'(\xi)$ at $\xi \in \partial\mathbb{D}$ ([4]) if there exist $\lim_{r\to 1-0} F(r\xi) \in \partial\mathbb{D}$ and $F'(\xi) := \lim_{r\to 1-0} F'(r\xi) \in \mathbb{C}$. By [2, Theorem 2],

$$|F'(\xi)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\xi-a_n|^2} + 2\int_0^{2\pi} \frac{d\mu(t)}{|e^{it}-\xi|^2},$$

so (5) with $N = 1$ and Carathéodory's theorem ([4, Sec. 298–299]) (cf. (3)) provide the existence of the angular derivative.

In order to formulate the next results, we need some information on fractional derivatives. For $f \in L(0,a)$, the Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by ([15, Chap. I, p. 33])

$$D^{-\alpha} f(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-x)^{\alpha-1} f(x)dx, \qquad r \in (0,a)$$

$$D^0 f(r) \equiv f(r), \qquad D^\alpha f(r) = \frac{d^p}{dr^p}\{D^{-(p-\alpha)} f(r)\}, \qquad \alpha \in (p-1,p], p \in \mathbb{N},$$

where $\Gamma(\alpha)$ is the Gamma function.
The Stolz angle with the vertex ζ is defined by

$$\mathcal{S}_\sigma(\zeta) = \{z \in \mathbb{D} : |1 - z\bar{\zeta}| \le \sigma(1-|z|)\}, \qquad \sigma \ge 1.$$

We denote $\mathcal{S}_\sigma^*(\xi) = \mathcal{S}_\sigma(\xi) \cap D(\xi, \frac{1}{2})$.

Theorem C [6]. *Let $0 \le \gamma < 1, \theta \in \mathbb{R}$, and $F \in H^\infty$. Necessary and sufficient that for every divisor F_* of F and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, F) > 0$ such that*

$$\sup_{z \in S_\sigma^*(e^{i\theta})} |D^{-\gamma} \arg F_*(z)| < K, \tag{6}$$

and that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|^{1-\gamma}} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it} - e^{i\theta}|^{1-\gamma}} < \infty \tag{7}$$

hold.

In view of Theorems B and C the following questions arise:

(i) Does a counterpart of Theorem B for fractional derivatives hold?

(ii) What are the necessary and sufficient conditions for the boundedness of $D^\alpha \arg F(z)$ for $F \in H^\infty$?

In this paper we give partial answers to these questions.

Let us denote

$$G(z) = \exp\left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right), \tag{8}$$

The following theorem yields the necessary and sufficient conditions for the local growth of $\arg G$ in terms of local properties of the boundary measure.

Theorem 1. *Let $\theta \in \mathbb{R}$, $\sigma > 1$, $\alpha > 0$. Given G (8), the value $|D^\alpha \arg G_*(re^{i\varphi})|$ is bounded in the Stolz angle $\mathcal{S}_\sigma(e^{i\theta})$ for each divisor G_* of G if and only if*

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{i\theta} - e^{it}|^{1+\alpha}} < \infty. \tag{9}$$

Corollary 2. *Let $\alpha > 0$. Given G (8), $\sup_{|z|<1} |D^\alpha \arg G_*(z)| < \infty$ for each divisor G_* of G if and only if*

$$\sup_\theta \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it} - e^{i\theta}|^{1+\alpha}} < \infty.$$

For an analytic function f in $\mathbb{D}$, we set

$$f^{[\alpha]}(re^{i\varphi}) = D^\alpha(r^\alpha f(re^{i\varphi})), \qquad \alpha > 0, r > 0.$$

This definition provides that $f^{[\alpha]}(z)$ is analytic in $\mathbb{D}$ ([8, Chapter IX]).

Theorem 3. *Let $\alpha \in (0,1)$. Let $F \in H^\infty$ be defined by (4). If*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|e^{i\theta}-a_n|^{1+\alpha}} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{i\theta}-e^{it}|^{1+\alpha}} < \infty, \tag{10}$$

then for every divisor F_* of F, $|F_*^{[\alpha]}(z)|$ is bounded in $\mathcal{S}_\sigma(e^{i\theta})$.

