

The Existence of Heteroclinic Travelling Waves in the Discrete Sine-Gordon Equation with Nonlinear Interaction on a 2D-Lattice

S. Bak

The article deals with the discrete sine-Gordon equation that describes an infinite system of nonlinearly coupled nonlinear oscillators on a 2D-lattice with the external potential $V(r) = K(1 - \cos r)$. The main result concerns the existence of heteroclinic travelling waves solutions. Sufficient conditions for the existence of these solutions are obtained by using the critical points method and concentration-compactness principle.

Key words: discrete sine-Gordon equation, nonlinear oscillators, 2D-lattice, heteroclinic travelling waves, critical points, concentration-compactness principle.

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1. Introduction

In the paper, we study the discrete sine-Gordon equation that describes the dynamics of an infinite system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice. Let $q_{n,m}$ be a generalized coordinate of the (n, m) -th oscillator at the time t . It is assumed that each oscillator interacts nonlinearly with its four nearest neighbors. The equation of motion of the system considered is of the form

$$\begin{aligned} \ddot{q}_{n,m} = & V'(q_{n+1,m} - q_{n,m}) - V'(q_{n,m} - q_{n-1,m}) + V'(q_{n,m+1} - q_{n,m}) \\ & - V'(q_{n,m} - q_{n,m-1}) - K \sin(q_{n,m}), \quad (n, m) \in \mathbb{Z}^2, \end{aligned} \quad (1)$$

where $K > 0$. Equations (1) form an infinite system of ordinary differential equations.

System (1) can be considered as a 2D version of the Frenkel–Kontorova model (see, e.g., [11]). Notice that this system represents a wide class of systems called lattice dynamical systems extensively studied in last decades. In this area of research, a great attention is paid to an important specific class of solutions called travelling waves solutions. A comprehensive presentation of the results on travelling waves for 1D Fermi–Pasta–Ulam lattices is given in [19]. The existence

of periodic travelling waves in the Fermi–Pasta–Ulam system on a 2D-lattice is studied in [4]. On the other hand, some results on the chains of oscillators are also known in the literature. In particular, in [14] they are obtained by means of bifurcation theory, while in [1] and [2] the existence of periodic and solitary travelling waves is studied by means of the critical point theory. In papers [3,10,12,13], travelling waves for infinite systems of linearly coupled oscillators on a 2D-lattice are studied. Paper [18] is devoted to periodic and homoclinic travelling waves for the infinite one-dimensional chain of nonlinearly coupled nonlinear particles. In [6], a result on the existence of subsonic periodic travelling waves for the system of nonlinearly coupled nonlinear oscillators on a 2D-lattice is obtained, and in [7], supersonic periodic travelling waves for these systems are studied. Paper [15] contains a result on the existence of heteroclinic travelling waves for the discrete sine-Gordon equation with linear interaction. In [16], periodic, homoclinic and heteroclinic travelling waves for such systems with nonlinear interaction are studied. In paper [5], a result on the existence of periodic travelling waves for the discrete sine-Gordon equation with nonlinear interaction on a 2D-lattice is obtained. [8] is devoted to the existence of heteroclinic travelling waves for the discrete sine-Gordon equation with linear interaction on a 2D-lattice.

2. The problem statement

A travelling wave solution of equation (1) is a function of the form

$$q_{n,m}(t) = u(n \cos \varphi + m \sin \varphi - ct),$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$c^2 u''(s) = V'(u(s + \cos \varphi) - u(s)) - V'(u(s) - u(s - \cos \varphi)) \\ + V'(u(s + \sin \varphi) - u(s)) - V'(u(s) - u(s - \sin \varphi)) - K \sin(u(s)). \quad (2)$$

The constant $c \neq 0$ is called the speed of the wave. If $c > 0$, then the wave moves to the right, otherwise to the left.

