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On the Long-Time Asymptotics for the Korteweg–de Vries Equation with Steplike Initial Data Associated with Rarefaction Waves

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We discuss an asymptotical behavior of the rarefaction wave for the KdV equation in the region behind the wave front. The first and the second terms of the asymptotical expansion for such a solution with respect to large time were derived without detailed analysis in [1]. In the present work, we correct the formula for the second term by investigating the corresponding parametrix problem. We also study an influence of the resonance on the asymptotical behavior of the solution.

Key words: KdV equation, rarefaction wave, parametrix problem.

Mathematical Subject Classification 2010: 37K40, 35Q53, 35Q15.

1. Introduction

This paper is a continuation of [1], where the long-time asymptotics of the Cauchy problem solution for the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

with steplike initial data $q(x,0) = q_0(x)$,

$$\begin{cases} q_0(x) \to 0, & \text{as} \quad x \to +\infty, \\ q_0(x) \to c^2, & \text{as} \quad x \to -\infty, \end{cases} \tag{2}$$

was studied. The initial profile of type (2) corresponds to the rarefaction wave. Its asymptotics is well understood on a physical level of rigor (see [10,12,15]). In [1], the asymptotics of the solution for (1), (2) is studied mathematically rigorously for the regions ahead of the back wave front by using the nonlinear steepest descent method [6]. As for the region behind the back wave front, in [1], the respective Riemann–Hilbert (RH) problem was reduced to a model RH problem in the nonresonant case. The structure of the transformations which were performed to get this model problem led to an assumption that the solution of (1), (2) should be asymptotically close to the respective background constant c^2, plus a decaying "radiation part" of order $O(t^{-1/2})$. Moreover, for this second term of the asymptotical expansion a formula was given which had the same form as for the decaying initial data, that is, when $q_0(x) \to 0$ as $x \to \pm\infty$.

The objectives of the present paper are: (a) to justify the asymptotical expansion for the solution of (1), (2) with respect to large t in the region $x < (-6c^2 - \epsilon)t$; (b) to check a possible influence of the resonance on the asymptotical expansion; (c) to clarify the formula for the second term.

We will assume that the initial profile (2) satisfies the condition

$$\int_0^{+\infty} \mathrm{e}^{(c+\kappa)x}(|q_0(x)| + |q_0(-x) - c^2|\, dx < \infty,\ x^4 q^{(i)}(x) \in L_1(\mathbb{R}),\ i = 1, \dots, 8, \quad (3)$$

where $\kappa > 0$ is a small number. Under this condition the solution of the Cauchy problem (1), (2) exists in the classical sense and it is unique in the domain $(x,t) \in \mathbb{R} \times [0,T]$ for any $T > 0$. Moreover, for each t, it tends to the background constants $0, c^2$ with at least the first finite moment of perturbation (cf. [8]). Note that condition (3) is more restrictive than the decay condition from [1]. In fact, as we will see later, condition (3) appears as a natural restriction in the domain behind the back wave front in the resonant case. We prove the following:

Theorem 1. *Let $q(x,t)$ be the solution of the Cauchy problem* (1)–(3)*. Then for arbitrary small $\epsilon > 0$ in the domain $x < (-6c^2 - \epsilon)t$ the following asymptotics is valid as $t \to \infty$:*

$$q(x,t) = c^2 + \sqrt{\frac{4\nu(a)a}{3t}} \sin(16ta^3 - \nu(a)\log(192ta^3) + \Delta(a)) + o(t^{-\gamma}) \quad (4)$$

for some $1/2 < \gamma < 1$. Here

$$a = \sqrt{-\frac{c^2}{2} - \frac{x}{12t}}, \quad \nu(a) = -\frac{1}{2\pi}\log\left(1 - |R(a)|^2\right), \quad (5)$$

$$\Delta(a) = \frac{\pi}{4} + \arg(R(a)) + \arg(\Gamma(\mathrm{i}\nu(a))) + \frac{1}{\pi}\int_{\mathbb{R}\setminus[-a,a]} \log\left(\frac{1 - |R(s)|^2}{1 - |R(a)|^2}\right)\frac{ds}{s-a},$$

$\Gamma(z)$ is the Gamma-function and $R(k)$ is the left reflection coefficient of the initial profile (3).

Note that the radiation part of formula (4) given by the *left* scattering data looks very similar to that of the decaying case [9]. However, the investigation of Riemann–Hilbert problems associated with the steplike initial profile has its own distinctive features.

To simplify the presentation we changed notations used in [1] and omitted some indices.

2. Statement of the RH Problem

First briefly recall some facts from the scattering theory of the Schrödinger operator with steplike potentials (see [4, 5, 7]). Let $q(x,t)$ be the solution of the Cauchy problem (1)–(3). Consider the underlying spectral problem for the operator $H(t) := -\frac{d^2}{dx^2} + q(x,t)$ on the whole axis:

$$(H(t)f)(x) = \lambda f(x), \qquad x \in \mathbb{R}, \tag{6}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. As is known, the spectrum of $H(t)$ consists of an absolutely continuous part $\mathbb{R}_+$ and a finite number of negative eigenvalues $-\kappa_1^2 < \cdots < -\kappa_N^2 < 0$. In turn, the continuous spectrum consists of a part $[0, c^2]$ of multiplicity one and a part $[c^2, \infty)$ of multiplicity two. Instead of λ in equation (6) another spectral parameter $k = \sqrt{\lambda - c^2}$ is used for sake of convenience. Here for the square root we choose the standard branch such that the function $k = k(\lambda)$ is a bijection between the domains $\mathbb{C}\backslash\mathbb{R}_+$ and $\mathfrak{D} := \mathbb{C}^+\backslash(0, \mathrm{i}c]$. The solutions of equation (6) will be considered as the functions of the parameter $k \in \overline{\mathfrak{D}} = \mathfrak{D} \cup \partial\mathfrak{D}$. In particular, equation (6) has two Jost solutions $\phi(k,x,t)$ and $\phi_1(k,x,t)$ satisfying the conditions

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}\sqrt{k^2+c^2}x}\phi_1(k,x,t) = \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}kx}\phi(k,x,t) = 1, \quad k \in \overline{\mathfrak{D}}.$$

The Jost solutions satisfy the scattering relation

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \quad k \in \mathbb{R},$$

where $T(k,t)$, $R(k,t)$ are the left transmission and reflection coefficients. For the transmission coefficient, the following formula is valid: $T(k,t) = 2\mathrm{i}kW^{-1}(k,t)$, where $W(k,t) := \langle\phi_1, \phi\rangle(k,t)$ is the Wronskian of the Jost solutions. As a function of k, the Wronskian $W(k,t)$ is a holomorphic function in the domain $\mathfrak{D}$, it has continuous limit values on the boundary $\partial\mathfrak{D}$ and never vanishes on $\partial\mathfrak{D}$, except possibly at the point $k = \mathrm{i}c$. At this point there are two options:

(a) If $W(\mathrm{i}c, 0) \neq 0$, then $W(\mathrm{i}c, t) \neq 0$ for any t. In this case, we say that at the point $\mathrm{i}c$ there is no resonance. It is a general situation.