Corollary 4. *Let $F \in H^\infty$, $\alpha > 0$. If*

$$\sup_\theta \left\{ \sum_{n=1}^{\infty} \frac{1-|a_n|}{|e^{i\theta}-a_n|^{1+\alpha}} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it}-e^{i\theta}|^{1+\alpha}} \right\} < \infty, \tag{11}$$

then for every divisor F_* of F, $\sup_{z\in\mathbb{D}} |F_*^{[\alpha]}(z)| < \infty$.

Note that in the limit case $\alpha = 0$, the assertion of Theorem 3 would be a bit weaker than a generalization of the sufficiency part of Theorem A ([1, Lemma 3]), we have the boundedness in the Stolz angles instead of the existence of the radial limit. On the other hand, similarly to Theorem B, in the case $\alpha = 1$, we would have boundedness of $F'(z)$ but in the Stolz angles as well. It seems plausible that the converse statement to Theorem 3 is true, but we were not able to prove it. Nevertheless, we show that the statement of Theorem 3 is sharp in Example 1.

2. Proof of the Theorems

P r o o f of Theorem 1. *Sufficiency.* Without loss of generality, we may assume that $\theta = 0$. Let us denote

$$S(re^{i\varphi}) = \frac{e^{it}+re^{i\varphi}}{e^{it}-re^{i\varphi}}.$$

Then

$$S_r^{(n)}(re^{i\varphi}) = \frac{2n!e^{i(t-\varphi)}}{(e^{i(t-\varphi)}-r)^{n+1}}, \quad n \in \mathbb{N}.$$

According to the definition of G_*, we have

$$\arg G_*(z) = -\mathrm{Im}\left(\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu_*(t) \right), \tag{12}$$

where $\mu_* \prec \mu$.

For $f^{(p)} \in L(0,l)$, the following equality holds ([8, Chapter IX, p. 572], [15, Chapter I, p. 39]):

$$D^{\alpha} f(x) = \sum_{k=0}^{p-1} \frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)} x^{k-\alpha} + \frac{1}{\Gamma(p-\alpha)} \int_0^x (x-t)^{p-\alpha-1} f^{(p)}(t)dt, \qquad (13)$$

$p-1 < \alpha \le p, p \in \mathbb{N}$.
Applying (13) to $\arg G_*$, we obtain

$$D^{\alpha} \arg G_*(re^{i\varphi}) = -\sum_{k=0}^{p-1} \frac{2k! \sin k(\varphi - t)}{\Gamma(1+k-\alpha)} r^{k-\alpha}$$

$$-\frac{1}{\Gamma(p-\alpha)} \int_0^r (r-x)^{p-\alpha-1} \int_{-\pi}^{\pi} \operatorname{Im}\left(\frac{2p! e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - x)^{p+1}} \right) d\mu_*(t)dx,$$

$$\left| D^{\alpha} \arg G_*(re^{i\varphi}) \right| \le \sum_{k=0}^{p-1} \frac{2k! r^{k-\alpha}}{\Gamma(1+k-\alpha)}$$

$$+\frac{2p!}{\Gamma(p-\alpha)} \int_0^r \int_{-\pi}^{\pi} \frac{(r-x)^{p-\alpha-1}}{|e^{i(t-\varphi)} - x|^{p+1}} d\mu_*(t)dx.$$

In order to finish the proof, we need the following lemma.

Lemma A [10]. *Let $0 \le \gamma < \alpha < \infty$. Then there exists a constant $C(\gamma, \alpha)$ such that*

$$D^{-\gamma} \frac{1}{|1-r\xi|^{\alpha}} \le \frac{C(\alpha,\gamma)}{|1-r\xi|^{\alpha-\gamma}}, \quad \xi \in \overline{\mathbb{D}}, 0 < r < 1.$$