An important role is played by the quantity c_1 defined by the equation

$$c_1^2 := 2 \sup_{|r| < 6\pi} \left| \frac{V(r)}{r^2} \right|.$$

We consider the case of heteroclinic travelling waves. The profile function of this wave satisfies the conditions:

$$\lim_{s \rightarrow -\infty} u(s) = -\pi \quad \text{and} \quad \lim_{s \rightarrow +\infty} u(s) = \pi. \quad (3)$$

In what follows, a solution of equation (2) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying equation (2) for all $s \in \mathbb{R}$.

3. Variational setting

To equation (2), we associate the functional

$$J(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 - V(u(s + \cos \varphi) - u(s)) - V(u(s + \sin \varphi) - u(s)) + K(1 + \cos(u(s))) \right] ds, \quad (4)$$

defined on the Hilbert space

$$E := \{u \in H_{\text{loc}}^1(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$$

with the scalar product

$$(u, v)_E = u(0)v(0) + \int_{-\infty}^{+\infty} u'(s)v'(s) ds.$$

It is not so difficult to verify that the critical points of the functional J are the solutions of equation (2).

Now we introduce the following notation:

$$\begin{aligned} \mathcal{M}_{-\pi, \pi} &= \{u \in E : u(-\infty) = -\pi, u(+\infty) = \pi\}, \\ Au(s) &:= u(s + \cos \varphi) - u(s), \\ Bu(s) &:= u(s + \sin \varphi) - u(s). \end{aligned}$$

According to Lemma 3.1 from [10],

$$\begin{aligned} \|Au(s)\|_{L^2(\mathbb{R})} &\leq |\cos \varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, & u \in E, \\ \|Bu(s)\|_{L^2(\mathbb{R})} &\leq |\sin \varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, & u \in E. \end{aligned}$$

Then the functional J can be expressed in the form

$$J(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 - V(Au(s)) - V(Bu(s)) + K(1 + \cos(u(s))) \right] ds. \quad (5)$$

Throughout the paper we will assume that the interaction potential $V(r)$ satisfies the following conditions:

- (i) $V(r) \in C^1(\mathbb{R})$, $V(0) = 0$ and $V(r) \geq 0$ for all $r \in \mathbb{R}$;
- (ii) $\lim_{r \rightarrow \pm\infty} V(r) = +\infty$;
- (iii) there exists finite $\lim_{r \rightarrow 0} \left| \frac{V(r)}{r^2} \right|$;
- (iv) the wave speed c satisfies $c^2 > c_1^2$.

The following lemma can be obtained by a straightforward calculation (see [15] for details).

Lemma 3.1. *Let $v_0 : \mathbb{R} \rightarrow [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(s) = -\pi$ for $s < -1$ and $v_0(s) = \pi$ for $s > 1$. Define the functional $\Psi : H^1(\mathbb{R}) \rightarrow \mathbb{R}$ by*

$$\Psi(v) := J(v_0 + v)$$

and suppose that assumptions (i)–(iv) are satisfied. Then the following holds:

- (i₁) $\Psi(v) < +\infty$ for all $v \in H^1(\mathbb{R})$ (equivalently, $J(u) < +\infty$ for all u of the form $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$);
- (ii₁) $J(u) = +\infty$ for all $u \in \mathcal{M}_{-\pi, \pi}$ which are not of the form $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$. In particular, a minimizer u of J on $\mathcal{M}_{-\pi, \pi}$ can be expressed as $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$;
- (iii₁) $\Psi \in C^1$ on $H^1(\mathbb{R})$;
- (iv₁) let $v \in H^1(\mathbb{R})$ be a critical point of Ψ and set $u := v_0 + v$. Then $u, v \in C^2(\mathbb{R})$, and u is a solution of (2) with boundary conditions (3).