(b) If $W(\mathrm{i}c,0)=0$ (i.e., $W(\mathrm{i}c,t)=0$ for any t), then we deal with the resonance at the point $\mathrm{i}c$. Note (cf. [7]) that in this case,

$$W(k,t)=C\sqrt{k-\mathrm{i}c}\,(1+o(1)),\quad C=C(t)\neq 0.$$

For the operator $H(t)$, the point $\mathrm{i}c$ is the only point where the resonance can happen, that is why we associate the notion of the resonant or nonresonant cases with the solution $q(x,t)$.

Obviously, the transmission coefficient $T(k,t)$ has a meromorphic extension to the domain $\mathfrak{D}$ with simple poles at the points $\mathrm{i}\kappa_1,\dots,\mathrm{i}\kappa_N$. We set

$$\chi(k,t):=-\lim_{\varepsilon\to+0}\frac{\sqrt{(k+\epsilon)^2+c^2}}{k}|T(k+\varepsilon,t)|^2,\quad k\in[0,\mathrm{i}c].$$

This function is purely imaginary. Moreover,

$$\chi(k,t)=\mathrm{i}\,|\chi(k,t)|,\quad k\in[0,\mathrm{i}c].$$

It is continuous on the set $[0,\mathrm{i}c)$ with $\chi(0,t)=0$. In the nonresonant case,

$$\chi(k,t)=C(t)\sqrt{k-\mathrm{i}c}(1+o(1)),\quad k\to\mathrm{i}c,\quad C(t)\neq 0. \tag{7}$$

In the resonant case, the function $\chi(k,t)$ has a singularity

$$\chi(k,t)=\frac{C(t)}{\sqrt{k-\mathrm{i}c}}(1+o(1)),\quad k\to\mathrm{i}c,\quad C(t)\neq 0. \tag{8}$$

Next, it is evident that the Jost solutions $\phi(\mathrm{i}\kappa_j,x,t)$ are the eigenfunctions of the operator $H(t)$. Denote the inverse squares of the norms as

$$\gamma_j(t)=\left(\int_{\mathbb{R}}\phi^2(\mathrm{i}\kappa_j,x,t)\,dx\right)^{-1}.$$

The functions $R(k,t)$, $k\in\mathbb{R}$, and $\chi(k,t)$, $k\in[0,\mathrm{i}c]$, and also the quantities $-\kappa_j^2$, $\gamma_j(t)$, $j=1,\dots,N$ are the left scattering data of the operator $H(t)$. Their evolution due to the KdV flow is given by formulas (cf. [11]):

$$\gamma_j(t)=\gamma_j\mathrm{e}^{-8\kappa_j^3t+12c^2\kappa_jt}, \tag{9}$$

$$\chi(\lambda,t)=\chi(k)\mathrm{e}^{-8\mathrm{i}tk^3-12\mathrm{i}tkc^2}, \tag{10}$$

$$R(\lambda,t)=R(k)\mathrm{e}^{-8\mathrm{i}tk^2-12\mathrm{i}tkc^2}, \tag{11}$$

where we denoted $\chi(k)=\chi(k,0)$, $R(k)=R(k,0)$, and $\gamma_j=\gamma_j(0)$. By means of the Inverse Scattering Transform, the solution $q(x,t)$ of problem (1)–(3) can be uniquely recovered from the left initial scattering data (see [7]),

$$\{R(k),\quad k\in\mathbb{R};\quad \chi(k),\quad k\in[0,\mathrm{i}c];\quad -\kappa_j^2,\quad \gamma_j>0,\quad j=1,\dots,N\}.$$

The properties of the left scattering data listed above allow us to formulate a vector RH problem. Namely, in $\mathfrak{D}$ we introduce a meromorphic vector function (variables x and t are treated as parameters)

$$\tilde{m}(k) = (\tilde{m}_1(k), \tilde{m}_2(k)) = \Big(T(k,t)\phi_1(k,x,t)\mathrm{e}^{-\mathrm{i}kx}, \quad \phi(k,x,t)\mathrm{e}^{\mathrm{i}kx}\Big). \tag{12}$$

This function has the following expansion as $k \to \infty$ (cf. [1]):

$$\tilde{m}(k) = \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{2\mathrm{i}k}\left(\int_{-\infty}^{x}(q(y,t)-c^2)dy\right)\begin{pmatrix}1 & -1\end{pmatrix} + O\left(\frac{1}{k^2}\right), \tag{13}$$

and therefore $\tilde{m}$ is bounded at infinity. The only singularities of this vector function in $\mathfrak{D}$ are the poles of its first component $\tilde{m}_1(k)$ at the points $\mathrm{i}\kappa_j$. Beyond these poles the function $\tilde{m}$ is continuous up to the boundary $\partial\mathfrak{D}$ except, probably, at the point $\mathrm{i}c$ in the resonant case. Let us extend $\tilde{m}$ to the domain $\mathfrak{D}^* = \{k : -k \in \mathfrak{D}\}$ by the symmetry condition $\tilde{m}(-k) = \tilde{m}(k)\sigma_1$, where $\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is the first Pauli matrix. After this extension the second component of the vector function $\tilde{m}(k)$ has poles at the points $-\mathrm{i}\kappa_j$. Also, $\tilde{m}(k)$ has jumps along the real axis and along the segment $[\mathrm{i}c, -\mathrm{i}c]$.

Introduce a cross-shaped contour $\widetilde{\Sigma} := \mathbb{R} \cup [\mathrm{i}c, -\mathrm{i}c]$ with a natural orientation from minus to plus infinity on $\mathbb{R}$, and from up to down on $[\mathrm{i}c, -\mathrm{i}c]$. Denote by $\tilde{m}_+(k)$ (respectively $\tilde{m}_-(k)$) the limiting nontangential values of $\tilde{m}(k)$ from the right (respectively left) in the contour direction.

To simplify notations throughout of this paper along with the first Pauli matrix σ_1, we use the third Pauli matrix $\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and three more matrices:

$$\mathbb{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbb{J} := \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad \mathbb{J}^\dagger := \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \sigma_1\mathbb{J}\sigma_1. \tag{14}$$

Let now $\mathbb{T}_j$ (respectively $\mathbb{T}_j^*$) be circles centered at $\mathrm{i}\kappa_j$ (respectively $-\mathrm{i}\kappa_j$) with radii $0 < \delta < \frac{1}{4}\min_{j=1}^N |\kappa_j - \kappa_{j-1}|$, $\kappa_0 := 0$. Choose $\delta > 0$ so small that the discs $|k - \mathrm{i}\kappa_j| < \delta$ lie inside the upper half-plane and do not intersect any of the other contours, moreover $\kappa_1 - \delta > \kappa + c$, where κ is the same as in estimate (3). The small circles $\mathbb{T}_j$ around $\mathrm{i}\kappa_j$ are oriented counterclockwise, and the circles $\mathbb{T}_j^*$ around $-\mathrm{i}\kappa_j$ are oriented clockwise.