Using Lemma A and the fact that $z \in \mathcal{S}_{\sigma}(e^{i\theta})$, we obtain

$$\left| D^{\alpha} \arg G_*(re^{i\varphi}) \right| \le \sum_{k=0}^{p-1} \frac{2k! r^{k-\alpha}}{\Gamma(1+k-\alpha)} + C(\alpha) \int_{-\pi}^{\pi} \frac{d\mu_*(t)}{|e^{i(t-\varphi)} - r|^{\alpha+1}}$$

$$\le C(\alpha) + C(\alpha) \int_{-\pi}^{\pi} \frac{d\mu_*(t)}{|e^{i\theta} - e^{it}|^{\alpha+1}} < \infty.$$

Necessity. Since $\frac{1}{|1-e^{it}|}$ is bounded outside $[-\varepsilon, \varepsilon]$, we consider the integral (9) only on the interval $[0, \varepsilon]$, where $\varepsilon > 0$ will be specified later. Convergence of the integral on $[-\varepsilon, 0]$ can be shown in a similar way. Let us estimate

$$\arg S_r^{(n)}(re^{i\varphi}) = \arg\left(\frac{2n! e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - r)^{n+1}} \right), \qquad n \in \mathbb{N}.$$

We consider $z = re^{i\varphi} \in \mathbb{D}$ such that $\arg(1-z) = \sigma_0 \in \left(\frac{(4n+1)\pi}{4(2n+1)}, \frac{\pi}{2}\right)$. We choose $\varepsilon > 0$ satisfying $\varepsilon < \frac{\pi}{4(2n+1)}$ and $|e^{i\varepsilon} - 1| < \frac{1}{8}$. Let $0 < t < \varepsilon$. By the construction, we have

$$\sigma_0 < \arg(e^{it} - re^{i\varphi}) < \frac{\pi}{2} + \frac{\varepsilon}{2},$$

thus

$$\sigma_0 - \varphi < \arg(e^{i(t-\varphi)} - r) < \frac{\pi}{2} + \frac{\varepsilon}{2} - \varphi.$$

Since $z \in \mathcal{S}_\sigma(e^{i\theta})$ for some $\sigma > 0$, we have

$$\varphi = \arg z = O(1-|z|) \Rightarrow \varphi \sim (r-1)\tan\sigma_0,\ z \to 1, \arg(1-z) = \sigma_0.$$

So we can assume that $\frac{-\varepsilon}{n+1} < \varphi < 0$ as $r \to 1-0$. Then, denoting

$$\gamma(t) = \gamma_{n,r}(t) := \arg\left(e^{i(t-\varphi)}\right) - (n+1)\arg\left(e^{i(t-\varphi)} - r\right), \tag{14}$$

we obtain

$$2\varepsilon - \sigma_0(n+1) > \gamma(t) > -(n+1)\Big(\frac{\pi}{2} + \varepsilon\Big). \tag{15}$$

Due to the choice of σ_0 and ε for $n = 2k, k \in \mathbb{N}$, $n-1 < \alpha \le n$, we get

$$\Big(-(n+1)\Big(\frac{\pi}{2} + \varepsilon\Big), 2\varepsilon - (n+1)\sigma_0\Big) \Subset (-\pi(k+1), -\pi k).$$

For $n = 2k-1, k \in \mathbb{N}$, $n-1 < \alpha \le n$, we consider

$$\arg S_r^{(n+1)}(re^{i\varphi}) = \arg\left(\frac{2(n+1)!e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - r)^{n+2}}\right), \qquad n \in \mathbb{N}.$$

Using the similar estimates, we obtain

$$\gamma_{n+1,r}(t) \in \Big(-(n+2)\Big(\frac{\pi}{2} + \varepsilon\Big), 2\varepsilon - (n+2)\sigma_0\Big) \Subset (-\pi(k+1), -\pi k).$$

It follows from the previous inclusion that $\sin\gamma(t)$ keeps the sign for $t \in [0, \varepsilon]$. Let χ_E be the characteristic function of a set E. Let us denote $\mu_* = \mu\chi_{[0,\varepsilon]}$. We deduce

$$\big|\arg G_*^{(n)}(re^{i\varphi})\big| = \left|\mathrm{Im}\left(\int\limits_0^\varepsilon \frac{2n!e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - r)^{n+1}} d\mu(t)\right)\right|$$

$$\ge \int\limits_0^\varepsilon \frac{2n!}{|e^{i(t-\varphi)} - r|^{n+1}} |\sin\gamma(t)| d\mu(t).$$