Let F be a non-negative function in $C^\infty(\mathbb{R})$ such that

$$\begin{cases} F(r) = 0, & \text{if } |r| \leq \frac{5\pi}{2}, \\ F(r) \geq 4 \left| \int_0^{2r} |V'(x)| dx \right| \text{ and } F(r) \geq 2K, & \text{if } |r| \geq 3\pi, \\ \frac{1}{2} \leq 1 + \cos r + \frac{1}{2K} F(r), & \text{if } |r| \in \left(\frac{5\pi}{2}, 3\pi \right). \end{cases} \quad (6)$$

Now we define the modified functional $\tilde{J} : E \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\begin{aligned} \tilde{J}(u) := \int_{-\infty}^{+\infty} & \left[\frac{c^2}{2} (u'(s))^2 - V(Au(s)) - V(Bu(s)) \right. \\ & \left. + K(1 + \cos(u(s))) + F(u(s)) \right] ds. \end{aligned} \quad (7)$$

Remark 3.2. Obviously, $\tilde{J}(u) = J(u)$ for all $u \in E$ with norm

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{5}{2}\pi.$$

Now we denote the modified potential of interaction by

$$\tilde{V}(r) = \left| \int_0^r |V'(x)| dx \right|.$$

Then from (6) for all $|r| \geq 3\pi$, we have

$$V(2r) \leq \tilde{V}(2r) \leq \frac{1}{4}F(r). \quad (8)$$

Hence, by (ii), $F(r) \rightarrow +\infty$ for $r \rightarrow \pm\infty$.

The lemma below can be found in [16, Lemma 2.5].

Lemma 3.3. *Let $W \in C^1(\mathbb{R})$ be such that $W(\pm\pi) = 0$ and $W(\xi) > 0$ for $|\xi| < \pi$, and let*

$$I(u) := \int_{-\infty}^{+\infty} [(u'(s))^2 + W(u(s))] ds.$$

Then the minimum of I on $\mathcal{M}_{-\pi,\pi}$ is attained and

$$\min_{u \in \mathcal{M}_{-\pi,\pi}} I(u) = 2 \int_{-\pi}^{\pi} \sqrt{W(\xi)} d\xi =: \vartheta.$$

Moreover, with the same ϑ ,

$$\inf_{T>0} \inf_{u \in H^1(-T,T)} \left\{ \int_{-T}^T [(u'(s))^2 + W(u(s))] ds : u(-T) = -\pi, u(T) = \pi \right\} = \vartheta.$$

Lemma 3.4. *Assume conditions (i)–(iv) hold. Then for all $u \in E$,*

$$\tilde{J}(u) \geq \int_{-\infty}^{+\infty} \left[\frac{c^2 - c_1^2}{2} (u'(s))^2 + K(1 + \cos(u(s))) + \frac{1}{2} F(u(s)) \right] ds, \quad (9)$$

and the functional \tilde{J} is bounded from below on $\mathcal{M}_{-\pi,\pi}$. Moreover,

$$8\sqrt{(c^2 - c_1^2)K} < \inf_{u \in \mathcal{M}_{-\pi,\pi}} \tilde{J}(u) < 8c\sqrt{K}. \quad (10)$$

Proof. Since

$$\begin{aligned} |Au(s)| &\leq |u(s + \cos \varphi)| + |u(s)| \leq 2 \max\{|u(s + \cos \varphi)|, |u(s)|\}, \\ |Bu(s)| &\leq |u(s + \sin \varphi)| + |u(s)| \leq 2 \max\{|u(s + \sin \varphi)|, |u(s)|\}, \end{aligned}$$

then for every $k > 0$,

$$\begin{aligned} \{s \in \mathbb{R} : |Au(s)| > k\} &\subseteq \left\{ s \in \mathbb{R} : \max\{|u(s + \cos \varphi)|, |u(s)|\} > \frac{k}{2} \right\} \\ &\subseteq \left\{ s \in \mathbb{R} : |u(s + \cos \varphi)| > \frac{k}{2} \right\} \cup \left\{ s \in \mathbb{R} : |u(s)| > \frac{k}{2} \right\}, \\ \{s \in \mathbb{R} : |Bu(s)| > k\} &\subseteq \left\{ s \in \mathbb{R} : \max\{|u(s + \sin \varphi)|, |u(s)|\} > \frac{k}{2} \right\} \\ &\subseteq \left\{ s \in \mathbb{R} : |u(s + \sin \varphi)| > \frac{k}{2} \right\} \cup \left\{ s \in \mathbb{R} : |u(s)| > \frac{k}{2} \right\}. \end{aligned}$$