Introduce also the phase function $\Phi(k) = \Phi(k,x,t)$:

$$\Phi(k) = -4\mathrm{i}k^3 - 6\mathrm{i}c^2k - 12\mathrm{i}\xi k, \quad \xi = \frac{x}{12t}.$$

This function is odd in $\mathbb{C}$. Its stationary points are $\pm a$, where $a := \sqrt{-\frac{c^2}{2} - \xi}$. The signature table for $\operatorname{Re}\Phi(k)$ when $\xi < -\frac{c^2}{2}$ is shown in Figure 1.

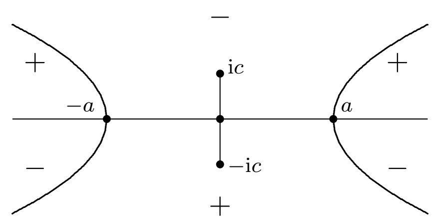

Fig. 1: The signature table of $\mathrm{Re}(\Phi(k))$.

Redefine now $\tilde m(k)$ inside $\mathbb{T}_j, \mathbb{T}_j^*,\ j=1,...,N$ according to

$$m(k)=\begin{cases}\tilde m(k)A_j(k), & |k-\mathrm{i}\kappa_j|<\delta,\\ \tilde m(k)\sigma_1 A_j^{-1}(-k)\sigma_1, & |k+\mathrm{i}\kappa_j|<\delta,\\ \tilde m(k), & \text{else,}\end{cases} \tag{15}$$

where

$$A_j(k)=\begin{pmatrix}1 & 0\\ -\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1\end{pmatrix}=\mathbb{I}-\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j}\mathbb{J}.$$

Thus $m(k)$ becomes holomorphic but with additional jumps along the circles $\mathbb{T}_j$, $\mathbb{T}_j^*$, $j=1,\dots,N$. Moreover, it preserves the asymptotics (13) of $\tilde m(k)$ as $k\to\infty$.

Theorem 2. *Let* $\{R(k),\ k\in\mathbb{R};\ \chi(k),\ k\in[0,\mathrm{i}c];\ (\kappa_j,\gamma_j),\ 1\le j\le N\}$ *be the left scattering data of the operator* $H(0)$*. Then the vector function* $m(k)=m(k,x,t)$*, defined by* (12), (15)*, is the unique solution of the following vector Riemann–Hilbert problem:*

Find a vector function $m(k)$ *which is holomorphic away from the contour* $\Sigma=\bigcup_{j=1}^N(\mathbb{T}_j\cup\mathbb{T}_j^*)\cup\mathbb{R}\cup[-\mathrm{i}c,\mathrm{i}c]$*, has continuous limiting values from both sides of the contour, except possibly of the points* $\pm\mathrm{i}c$*, and satisfies:*

A. *The jump condition* $m_+(k)=m_-(k)v(k)$*, where*

$$v(k)=\begin{cases}\begin{pmatrix}1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)}\\ R(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}, & k\in\mathbb{R},\\ \begin{pmatrix}1 & 0\\ \chi(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}=\mathbb{I}+\chi(k)\mathrm{e}^{2t\Phi(k)}\mathbb{J}, & k\in[\mathrm{i}c,0],\\ A_j(k), & k\in\mathbb{T}_j,\quad k=1,\dots,N,\\ \sigma_1 v^{-1}(-k)\sigma_1, & k\in\cup_{j=1}^N\mathbb{T}_j^*\cup[0,-\mathrm{i}c].\end{cases}$$

B. *The symmetry condition*

$$m(-k) = m(k)\sigma_1. \tag{16}$$

C. *The normalization condition* $\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

D. *In the vicinities of the points* $\pm\mathrm{i}c$*:*

(a) *if* $\chi(k)$ *satisfies* (7)*, then* $m(k)$ *is continuous at the points* $\pm\mathrm{i}c$*;*

(b) *if* $\chi(k)$ *satisfies* (8)*, then*

$$m(k) = \begin{pmatrix} \frac{C_1}{\sqrt{k-\mathrm{i}c}}, & C_2 \end{pmatrix}(1+o(1)), \quad k \to \mathrm{i}c, \quad C_1 \neq 0,$$

with a similar condition at $-\mathrm{i}c$ *due to* (16)*.*

Proof. The proof can be obtained combining the uniqueness result from [2] and a slightly modified for the resonant case proof of Theorem 2.5 from [1]. □

3. Reduction to the Model Problem

In this section, we describe some conjugation/deformation steps as $\xi < -c^2/2$ for the RH problem **A**–**D** which lead to an equivalent RH problem. The new RH problem will have a jump matrix close to the unitary matrix $\mathbb{I}$ for large time except of small vicinities of the points $\pm a$. A short description of these steps was proposed in Section 8 of [1]. We extend these steps taking into account the resonant case.

According to the signature table of the phase function (see Figure 1), the matrix $v(k)$ is exponentially close for large t to the identity matrix $\mathbb{I}$ on the segments $[-\mathrm{i}c, 0) \cup (0, \mathrm{i}c]$ and on the circles $\cup_{j=1}^{N}(\mathbb{T}_j \cup \mathbb{T}_j^*)$, but it is oscillatory with respect to t on the real axis. Besides one can have singularities of $v(k)$ at the points $\pm\mathrm{i}c$. As a first step, we apply the standard upper–lower and lower–upper factorizations (cf. [6, 9]) to the matrix $v(k)$ as $k \in \mathbb{R}$. To this end, we construct an analytic in the domain $\mathbb{C} \setminus ((-\infty, -a) \cup (a, \infty))$ function $d(k)$ satisfying the jump condition

$$d_+(k) = d_-(k)(1 - |R(k)|^2) \text{ for } k \in \mathbb{R} \setminus [-a, a],$$

and such that $d(-k) = d^{-1}(k)$ and $d(k) \to 1$ as $k \to \infty$. By the Sokhotski–Plemelj formula, this function is explicitly given by

$$d(k) = \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}\setminus[-a,a]} \frac{\log(1 - |R(s)|^2)}{s-k}\, ds\right). \tag{17}$$

Since the domain of integration is even and the function $\log(1-|R(s)|^2)$ is also even, then $d(-k)=d^{-1}(k)$. For $k\to\infty$, we have

$$d(k)=1-\frac{1}{2\pi \mathrm{i}k}\int_{\mathbb{R}\setminus[-a,a]}\log(1-|R(s)|^2)\,ds+O\left(\frac{1}{k^2}\right). \qquad (18)$$