We consider the function

$$G_n(z) = e^{P_n(z)}, \quad P_n(z) = a_0 + a_1 z + \ldots + a_{n-1}z^{n-1}, a_0, \ldots, a_{n-1} \in \mathbb{C}. \tag{16}$$

Since $D^\alpha \arg G_n(z)$ is uniformly continuous on $\overline{\mathbb{D}}$ and, consequently, bounded, without loss of generality, we can consider

$$D^\alpha \arg \frac{G_*(z)}{G_n(z)} = D^\alpha \arg G_*(z) - D^\alpha \arg G_n(z) \tag{17}$$

instead of $D^\alpha \arg G_*(z)$. Applying (13), we obtain

$$\left| D^\alpha \arg \frac{G_*(re^{i\varphi})}{G_n(re^{i\varphi})} \right| = \Bigg| - \sum_{p=0}^{n-1} \int_0^\varepsilon \frac{2p! r^{p-\alpha} \sin p(t-\varphi) d\mu(t)}{\Gamma(1+p-\alpha)}$$

$$- \int_0^r \frac{(r-x)^{n-\alpha-1}}{\Gamma(n-\alpha)} \int_0^\varepsilon \mathrm{Im}\left(\frac{2n! e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - x)^{n+1}} \right) d\mu(t) dx - D^\alpha \mathrm{Im} P_n(re^{i\varphi}) \Bigg|.$$

We choose the coefficients $a_0, \dots, a_{n-1}$ such that

$$D^\alpha \mathrm{Im} P_n(re^{i\varphi}) = - \sum_{p=0}^{n-1} \int_0^\varepsilon \frac{2p! r^{p-\alpha} \sin p(t-\varphi) d\mu(t)}{\Gamma(1+p-\alpha)}. \tag{18}$$

Since

$$D^\alpha \Big(\frac{x^\gamma}{\Gamma(1+\gamma)} \Big) = \frac{x^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}, \qquad \gamma > -1, \tag{19}$$

it is easy to check that

$$a_p = 2 \int_0^\varepsilon e^{-ipt} d\mu(t) \tag{20}$$

is a solution of (18). Thus,

$$\left| D^\alpha \arg \frac{G_*(re^{i\varphi})}{G_n(re^{i\varphi})} \right|$$

$$= \left| \frac{1}{\Gamma(n-\alpha)} \int_0^r (r-x)^{n-\alpha-1} \int_0^\varepsilon \mathrm{Im}\left(\frac{2n! e^{i(t-\varphi)}}{(e^{i(t-\varphi)} - x)^{n+1}} \right) d\mu(t) dx \right|$$

$$\geq \left| \frac{1}{\Gamma(n-\alpha)} \int_0^r (r-x)^{n-\alpha-1} \int_0^\varepsilon \frac{2n! \sin \gamma(t)}{|e^{i(t-\varphi)} - x|^{n+1}} d\mu(t) dx \right|.$$

In order to estimate the inner integral, we may assume that $|1-z| < \frac{1}{8}$. Since $|e^{i\varepsilon} - 1| < \frac{1}{8}$, we have $r > 2|z - e^{it}|, t \in [0, \varepsilon]$. For $r - x \leq 2|re^{i\varphi} - e^{it}|$, we deduce

$$|xe^{i\varphi} - e^{it}| \leq |re^{i\varphi} - xe^{i\varphi}| + |re^{i\varphi} - e^{it}| \leq |r-x| + |re^{i\varphi} - e^{it}| \leq 3|re^{i\varphi} - e^{it}|. \tag{21}$$

Using (21), we obtain

$$\left|D^{\alpha} \arg \frac{G_*(re^{i\varphi})}{G_n(re^{i\varphi})}\right|$$

$$\geq \left| C(\alpha) \int\limits_0^{\varepsilon} \int\limits_{r-2|e^{i(t-\varphi)}-r|}^{r-|e^{i(t-\varphi)}-r|} \frac{(r-x)^{n-\alpha-1} dx d\mu(t)}{|e^{i(t-\varphi)}-r|^{n+1}} \right|$$

$$\geq C(\alpha) \int\limits_0^{\varepsilon} \frac{d\mu(t)}{|e^{i(t-\varphi)}-r|^{\alpha+1}}.$$