Making use of (8) and the monotonicity of the potential \tilde{V} on $(-\infty, 0)$ and on $(0, +\infty)$, we have

$$\begin{aligned} \int_{\{s \in \mathbb{R} : |Au(s)| > 6\pi\}} V(Au(s)) ds &\leq \int_{\{s \in \mathbb{R} : |Au(s)| > 6\pi\}} \tilde{V}(Au(s)) ds \\ &\leq \int_{\{s \in \mathbb{R} : |Au(s)| > 6\pi\}} \tilde{V}(2 \max\{|u(s + \cos \varphi)|, |u(s)|\}) ds \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\{s \in \mathbb{R}: \max\{|u(s + \cos \varphi)|, |u(s)|\} > 3\pi\}} \frac{1}{4} F(\max\{|u(s + \cos \varphi)|, |u(s)|\}) ds \\
 &\leq 2 \int_{\{s \in \mathbb{R}: |u(s)| > 3\pi\}} \frac{1}{4} F(u(s)) ds \leq \frac{1}{2} \int_{-\infty}^{+\infty} F(u(s)) ds.
 \end{aligned} \tag{11}$$

Similarly,

$$\int_{\{s \in \mathbb{R}: |Bu(s)| > 6\pi\}} V(Bu(s)) ds \leq \frac{1}{2} \int_{-\infty}^{+\infty} F(u(s)) ds. \tag{12}$$

By the definition of c_1 , we obtain

$$\begin{aligned}
 \int_{\{s \in \mathbb{R}: |Au(s)| \leq 6\pi\}} V(Au(s)) ds &\leq \int_{\{s \in \mathbb{R}: |Au(s)| \leq 6\pi\}} \frac{c_1^2}{2} (Au(s))^2 ds \\
 &\leq \int_{-\infty}^{+\infty} \frac{c_1^2}{2} (Au(s))^2 ds, \\
 \int_{\{s \in \mathbb{R}: |Bu(s)| \leq 6\pi\}} V(Bu(s)) ds &\leq \int_{\{s \in \mathbb{R}: |Bu(s)| \leq 6\pi\}} \frac{c_1^2}{2} (Bu(s))^2 ds \\
 &\leq \int_{-\infty}^{+\infty} \frac{c_1^2}{2} (Bu(s))^2 ds.
 \end{aligned}$$

Then it follows from (11) and (12) that

$$\begin{aligned}
 \tilde{J}(u) &\geq \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 - \frac{c_1^2}{2} (Au(s))^2 - \frac{c_1^2}{2} (Bu(s))^2 \right. \\
 &\quad \left. + K(1 + \cos(u(s))) + F(u(s)) \right] ds \\
 &\quad - \int_{\{s \in \mathbb{R}: |Au(s)| > 6\pi\}} V(Au(s)) ds - \int_{\{s \in \mathbb{R}: |Bu(s)| > 6\pi\}} V(Bu(s)) ds \\
 &\geq \int_{-\infty}^{+\infty} \left[\frac{c^2 - c_1^2}{2} (u'(s))^2 + K(1 + \cos(u(s))) + \frac{1}{2} F(u(s)) \right] ds
 \end{aligned}$$

for all $u \in E$, and (9) holds true.