Put $m^{(1)}(k)=m(k)d(k)^{-\sigma_3}$. Evidently, $m^{(1)}(-k)=m^{(1)}(k)\sigma_1$. One can check that (see, e.g., [9]) $m^{(1)}(k)$ satisfies the jump condition $m_+^{(1)}(k)=m_-^{(1)}(k)v^{(1)}(k)$ with

$$v^{(1)}(k)=\begin{pmatrix}1-|R(k)|^2 & -\overline{R(k)}d^2(k)\mathrm{e}^{-2t\Phi(k)}\\ R(k)d^{-2}(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix},\quad k\in[-a,a],$$
$$v^{(1)}(k)=\begin{pmatrix}(1-|R(k)|^2)d_+^{-1}(k)d_-(k) & -\overline{R}(k)d_+(k)d_-(k)\mathrm{e}^{-2t\Phi(k)}\\ R(k)d_+^{-1}(k)d_-^{-1}(k) & d_-^{-1}(k)d_+(k)\end{pmatrix},$$
$$k\in\mathbb{R}\setminus[-a,a],$$
$$v^{(1)}(k)=d(k)^{\sigma_3}v(k)d(k)^{-\sigma_3},\quad k\in\cup_{j=1}^N(\mathbb{T}_j^U\cup\mathbb{T}_j^L)\cup[\mathrm{i}c,-\mathrm{i}c].$$

Recall that $\overline{R(k)}=R(-k)$ for $k\in\mathbb{R}$. Under condition (3), one can continue the function $R(k)$ in the vicinity of the contour $\widetilde{\Sigma}$. Introduce the domains Ω_l^*, Ω_l, Ω_r^*, Ω_r, Ω^*, and Ω together with their boundaries $\mathscr{C}_l^*$, $\mathscr{C}_l$, $\mathscr{C}_r^*$, $\mathscr{C}_r$, $\mathscr{C}^*$, and $\mathscr{C}$, which are contained in the strip $\{k:|\operatorname{Im}k|<c+\kappa\}$ as depicted in Figure 2. Using

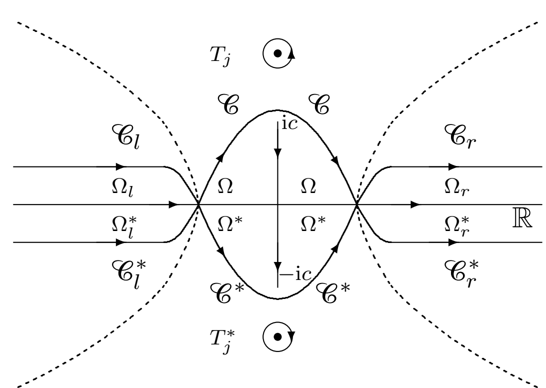

Fig. 2: Contour deformation in the domain $x<-6c^2t$.

(14), put

$$\begin{aligned}
B(k) :=& \mathbb{I} + R(k)d^{-2}(k)\mathrm{e}^{2t\Phi(k)}\mathbb{J}, \qquad k \in \Omega,\\
B^*(k) :=& \mathbb{I} - \overline{R(k)}d^{2}(k)\mathrm{e}^{-2t\Phi(k)}\mathbb{J}^\dagger, \qquad k \in \Omega^*,\\
A(k) :=& \mathbb{I} + \frac{\overline{R(k)}d^{2}(k)}{1-|R(k)|^2}\mathrm{e}^{-2t\Phi(k)}\mathbb{J}^\dagger, \quad k \in \Omega_r \cup \Omega_l,\\
A^*(k) :=& \mathbb{I} - \frac{R(k)d^{-2}(k)}{1-|R(k)|^2}\mathrm{e}^{2t\Phi(k)}\mathbb{J}, \qquad k \in \Omega_r^* \cup \Omega_l^*.
\end{aligned} \tag{19}$$

Then

$$v^{(1)}(k) = \begin{cases} B_-^*(k)B_+(k), & k \in [-a,a],\\ A_-^*(k)A_+(k), & k \in \mathbb{R}\setminus[-a,a]. \end{cases}$$

Redefine $m^{(1)}(k)$ according to

$$m^{(2)}(k) = m^{(1)}(k)\begin{cases} B(k), & k\in\Omega,\\ B^*(k), & k \in \Omega^*,\\ A(k), & k \in \Omega_l\cup\Omega_r,\\ A^*(k), & k\in \Omega_l^*\cup\Omega_r^*,\\ \mathbb{I}, & \text{else}. \end{cases} \tag{20}$$

Lemma 1. *The following formulas are valid:*

$$\begin{aligned}
&B_-(k)v^{(1)}(k)(B_+(k))^{-1} = \mathbb{I}, \quad k \in [\mathrm{i}c, 0],\\
&(B_-^*(k))^{-1}v^{(1)}(k)B_+^*(k) = \mathbb{I}, \quad k \in [0,-\mathrm{i}c].
\end{aligned}$$

Proof. We observe that for $k \in [\mathrm{i}c, 0]$:

$$B_-(k)v^{(1)}(k)B_+(k)^{-1} = \begin{pmatrix} 1 & 0\\ d(k)^{-2}(R_-(k)-R_+(k)+\chi(k))\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}.$$

As is known, under condition (3), the complex conjugated Jost solution $\overline{\phi}(k,x,0)$ can be continued analytically into a strip. Denote this continuation as $\breve{\phi}(k,x,0)$. It does not have a jump along the interval $[\mathrm{i}c, 0]$. Then the continuation of $R(k)$ can be represented via Wronskians in a usual way (cf. [7]). If $\phi_1(k,x) := \lim_{\epsilon\to+0}\phi_1(k+\epsilon,x,0)$, then

$$R_-(k) = -\frac{\langle\overline{\phi}_1,\breve{\phi}\rangle}{\langle\overline{\phi}_1,\phi\rangle}, \quad R_+(k) = -\frac{\langle\phi_1,\breve{\phi}\rangle}{\langle\phi_1,\phi\rangle}, \quad \chi(k) = -\frac{\langle\phi,\breve{\phi}\rangle}{\langle\overline{\phi}_1,\breve{\phi}\rangle}\frac{\langle\phi_1,\overline{\phi}_1\rangle}{\langle\phi_1,\phi\rangle},$$

where $\langle f, g\rangle$ is the usual Wronskian of two solutions of (6). Applying the Plücker identity (cf. [14]),

$$\langle f_1,f_2\rangle\langle f_3,f_4\rangle + \langle f_1,f_3\rangle\langle f_4,f_2\rangle + \langle f_1,f_4\rangle\langle f_2,f_3\rangle \equiv 0,$$

to the functions $f_1 = \phi_1, f_2 = \phi,\ f_3 = \overline{\phi}_1,\ f_4 = \breve{\phi}$, we get

$$R_-(k) - R_+(k) + \chi(k) \equiv 0, \tag{21}$$

which proves the first identity of the lemma. The second identity can be proved in the same way. □

Note that equality (21) and transformation (20) imply that in both, the resonant and the nonresonant cases, the vector function $m^{(2)}(k)$ does not have a jump along the interval $[\mathrm{i}c, -\mathrm{i}c]$. Therefore the final asymptotics will not depend on the resonance.