Tending z to 1, using Fatou's lemma and the boundedness of $D^{\alpha} \arg G_*(z)$, we conclude that

$$C \geq C(\alpha) \int\limits_0^{\varepsilon} \frac{d\mu(t)}{|e^{it}-1|^{\alpha+1}}.$$

For $n = 2k-1, k \in \mathbb{N}$, we set

$$G_n(z) = e^{P_n(z)}, \quad P_n(z) = \sum_{p=0}^{n} a_p z^p, \tag{22}$$

where a_p are defined by (20). Integrating (13) by parts, we get

$$D^{\alpha} f(x) = \sum_{k=0}^{p} \frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)} x^{k-\alpha} + \frac{1}{\Gamma(p-\alpha+1)} \int\limits_0^x (x-t)^{p-\alpha} f^{(p+1)}(t) dt.$$

Then

$$\left|D^{\alpha} \arg \frac{G_*(re^{i\varphi})}{G_n(re^{i\varphi})}\right|$$

$$= \left| \int\limits_0^r \frac{(r-x)^{n-\alpha}}{\Gamma(n-\alpha)} \int\limits_0^{\varepsilon} \operatorname{Im}\left(\frac{2(n+1)! \sin \gamma_{n+1,r}(t)}{|e^{i(t-\varphi)}-x|^{n+2}} \right) d\mu(t) dx \right|.$$

The rest of the proof repeats that for the case $n = 2k$. ■

P r o o f of Theorem 3. Let (a_n^*) be the zero sequence of F_*. Let us calculate the derivative of $r^{\alpha} F_*(re^{i\varphi})$

$$\frac{\partial}{\partial r}(r^{\alpha} F_*(re^{i\varphi})) = \alpha r^{\alpha-1} F_*(re^{i\varphi}) - r^{\alpha} F_*(re^{i\varphi}) \int\limits_{-\pi}^{\pi} \frac{2e^{i\varphi} e^{it}}{(e^{it}-re^{i\varphi})^2} dm_*(t)$$

$$+r^{\alpha}F_*(re^{i\varphi})\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|^2}{(re^{i\varphi}-a_n{}^*)(1-\overline{a}_n{}^*re^{i\varphi})}.$$

Using (13) with $p=1$, we obtain

$$D^{\alpha}(r^{\alpha}F_*(re^{i\varphi}))=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^r(r-x)^{-\alpha}F_*(xe^{i\varphi})\Bigg(\alpha x^{\alpha-1}$$

$$-x^{\alpha}\int\limits_{-\pi}^{\pi}\frac{2e^{i\varphi}e^{it}}{(e^{it}-xe^{i\varphi})^2}dm_*(t)+x^{\alpha}\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|^2}{(xe^{i\varphi}-a_n{}^*)(1-\overline{a}_n{}^*xe^{i\varphi})}\Bigg).$$

Since $F_*\in H^{\infty}$, we have

$$|F_*^{[\alpha]}(re^{i\varphi})|\le\frac{C}{\Gamma(1-\alpha)}\Bigg(\int\limits_0^r(r-x)^{-\alpha}\alpha x^{\alpha-1}dx$$

$$+\int\limits_0^r(r-x)^{-\alpha}x^{\alpha}\int_{-\pi}^{\pi}\frac{2}{|e^{it}-xe^{i\varphi}|^2}dm_*(t)dx$$

$$+\int\limits_0^r(r-x)^{-\alpha}x^{\alpha}\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|^2}{|1-\overline{a}_n{}^*xe^{i\varphi}|^2}dx\Bigg).$$

It follows from the proof of Theorem 1 and (19) that it is sufficient to estimate

$$\frac{r^{\alpha}}{\Gamma(1-\alpha)}\int\limits_0^r(r-x)^{-\alpha}\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|^2}{|1-\overline{a}_n{}^*xe^{i\varphi}|^2}dx.$$