Applying Lemma 3.3 to the functional

$$I_1(u) = \frac{c^2 - c_1^2}{2} \int_{-\infty}^{+\infty} [(u'(s))^2 + W_1(u(s))] ds,$$

where

$$W_1(x) := \frac{2K}{c^2 - c_1^2} [1 + \cos x + \frac{1}{2K} F(x)],$$

and making use of (9), we obtain

$$\begin{aligned}
 \inf_{u \in \mathcal{M}_{-\pi, \pi}} \tilde{J}(u) &\geq (c^2 - c_1^2) \left| \int_{-\pi}^{\pi} \sqrt{W_1(x)} dx \right| \\
 &= \sqrt{2(c^2 - c_1^2)K} \left| \int_{-\pi}^{\pi} \sqrt{1 + \cos x + 0} dx \right| = 8\sqrt{(c^2 - c_1^2)K}.
 \end{aligned}$$

Furthermore, since $V \geq 0$, we have

$$\tilde{J}(u) \leq \frac{c^2}{2} \int_{-\infty}^{+\infty} \left[(u'(s))^2 + \frac{2}{c^2} \left(K(1 + \cos(u(s))) + \frac{3}{2} F(u(s)) \right) \right] ds.$$

Now, we apply Lemma 3.3 to the functional

$$I_2(u) = \frac{c^2 - c_1^2}{2} \int_{-\infty}^{+\infty} [(u'(s))^2 + W_2(u(s))] ds,$$

where

$$W_2(x) := \frac{2K}{c^2} [1 + \cos x + \frac{3}{2K} F(x)].$$

As a consequence, we obtain

$$\inf_{u \in \mathcal{M}_{-\pi, \pi}} \tilde{J}(u) \leq c^2 \left| \int_{-\pi}^{\pi} \sqrt{W_2(x)} dx \right| < 8c\sqrt{K},$$

from which inequalities (10) follow. \square

The following lemma can be proved in the same way as Lemma 2.7 from [16].

Lemma 3.5. *Assume conditions (i)–(iv) hold. Let $\tilde{u} \in \mathcal{M}_{-\pi, \pi}$ be a minimizer of \tilde{J} on $\mathcal{M}_{-\pi, \pi}$, then*

$$\|\tilde{u}\|_{L^\infty(\mathbb{R})} \leq \frac{3}{2}\pi + \delta,$$

where

$$\delta := \frac{4c_1^2}{c^2 - c_1^2 + c\sqrt{c^2 - c_1^2}}. \quad (13)$$

In particular, if the speed c is large enough to ensure $\delta < \pi$, then $\|\tilde{u}\|_{L^\infty(\mathbb{R})} \leq \frac{5}{2}\pi$.

4. Main result

In order to prove the main result, we need the following version of the concentration-compactness principle obtained in [15, Lemma 4.1] (see [16, 17, 19] for other versions of this principle).

Given $T > 1$ and $\eta \in \mathbb{R}$, we define a truncated version of \tilde{J} by

$$\begin{aligned} \tilde{J}_T(u, \eta) := & \int_0^1 \int_{\eta-T+\tau}^{\eta+T-1+\tau} \frac{c^2}{2} (u'(s))^2 ds d\tau - \int_{\eta-T}^{\eta+T-1} V(Au(s)) ds \\ & - \int_{\eta-T}^{\eta+T-1} V(Bu(s)) ds + \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} \left[K(1 + \cos(u(s))) + \frac{3}{2} F(u(s)) \right] ds. \end{aligned}$$

Lemma 4.1 (Concentration-compactness). *Assume conditions (i)–(iv) hold. Let $(u_n) \subset \mathcal{M}_{-\pi, \pi}$ be a minimizing sequence for \tilde{J} on $\mathcal{M}_{-\pi, \pi}$, and let c be large enough to ensure $\delta < \pi$ for δ defined in (13). Then there exists a subsequence, still denoted by (u_n) , such that one of the following holds:*

(i₂) (concentration) there is a sequence $(\eta_n) \subset \mathbb{R}$ such that for all small enough $\varepsilon > 0$ there exists $T > 0$ such that

$$|\tilde{J}(u_n) - \tilde{J}_T(u_n, \eta_n)| < \varepsilon$$

for every $n \in \mathbb{N}$;

(ii₂) (vanishing) for all $T > 0$,

$$\limsup_{n \rightarrow \infty} \sup_{\eta \in \mathbb{R}} \tilde{J}_T(u_n, \eta) = 0;$$