By use of Lemma 1, we conclude that the vector function $m^{(2)}(k)$ satisfies the jump $m_+^{(2)}(k) = m_-^{(2)}(k)v^{(2)}(k)$ with

$$v^{(2)}(k) = \begin{cases} B(k), & k \in \mathscr{C}, \\ B^*(k), & k \in \mathscr{C}^*, \\ A(k), & k \in \mathscr{C}_l \cup \mathscr{C}_r, \\ A^*(k), & k \in \mathscr{C}_l^* \cup \mathscr{C}_r^*, \\ v^{(1)}(k), & k \in \cup_{j=1}^N (\mathbb{T}_j \cup \mathbb{T}_j^*). \end{cases} \tag{22}$$

Thus the matrix $v^{(2)}(k)$ has the structure

$$v^{(2)}(k) = \mathbb{I} + \begin{cases} F_1(k), & k \in \cup_{j=1}^N (\mathbb{T}_j \cup \mathbb{T}_j^*), \\ F_2(k), & k \in \mathscr{C}_l \cup \mathscr{C} \cup \mathscr{C}_r \cup \mathscr{C}_l^* \cup \mathscr{C}^* \cup \mathscr{C}_r^*, \end{cases}$$

with the matrices $F_{1,2}(k)$ admitting the estimates

$$\|F_1(k)\| \le C\mathrm{e}^{-Ct}, \quad \|F_2(k)\| \le C(a)\mathrm{e}^{-t\mu(|k^2-a^2|)}, \tag{23}$$

where $\|\cdot\|$ is any norm of a matrix 2×2, $C > 0$, $C(a) > 0$ and $\mu(s)$, $s \in \mathbb{R}_+$, is a strictly increasing continuous function with $\mu(0) = 0$ and $\mu(s) = O(s^{3/4})$ as $s \to \infty$. Note that the vector function $m^{(2)}(k)$ has no jump along the contour $\widetilde{\Sigma}$, and, therefore, the effect of resonance is not noticeable for $\xi < -c^2/2$. Due to (23), we can conclude that $m^{(2)}(k) \sim \begin{pmatrix} 1 & 1 \end{pmatrix}$ as $k \to \infty$. As it will be shown in the next section, an error term has the structure $\begin{pmatrix} 1 & -1 \end{pmatrix} O(k^{-1})O(t^{-1/2})$.

Recall that for large imaginary k with $|k| > \kappa_1 + 1$, we have $\tilde{m}(k) = m^{(2)}(k)d(k)^{\sigma_3}$ with $d(k)$ defined by (17). By use of (18), one can expect that for $k \to \infty$,

$$\tilde{m}_1(k) = m_1(k) \sim d(k) = 1 - \frac{\int_{\mathbb{R}\setminus[-a,a]} \log(1 - |R(s)|^2)ds}{2\pi \mathrm{i} k} + \frac{g(x,t)}{k} + O\left(\frac{1}{k^2}\right),$$

where $g(x,t) = o(1), g_x(x,t) = o(1)$ as $t \to \infty$ uniformly with respect to x. The function $g(x,t)$ appears due to the effect of parametrix in small vicinities of the points $\pm a$. A formula for this function will be obtained in Section 4. Next, by (13),

$$q(x,t) = \frac{\partial}{\partial x} \lim_{k\to\infty} 2\mathrm{i}k\,(\tilde{m}_1(k) - 1). \tag{24}$$

Since $\frac{\partial}{\partial x} a(\xi) = O(t^{-1})$, then it follows from (3) that after differentiation the integral from the right-hand side will be of order $O(t^{-1})$. Respectively,

$$q(x,t) = c^2 + o(1), \quad \text{as } t \to \infty.$$

Thus the leading term is equal to c^2 as expected. In the next section we will show that the effect of the parametrix points implies in fact the term of order $O\left(t^{-1/2}\right)$.

4. The Parametrix Problem

We use the same approaches as in [6,13], but for the vector RH problem as in [9]. Following these approaches, we start with investigation in more details of the behavior of the jump matrix $v^{(2)}(k)$ near the point $-a$. Represent (17) as

$$\log d(k) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}\setminus[-a,a]} \log \frac{1 - |R(s)|^2}{1 - |R(-a)|^2} \frac{ds}{s-k} + \frac{\log(1 - |R(-a)|^2)}{2\pi\mathrm{i}} \int_{\mathbb{R}\setminus[-a,a]} \frac{ds}{s-k}.$$

Since $\int_{\mathbb{R}\setminus[-a,a]} \frac{ds}{s-k} = \log\frac{k+a}{a-k}$, then

$$d(k) = \left(\frac{k+a}{a-k}\right)^{\mathrm{i}\nu} \mathrm{e}^{\eta(k)},$$

where

$$\nu := \nu(a) = -\frac{1}{2\pi} \log(1 - |R(-a)|^2),$$
$$\eta(k) := \eta(k,a) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}\setminus[-a,a]} \log \frac{1 - |R(s)|^2}{1 - |R(-a)|^2} \frac{ds}{s-k}. \tag{25}$$

Let $\mathbb{D}_\rho(-a)$ be a circle of the radius $0 < \rho < \inf\{\frac{1}{4}, \frac{a}{4}\}$ centered at the point $-a$. Without loss of generality, one can assume that inside the domain $\mathbb{D}_\rho(-a)$ the contours $\mathscr{C}(\rho) := \mathscr{C} \cap \mathbb{D}_\rho(-a)$, $\mathscr{C}^*(\rho) := \mathscr{C}^* \cap \mathbb{D}_\rho(-a)$, $\mathscr{C}_l(\rho) := \mathscr{C}_l \cap \mathbb{D}_\rho(-a)$, $\mathscr{C}_l^*(\rho) := \mathscr{C}_l^* \cap \mathbb{D}_\rho(-a)$ are the parts of rays $\{-a + s\mathrm{e}^{\mathrm{i}(2n+1)\pi/4},\ s \in \mathbb{R}_+\}$, and they have orientations as depicted in Figure 3. Put

$$\Gamma_\rho(-a) := \mathscr{C}(\rho) \cup \mathscr{C}^*(\rho) \cup \mathscr{C}_l(\rho) \cup \mathscr{C}_l^*(\rho).$$

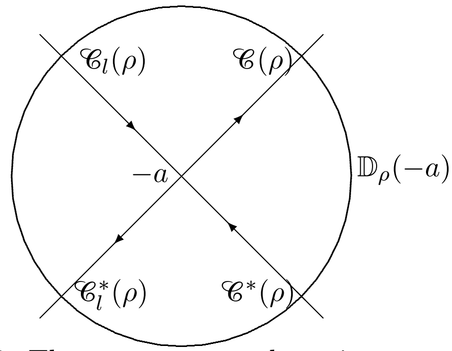

Fig. 3: The contour near the point $-a$.

Lemma 2. *The following inequalities hold for all $k \in \Gamma_\rho(-a)$ and $a > \epsilon$, where ϵ is a constant from Theorem 1 :*

$$\left|\mathrm{e}^{-2\eta(k)} - \mathrm{e}^{-2\eta(-a)}\right| \le C|k+a|(1 + |\log|k+a||), \tag{26}$$

$$\left|1 - \mathrm{e}^{-2\mathrm{i}\nu\log\frac{k-a}{2a}}\right| \le Ca^{-1}|k+a|, \tag{27}$$

where the constant $C = C(\epsilon)$ does not depend on ξ and k.