We have

$$|1-\overline{a}_nre^{i\varphi}|=r\Big|\frac{1}{r}-\overline{a}_ne^{i\varphi}\Big|>r|1-\overline{a}_ne^{i\varphi}|\ge r|a_n-e^{i\varphi}|,0<r<1. \tag{23}$$

Using the fact that $z\in\mathcal{S}_{\sigma}(e^{i\theta})$, (23) and applying Lemma 1, we deduce

$$|F_*^{[\alpha]}(re^{i\varphi})|\le C+C(\alpha)r^{\alpha}\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|^2}{|1-\overline{a}_n{}^*xe^{i\varphi}|^{1+\alpha}}$$

$$\le C+\frac{C(\alpha)}{r}\sum_{n=1}^{\infty}\frac{1-|a_n{}^*|}{|e^{i\theta}-a_n{}^*|^{1+\alpha}}<\infty,\frac{1}{2}\le r<1.$$

For $r < \frac{1}{2}$ the boundedness is obvious. ■

Example 1. Let $\alpha \in (0,1), \gamma > 1$. We show that the statement of Theorem 3 is sharp. Let μ be an absolutely continuous measure with the density

$$p(t) = \begin{cases} \gamma|t|^{\gamma-1}, & |t| \le \frac{\pi}{4}, \\ 0, & |t| \in (\frac{\pi}{4}, \pi]. \end{cases}$$

We prove that if

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{i\theta} - e^{it}|^{1+\alpha}} \tag{24}$$

is divergent, then $\left|G^{[\alpha]}(z)\right|$ is unbounded, where G is of the form (8). Without loss of generality, we may assume that $\theta = 0, \varphi = 0$. Since $|e^{it} - 1| \sim t$ as $t \downarrow 0$, the integral (24) is divergent for $\gamma \le 1 + \alpha$. Let us calculate the derivative of $r^\alpha G(r)$,

$$\frac{\partial}{\partial r}(r^\alpha G(r)) = \alpha r^{\alpha-1} G(r) - r^\alpha G(r)\gamma \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2e^{it}}{(e^{it} - r)^2} |t|^{\gamma-1} dt.$$

Using (13) with $p = 1$, we obtain

$$D^\alpha\big(r^\alpha G(r)\big) = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-x)^{-\alpha} G(x) \Bigg(\alpha x^{\alpha-1}$$

$$-x^\alpha \gamma \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(2(1+x^2)\cos t - 4x - 2i(1-x^2)\sin t)|t|^{\gamma-1} dt}{|e^{it} - x|^4} \Bigg) dx.$$

Since $p(t)$ is continuous at 0, we get ([11, Chapter IX, p. 369])

$$\big|G(r)\big| \to \exp\{-2\pi p(0)\} = 1, \quad r \to 1 - 0.$$

Using (19), we deduce

$$\big|D^\alpha(r^\alpha G(r))\big| \ge \frac{2\gamma}{\Gamma(1-\alpha)} \int_0^r (r-x)^{-\alpha} x^\alpha \int_0^{\frac{\pi}{4}} \frac{2\sin t(1-x^2)t^{\gamma-1} dt}{|e^{it} - x|^4} \, dx$$

$$-C(\alpha) \ge C(\alpha,\gamma) \int_0^r (r-x)^{-\alpha}(1-x) \int_0^{1-x} \frac{t^\gamma dt}{(1-x)^4} \, dx - C(\alpha)$$

$$\geq C(\alpha,\gamma)\int\limits_0^r \frac{(r-x)^{-\alpha}dx}{(1-x)^{2-\gamma}} - C(\alpha) \geq C(\alpha,\gamma)\int\limits_0^r \frac{dx}{(1-x)^{2-\gamma+\alpha}} - C(\alpha)$$

$$\geq \begin{cases} \frac{C}{(1-r)^{1-\gamma+\alpha}} - C, & \gamma < 1+\alpha, \\ C\ln\frac{1}{1-r} - C, & \gamma = 1+\alpha. \end{cases}$$

Thus, $\left|G^{[\alpha]}(z)\right|$ is unbounded for $\gamma \leq 1+\alpha$.

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