(iii₂) (dichotomy) there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ there are $(f_n), (g_n) \subset E$ such that

$$\begin{aligned} |u_n - (f_n + g_n - \pi)| \leq \varepsilon, \quad |\tilde{J}(u_n) - (\tilde{J}(f_n) + \tilde{J}(g_n))| \leq \varepsilon, \\ \lim_{n \rightarrow \infty} \text{dist}(\text{supp}(f'_n), \text{supp}(g'_n)) = +\infty, \quad \lim_{n \rightarrow \infty} \tilde{J}(f_n) = \alpha, \quad \lim_{n \rightarrow \infty} \tilde{J}(g_n) = \beta, \end{aligned}$$

for some $0 < \alpha, \beta < \inf_{u \in \mathcal{M}_{-\pi, \pi}} \tilde{J}(u)$ (π is needed in the first inequality to ensure $J(f_n) < +\infty$ and $J(g_n) < +\infty$).

Lemma 4.2. Under the assumptions of Lemma 4.1, the functional \tilde{J} has a minimizer on $\mathcal{M}_{-\pi, \pi}$.

Proof. By Lemma 3.4, the functional \tilde{J} is bounded from below on $\mathcal{M}_{-\pi, \pi}$. Let $(u_n) \subset \mathcal{M}_{-\pi, \pi}$ be a minimizing sequence. Then, by Lemma 4.1, the subsequence exists, still denoted by (u_n) , which satisfies either of the following criteria: concentration, vanishing or dichotomy.

Vanishing is impossible (see the proof of Lemma 5.1 in [15]).

We will show that dichotomy is also impossible. Indeed, as $f_n, g_n \in E$ and $\tilde{J}(f_n), \tilde{J}(g_n) < +\infty$, the analogous statement of Lemma 3.1 (with J replaced by \tilde{J}) shows that $f_n(\pm\infty), g_n(\pm\infty) \in \{\pm\pi\}$. Since $f_n + g_n - \pi \in \mathcal{M}_{-\pi, \pi}$, then only $f_n(-\infty) = f_n(+\infty)$ or only $g_n(-\infty) = g_n(+\infty)$. In the first case, we set $\tilde{u}_n := g_n$ and in the second case, $\tilde{u}_n := f_n$. Then $(\tilde{u}_n) \subset \mathcal{M}_{-\pi, \pi}$ and, by (iii₂), possibly after passing to a subsequence, we have

$$\lim_{n \rightarrow \infty} \tilde{J}(\tilde{u}_n) < \inf_{u \in \mathcal{M}_{-\pi, \pi}} \tilde{J}(u) = \lim_{n \rightarrow \infty} \tilde{J}(u_n).$$

We obtained a contradiction to the assumption that $(u_n) \subset \mathcal{M}_{-\pi, \pi}$ is a minimizing sequence of \tilde{J} .

Thus (i₂) holds. Hence, given $\varepsilon > 0$, there exists a sequence $(\eta_n) \subset \mathbb{R}$ and $T_0 > 0$ such that

$$|\tilde{J}(u_n) - \tilde{J}_{T_0}(u_n, \eta_n)| < \varepsilon.$$

Let $w_n(s) = u_n(\eta_n + s)$. The sequence (w_n) is bounded in E . Indeed, by (9),

$$\|w'_n\|_{L^2(\mathbb{R})} = \|u'_n\|_{L^2(\mathbb{R})} \leq \frac{2}{c^2 - c_1^2} J(u_n),$$

and by Lemma 3.5,

$$|w_n(0)| \leq \frac{3}{2}\pi + \delta.$$

Hence, (w_n) contains a subsequence, still denoted by (w_n) , that converges weakly to some limit $u \in E$. The convergence is uniform on $[-T_0, T_0]$, and

$$\|u'\|_{L^2(-T_0, T_0)} \leq \liminf_{n \rightarrow \infty} \|w_n'\|_{L^2(-T_0, T_0)}.$$