Proof. We give the proof for $k \in \mathscr{C}(\rho)$. The other cases are similar. First we show that

$$|\eta(k) - \eta(-a)| \le C|k+a|(1 + |\log|k+a||), \quad a \in I, \quad k \in \mathscr{C}(\rho). \tag{28}$$

Divide the domain of integration in (25) into three parts $[-\infty, -2a]$, $[-2a, -a]$, $[a, \infty]$, and denote by $I_1(k)$, $I_2(k)$, $I_3(k)$ the respective integrals. For $k \in \mathscr{C}(\rho)$, the following estimates are straightforward:

$$|I_1(k) - I_1(-a)| \le C|k+a|, \quad |I_3(k) - I_3(-a)| \le C|k+a|. \tag{29}$$

Integrating $2\pi\mathrm{i}I_2(\xi,k)$ by parts:

$$\int_{-2a}^{a} \log\frac{1-|R(s)|^2}{1-|R(-a)|^2}\frac{ds}{s-k} = -\log\frac{1-|R(-2a)|^2}{1-|R(-a)|^2}\log(-2a-k) - \int_{-2a}^{a}\log(s-k)\,d\log(1-|R(s)|^2),$$

we get

$$|I_2(k) - I_2(-a)| = \frac{1}{2\pi}\left|\log\frac{k+2a}{a}\log\frac{1-|R(-2a)|^2}{1-|R(-a)|^2}\right.$$

$$+\int_{-2a}^{-a}\log\frac{s-k}{s+a}\,d\log(1-|R(s)|^2)\Bigg|.$$

Since $|R(s)|\le C(\epsilon)<1$ as $|s|>\epsilon$, then

$$|I_2(k)-I_2(-a)|\le C(\epsilon)\left(\left|\int_{-2a}^{-a}\left|\log\frac{s-k}{s+a}\right|\,ds\right|+|k+a|\right).$$

The change of variables $v=-|k+a|/(s+a)$ gives

$$\left|\int_{-2a}^{-a}\log\left|\frac{s-k}{s+a}\right|ds\right|=|k+a|\left|\int_{\frac{|k+a|}{a}}^{\infty}\log(1+v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})\frac{dv}{v^2}\right|,$$

where we took into account that $k\in\mathscr{C}(\rho)$. Combining this estimate with the estimate

$$|\log|1+v\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}||\le C\begin{cases}v, & 0\le v\le 2\\ \log v, & 2\le v\le\infty\end{cases}$$

and with (29), we get (28). Next, by Lemma 23.2 from [3], we get

$$\sup_{\xi<(-c^2/2-\epsilon)}\ \sup_{k\in\mathbb{C}\setminus\mathbb{R}}|\eta(k)|<\infty.$$

Using this, (28) and inequality $|\mathrm{e}^w-1|\le|w|\max(1,\mathrm{e}^{\mathrm{Re}\,w}),\quad w\in\mathbb{C}$, we get (26) and also

$$\left|1-\mathrm{e}^{-2\nu\log\frac{k+a}{2a}}\right|\le\left|2\nu\log\frac{k+a}{2a}\right|\mathrm{e}^{|\,\mathrm{Re}(2\mathrm{i}\nu\log\frac{k+a}{2a})|}$$
$$\le C\left|\log\left(1+\frac{k-a}{2a}\right)\right|\le Ca^{-1}|k+a|.$$

This proves (27). □

Introduce a local parameter $z=\sqrt{48a}(k+a)$. Then $z\in\mathbb{D}_{\rho_1}$, where $\mathbb{D}_{\rho_1}$ is the circle of the radius $\rho_1=\sqrt{48a}\,\rho$ centered at 0. The contour $\Gamma_\rho(-a)$ in terms of the variable z will have notation Γ_{ρ_1}, and for the constituents of this contour we will keep notations $\mathscr{C}$, $\mathscr{C}^*$, $\mathscr{C}_l$, $\mathscr{C}_l^*$. Taking into account (5), put $\varphi(z):=-8\mathrm{i}a^3+\frac{\mathrm{i}}{4}z^2$,

$$r_1(z):=\tilde{R}(z)\mathrm{e}^{-2\tilde{\eta}(z)}\mathrm{e}^{2\mathrm{i}\nu\log(2a\sqrt{48a}-z)},$$
$$r_2(z):=\frac{\overline{\tilde{R}(z)}}{1-|\tilde{R}(z)|^2}\mathrm{e}^{2\tilde{\eta}(z)}\mathrm{e}^{-2\mathrm{i}\nu\log(2a\sqrt{48a}-z)},$$
$$r_3(z):=\frac{\tilde{R}(z)}{1-|\tilde{R}(z)|^2}\mathrm{e}^{-2\tilde{\eta}(z)}\mathrm{e}^{2\mathrm{i}\nu\log(2a\sqrt{48a}-z)},$$

$$r_4(z) := \overline{\tilde{R}(z)} e^{2\tilde{\eta}(z)} e^{-2i\nu \log(2a\sqrt{48a}-z)}.$$

where $\tilde{R}(z) := R(k(z))$, $\tilde{\eta}(z) := \eta(k(z))$. The phase function is represented as

$$\tilde{\Phi}(z) := \Phi(k(z)) = \varphi(z) - \frac{iz^3}{12a\sqrt{48a}}.$$

From (19) and (22), it follows that the jump matrix $v^{(2)}(k)$ as a function of the variable $z \in \Gamma_{\rho_1}$ has the form

$$\tilde{v}^{(2)}(z) = \mathbb{I} + \begin{cases} r_1(z) z^{-2i\nu} e^{2t\tilde{\Phi}(z)} \mathbb{J}, & z \in \mathscr{C}, \\ -r_2(z) z^{2i\nu} e^{-2t\tilde{\Phi}(z)} \mathbb{J}^\dagger, & z \in \mathscr{C}_l, \\ -r_3(z) z^{-2i\nu} e^{2t\tilde{\Phi}(z)} \mathbb{J}, & z \in \mathscr{C}_l^*, \\ r_4(z) z^{2i\nu} e^{-2t\tilde{\Phi}(z)} \mathbb{J}^\dagger, & z \in \mathscr{C}^*. \end{cases} \tag{30}$$