Since the functions $V(u)$, $1 + \cos u$ and $F(u)$ belong to $C^1(\mathbb{R})$ and therefore are Lipschitz continuous for $|u| \leq \frac{3}{2}\pi + \delta$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\left| \left(\tilde{J}(u) - \frac{c^2}{2} \|u'\|_{L^2(\mathbb{R})} \right) - \left(\tilde{J}_{T_0}(w_n) - \frac{c^2}{2} \|u'\|_{L^2(-T_0, T_0)} \right) \right| \leq \varepsilon.$$

In fact, this inequality holds for all $T > T_0$ instead of T_0 . By Lemma 3.1, $u \in \mathcal{M}_{-\pi, \pi}$. Furthermore, as $T \mapsto \tilde{J}_T(w_n, 0)$ is non-decreasing for every $n \in \mathbb{N}$, we obtain that $\tilde{J}_T(w_n, 0) \leq \tilde{J}(w_n)$. Then,

$$\begin{aligned} \tilde{J}(u) &= \lim_{T \rightarrow \infty} \tilde{J}_T(u, 0) \leq \lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} \tilde{J}_T(w_n, 0) \\ &\leq \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \tilde{J}(w_n) = \lim_{n \rightarrow \infty} \tilde{J}(w_n) = \lim_{n \rightarrow \infty} \tilde{J}(u_n), \end{aligned}$$

and thus u is a minimizer of the functional \tilde{J} on $\mathcal{M}_{-\pi, \pi}$. \square

The following theorem is the main result of the paper.

Theorem 4.3. *Assume conditions (i)–(iv) hold. Suppose that c is large enough to ensure $\delta < \pi$ for δ defined by (13). Then equation (2) has a solution u that satisfies boundary conditions (3).*

Proof. By Lemma 3.1, the modified functional \tilde{J} has a minimizer $u_* \in \mathcal{M}_{-\pi, \pi}$. We have to show that u_* is a solution of equation (2) with boundary conditions (3). We define the functional $\tilde{\Psi}$ similarly to Ψ but in terms of \tilde{J} . Then the function $v_* = u_* - v_0$ minimizes $\tilde{\Psi}$ on $H^1(\mathbb{R})$. Since the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is continuous, we have that

$$\|v_0 + v\|_{L^\infty(\mathbb{R})} < \frac{5}{2}\pi$$

for all v in the neighborhood $\Delta \subset H^1(\mathbb{R})$ of v_* . Then, by Remark 3.2, for all $v \in \Delta$,

$$\Psi(v) = J(v_0 + v) = \tilde{J}(v_0 + v) = \tilde{\Psi}(v),$$

and v_* minimizes Ψ as well as $\tilde{\Psi}$ in Δ . In particular, v_* is a local minimizer of the functional Ψ on $H^1(\mathbb{R})$, i.e., v_* is a critical point of Ψ . Hence, by Lemma 3.1 (iv₁), $u_* = v_0 + v_*$ is the solution of equation (2) that satisfies boundary conditions (3). \square

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S. Bak,

Vinnitsia Mykhailo Kotsiubynskyi State Pedagogical University, 32 Ostrozkogo St., Vinnitsia, 21001, Ukraine,

E-mail: sergiy.bak@gmail.com

Існування гетероклінічних рухомих хвиль в дискретному рівнянні синус-Гордона на двовимірній ґратці

С. Бак

Статтю присвячено дискретному рівнянню синус-Гордона, яке описує нескінченну систему нелінійно зв'язаних нелінійних осциляторів на двовимірній ґратці із зовнішнім потенціалом $V(r) = K(1 - \cos r)$. Основний результат стосується існування розв'язків у вигляді гетероклінічних рухомих хвиль. За допомогою методу критичних точок і принципу концентрованої компактності отримано достатні умови існування таких розв'язків.

Ключові слова: дискретне рівняння синус-Гордона, нелінійні осцилятори, двовимірна ґратка, гетероклінічні рухомі хвилі, критичні точки, принцип концентрованої компактності.