Put now

$$f := f(a) = R(-a) e^{-2\eta(-a)} e^{2i\nu(a) \log(2a\sqrt{48a})}. \tag{31}$$

Since $\nu \in \mathbb{R}$ and $\eta(-a) \in i\mathbb{R}$, then $|f| = |R(-a)|$. From Lemma 2, it follows that for $z \in \mathbb{D}_{\rho_1}$ the functions $\{r_j(z)\}_1^4$ satisfy the inequalities:

$$\begin{aligned} |r_1(z) - f| &\le C(\epsilon)|z|^\alpha, && z \in \mathscr{C}, \\ \left| r_2(z) - \frac{\overline{f}}{1-|f|^2} \right| &\le C(\epsilon)|z|^\alpha, && z \in \mathscr{C}_l, \\ \left| r_3(z) - \frac{f}{1-|f|^2} \right| &\le C(\epsilon)|z|^\alpha, && z \in \mathscr{C}_l^*, \\ \left| r_4(z) - \overline{f} \right| &\le C(\epsilon)|z|^\alpha, && z \in \mathscr{C}^*, \end{aligned} \tag{32}$$

where $\alpha < 1$ can be chosen arbitrary close to 1. Now we are ready to formulate an auxiliary RH problem in the domain $\mathbb{D}_{\rho_1}$, which is called the parametrix problem. We are looking for a holomorphic in $\mathbb{D}_{\rho_1} \setminus \Gamma_{\rho_1}$ matrix function $M^{\mathrm{par}}(z)$ satisfying the jump condition

$$M_+^{\mathrm{par}}(z) = M_-^{\mathrm{par}}(z) v^{\mathrm{par}}(z), \quad z \in \Gamma_{\rho_1}, \quad \text{with} \tag{33}$$

$$v^{\mathrm{par}}(z) := \mathbb{I} + \begin{cases} f z^{-2i\nu} e^{2t\varphi(z)} \mathbb{J}, & z \in \mathscr{C}, \\ \overline{f} z^{2i\nu} e^{-2t\varphi(z)} \mathbb{J}^\dagger, & z \in \mathscr{C}^*, \\ -\frac{\overline{f}}{1-|f|^2} z^{2i\nu} e^{-2t\varphi(z)} \mathbb{J}^\dagger, & z \in \mathscr{C}_l, \\ -\frac{f}{1-|f|^2} z^{-2i\nu} e^{2t\varphi(z)} \mathbb{J}, & z \in \mathscr{C}_l^*, \end{cases} \tag{34}$$

and the boundary condition $M^{\mathrm{par}}(z) \sim \mathbb{I}$, as $z \in \partial\mathbb{D}_{\rho_1}$.

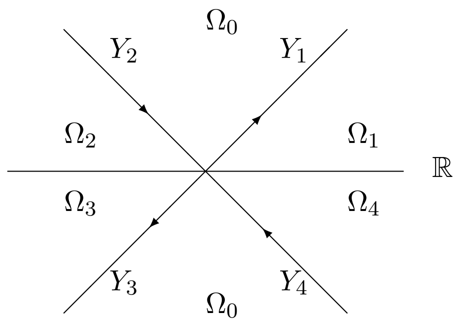

Fig. 4: The sets Ω_j and the rays Y_i, $j = 1, \ldots, 4$.

This problem was solved in [9, 13]. We recall briefly the main steps in the construction of its solution. Denote $\zeta = \sqrt{t}\, z$. We study the parametrix problem solution for large t. Consider first another auxiliary matrix RH problem in the domain $\mathbb{C}\backslash Y$, where $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ and $Y_i = \{s e^{i(2n+1)\pi/4},\ s \in \mathbb{R}_+\}$ are the contours as depicted in Figure 4. Let $M^Y(\zeta)$ solve the following problem:

$$M^Y(\zeta) \to \mathbb{I}, \qquad \zeta \to \infty, \tag{35}$$

$$M^Y_+(\zeta) = M^Y_-(\zeta) v^Y(\zeta), \qquad \zeta \in Y, \tag{36}$$

where the jump matrix $v^Y(\zeta)$ is defined by

$$v^Y(\zeta) := \mathbb{I} + \begin{cases} f\zeta^{-2i\nu} e^{\frac{i\zeta^2}{2}} \mathbb{J}, & \zeta \in Y_1, \\ -\frac{\overline{f}}{1-|f|^2} \zeta^{2i\nu} e^{\frac{-i\zeta^2}{2}} \mathbb{J}^\dagger, & \zeta \in Y_2, \\ -\frac{f}{1-|f|^2} \zeta^{-2i\nu} e^{\frac{i\zeta^2}{2}} \mathbb{J}, & \zeta \in Y_3, \\ \overline{f} \zeta^{2i\nu} e^{-\frac{i\zeta^2}{2}} \mathbb{J}^\dagger, & \zeta \in Y_4. \end{cases} \tag{37}$$

Following [13], define a sectionally analytic function $\tilde{M}^Y(\zeta)$ by

$$\tilde{M}^Y(\zeta) := \begin{pmatrix} \psi_{11}(\zeta) & \frac{(\frac{d}{d\zeta} - \frac{i\zeta}{2})\psi_{22}(\zeta)}{\overline{\beta}} \\ \frac{(\frac{d}{d\zeta} + \frac{i\zeta}{2})\psi_{11}(\zeta)}{\beta} & \psi_{22}(\zeta) \end{pmatrix}, \qquad \zeta \in \mathbb{C}\backslash\mathbb{R},$$

where $\beta = \beta(a)$ is given by

$$\beta := \sqrt{\nu(a)} e^{i(\frac{\pi}{4} - \arg f(a) + \arg \Gamma(i\nu(a)))}, \tag{38}$$

and the functions ψ_{11}, ψ_{22} are defined by

$$\psi_{11}(\zeta) = \begin{cases} e^{-\frac{3\pi\nu}{4}} D_{i\nu}(e^{-\frac{3i\pi}{4}}\zeta), & \operatorname{Im}\zeta > 0, \\ e^{\frac{\pi\nu}{4}} D_{i\nu}(e^{\frac{i\pi}{4}}\zeta), & \operatorname{Im}\zeta < 0, \end{cases}$$

$$\psi_{22}(\zeta) = \begin{cases} \mathrm{e}^{\frac{\pi\nu}{4}} D_{-\mathrm{i}\nu}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta > 0, \\ \mathrm{e}^{-\frac{3\pi\nu}{4}} D_{-\mathrm{i}\nu}(\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta < 0. \end{cases}$$

Here $D_s(z)$ denotes the parabolic cylinder function. Then (cf. [13]) the solution $M^Y(\zeta)$ of the matrix RH problem (35)–(37) is the following:

$$M^Y(\zeta) = \tilde{M}^Y(\zeta) D_j(\zeta), \quad \zeta \in \Omega_j, \quad j = 0, \dots, 4,$$

where $D_0(\zeta) = \zeta^{-\mathrm{i}\nu\sigma_3} \mathrm{e}^{\frac{\mathrm{i}\zeta^2}{4}\sigma_3}$ and

$$D_1(\zeta) = (\mathbb{I} - f\mathbb{J}) D_0(\zeta), \qquad D_2(\zeta) = (\mathbb{I} + \frac{\overline{f}}{1 - |f|^2}\mathbb{J}^\dagger) D_0(\zeta),$$

$$D_3(\zeta) = (\mathbb{I} + \frac{f}{1 - |f|^2}\mathbb{J}) D_0(\zeta), \qquad D_4(\zeta) = (\mathbb{I} - \overline{f}\mathbb{J}^\dagger) D_0(\zeta).$$

The matrix $M^Y(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash Y$ and satisfies the jump condition $M^Y_+(\zeta) = M^Y_-(\zeta) v^Y(\zeta)$, where $v^Y(\zeta)$ is defined by (37). Also, $M^Y(\zeta)$ satisfies the asymptotic formula

$$M^Y(\zeta) = \mathbb{I} + \frac{\mathrm{i}}{\zeta} \begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty, \tag{39}$$

where $\beta = \beta(a)$ is defined by (38). Put $D(t) := \mathrm{e}^{8\mathrm{i}a^3 t\sigma_3} t^{-\mathrm{i}\nu\sigma_3/2}$ and introduce the matrix $M^{\mathrm{par}}(z)$ by the formula

$$M^{\mathrm{par}}(z) := D(t) M^Y(\sqrt{t}z) D(t).$$

It is straightforward to check that $M^{\mathrm{par}}(z)$ satisfies (33)–(34). Due to (39), it is close as $t \to \infty$ to the identity matrix on $\partial\mathbb{D}_{\rho_1}$.

Put now $M_{-a}(k) = M^{\mathrm{par}}(\sqrt{48a}(k+a))$. This function is holomorphic in $\mathbb{D}_\rho(-a) \setminus (\mathscr{C} \cup \mathscr{C}^* \cup \mathscr{C}_l \cup \mathscr{C}_l^*)$ has the jump with the matrix $v^{\mathrm{par}}(\sqrt{48a}(k+a))$. It is easy to see that the matrix $M_a(k) := \sigma_1 M_{-a}(k) \sigma_1$ solves the corresponding parametrix problem in the domain $\mathbb{D}_\rho(a) \setminus (\mathscr{C} \cup \mathscr{C}^* \cup \mathscr{C}_r \cup \mathscr{C}_r^*)$. Moreover, due to (39),

$$\begin{aligned} M_{-a}(k) &= \mathbb{I} + \frac{\mathrm{i}}{\sqrt{48at}(k+a)} \begin{pmatrix} 0 & -\beta \mathrm{e}^{16\mathrm{i}a^3 t} t^{-\mathrm{i}\nu} \\ \overline{\beta}\mathrm{e}^{-16\mathrm{i}a^3 t} t^{\mathrm{i}\nu} & 0 \end{pmatrix} \\ &\quad + O\left(\frac{1}{t}\right), \qquad k \in \partial\mathbb{D}_\rho(-a), \\ M_a(k) &= \mathbb{I} - \frac{\mathrm{i}}{\sqrt{48at}(k-a)} \begin{pmatrix} 0 & \overline{\beta}\mathrm{e}^{-16\mathrm{i}a^3 t} t^{\mathrm{i}\nu} \\ -\beta \mathrm{e}^{16\mathrm{i}a^3 t} t^{-\mathrm{i}\nu} & 0 \end{pmatrix} \end{aligned} \tag{40}$$

$$+ O\left(\frac{1}{t}\right), \qquad k \in \partial\mathbb{D}_\rho(a).$$

The completion of the asymptotical analysis repeats now almost literally the same considerations as in [13], Theorem 2.1. To describe them briefly, let us denote

$$\tilde{\Gamma} := \mathscr{C}_l \cup \mathscr{C}_l^* \cup \mathscr{C} \cup \mathscr{C}^* \cup \mathscr{C}_r \cup \mathscr{C}_r^* \cup \partial\mathbb{D}_\rho(-a) \cup \partial\mathbb{D}_\rho(a).$$

For the vector $m^{(2)}(k)$ corresponding to the jump matrix (30), put

$$\hat{m}(k) = \begin{cases} m^{(2)}(k)(M_{\mp a}(k))^{-1}, & |k \pm a| < \rho, \\ m^{(2)}(k), & \text{otherwise.} \end{cases}$$

Then the vector function $\hat{m}(k)$ is holomorphic in $\mathbb{C} \setminus \tilde{\Gamma}$, it satisfies the standard symmetry and the normalization conditions, that is, $\hat{m}(k) \to (1 \quad 1)$ as $k \to \infty$ and $\hat{m}(-k) = \hat{m}(k)\sigma_1$. Moreover, it has a jump on $\tilde{\Gamma}$ with the jump matrix

$$\hat{v}(k) = \begin{cases} (M_{\mp a}(k))_- \, v^{(2)}(k)(M_{\mp a}(k))_+^{-1}, & k \in \Gamma_\rho(\mp a), \\ (M_{\mp a}(k))^{-1}, & |k \pm a| = \rho, \\ v^{(2)}(k), & \text{otherwise.} \end{cases}$$

Now from Theorem 2.1 of [13], estimates (32), (40), and a trivial equality

$$\frac{1}{2\pi\mathrm{i}} \int_{|k \pm a| = \rho} \frac{dk}{k \pm a} = 1$$

(the integration is counterclockwise), it follows that

$$\begin{aligned} &\lim_{k \to \mathrm{i}\infty} 2\mathrm{i}k \left(\hat{m}(k) - \begin{pmatrix} 1 & 1 \end{pmatrix}\right) \\ &\quad = -\frac{1}{\pi} \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\int_{|k+a|=\rho} (M_{-a}(k) - \mathbb{I})\, dk \right. \\ &\qquad\qquad\qquad \left. + \int_{|k-a|=\rho} (M_a(k) - \mathbb{I})\, dk \right) + O(t^{-\frac{1+\alpha}{2}}) \\ &\quad = \frac{2}{\sqrt{48at}} (1, \, -1) \left(\beta \mathrm{e}^{16\mathrm{i}a^3 t - \mathrm{i}\nu \log t} + \overline{\beta} \mathrm{e}^{-16\mathrm{i}a^3 t + \mathrm{i}\nu \log t} \right) + O(t^{-\frac{1+\alpha}{2}}) \\ &\quad = \frac{\sqrt{\nu(a)}}{\sqrt{3at}} \cos\left(16a^3 t - \nu(a) \log t - \mathrm{i} \log \frac{\beta(a)}{\sqrt{\nu(a)}} \right) + O(t^{-\frac{1+\alpha}{2}}), \end{aligned} \tag{41}$$

where the term $O(t^{-\frac{1+\alpha}{2}})$ can be differentiated with respect to x, and the derivative has the same order $O(t^{-\frac{1+\alpha}{2}})$ as $t \to \infty$ uniformly with respect to ξ in the domain $\xi < -\frac{c^2}{2} - \epsilon$ (cf. [13], Theorem 2.1). Next, by (5), we have $\frac{\partial a}{\partial x} = -\frac{1}{24at}$. Combining this with (24), (31), (38) and (41), we get

$$q(x,t) = c^2 + \sqrt{\frac{4\nu(a)a}{3t}} \sin\left(16ta^3 - \nu(a)\log t + \frac{\pi}{4} - \arg f(a) + \arg\Gamma(\mathrm{i}\nu)\right)$$

with

$$\arg f(a) = \nu(a)\log(192a^3) + \arg R(-a) + \frac{1}{\pi}\int_{\mathbb{R}\setminus[-a,a]} \log\left(\frac{1-|R(s)|^2}{1-|R(a)|^2}\right)\frac{ds}{s+a}.$$

The result of Theorem 1 is now immediate from $\arg R(-a) = -\arg R(a)$ and the oddness of the last integral with respect to a.

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