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Nonlinear Dynamics of Solitons for the Vector Modified Korteweg–de Vries Equation

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The vector generalization of the modified Korteweg–de Vries equation is considered and the inverse scattering transform for solving this equation is developed. The solitons and the breather solutions are constructed and the processes of their interactions are studied. It is shown that along with one-component soliton solutions, there are three-component solutions which have essentially a three-component structure.

Key words: vector mKdV, inverse scattering transform, soliton, collision.

Mathematical Subject Classification 2010: 35Q51.

1. Introduction

The Korteweg–de Vries equation (KdV)

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \tag{1.1}$$

is a classical equation in the theory of nonlinear waves. It was integrated for the first time in [7] by the inverse scattering transform method. Later the same method was applied to integrate other physically interesting equations. Their soliton and multi-soliton solutions were found and interactions of these solutions were studied (see, e.g., [1, 2, 4–6, 10, 12, 16]).

Among these equations there is the modified Korteweg–de Vries (mKdV) equation

$$\partial_t u + \alpha u^2 \partial_x u + \partial_x^3 u = 0. \tag{1.2}$$

The mKdV equation, like the KdV equation, describes nonlinear waves in media with dispersion, and it is completely integrable. The dynamics of solutions for this equation depends essentially on the sign of nonlinearity α, $\pm\alpha > 0$. In particular, the soliton solutions exist for $\alpha > 0$ only.

Along with the scalar mKdV equation (1.2), the vector generalizations of this equation

$$\partial_t \mathbf{u} + 24|\mathbf{u}|^2 \partial_x \mathbf{u} + \partial_x^3 \mathbf{u} = 0,$$

$$\partial_t \mathbf{u} + 12\partial_x(|\mathbf{u}|\mathbf{u})|\mathbf{u}| + \partial_x^3 \mathbf{u} = 0, \qquad \mathbf{u} \in \mathbb{R}^n, \ n \geq 2,$$

are also explored (see, e.g., [2, 3, 13–15]). It should be mentioned that not all of these generalizations are completely integrable.

In this paper, we consider another vector generalization of the mKdV equation (1.2),

$$\partial_t \mathbf{u} + 6\mathbf{u} \times \partial_x^2 \mathbf{u} + 6\left(\mathbf{u}\partial_x|\mathbf{u}|^2 - |\mathbf{u}|^2\partial_x \mathbf{u}\right) + \partial_x^3 \mathbf{u} = 0, \quad \mathbf{u} \in \mathbb{R}^3, \tag{1.3}$$

where $\mathbf{u}(x,t)$ is a three-component vector function of the variables $x \in \mathbb{R}$, $t \in \mathbb{R}$, and the sign "$\times$" means the vector product in $\mathbb{R}^3$. It turns out that this equation is completely integrable. Notice that equation (1.3) is a polynomial vector equation, but it does not belong to the integrable polynomial equations studied in [13].

A more general equation of this type,

$$\partial_t \mathbf{u} + a\mathbf{u} \times \partial_x^2 \mathbf{u} + b\left(\mathbf{u}\partial_x|\mathbf{u}|^2 - |\mathbf{u}|^2\partial_x \mathbf{u}\right) + c\partial_x^3 \mathbf{u} = 0,$$

with constant coefficients a, b, c satisfying $a^2 = 6bc$, $b > 0$, can be reduced to the canonical form (1.3) by a simple change of variables. The first two summands of these equations are the same as in the Ginzburg–Landau equation. They appeared in the modelling of nonlinear magnetization waves in ferromagnetic media [9, 11].

The goals of this paper are to develop the inverse scattering transform for equation (1.3), to construct the explicit formulas for some soliton solutions and to study their interactions.

2. Reduction of equation (1.3) to the matrix equation

In [8], it was mentioned that equation (1.3) can be reduced to the matrix completely integrable equation. Namely, let

$$\mathbf{u}(x,t) = (u_1(x,t,), u_2(x,t,), u_3(x,t,)) \tag{2.1}$$

be a real-valued solution of equation (1.3). Put

$$\Phi(x,t) = -i(u_1(x,t)\sigma_1 + u_2(x,t)\sigma_2 + u_3(x,t)\sigma_3), \tag{2.2}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, and $i = \sqrt{-1}$. By a straightforward computation, one can check that matrix-function (2.2) satisfies the differential equation

$$\partial_t \Phi + 3\left[\Phi, \partial_x^2 \Phi\right] - 6\Phi(\partial_x \Phi)\Phi - \partial_x^3 \Phi = 0, \tag{2.3}$$

where $[\cdot,\cdot]$ is the commutator. On the other hand, if the complex valued matrix solution of equation (2.3) satisfies the conditions

$$\Phi^* = -\Phi, \quad \operatorname{tr}\Phi = 0, \tag{2.4}$$

then this solution can be represented as in (2.2) with real functions $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$. Moreover, the vector (2.1) with these functions as components satisfies equation (1.3). In (2.4), the star denotes the complex conjugation and transpose, and "tr" is the trace of the matrix.

Conditions (2.4) are invariant under the action of the semigroup generated by equation (2.3). The following lemma holds.

Lemma 2.1. *Let $\Phi(x,t)$ be the solution of the Cauchy problem for equation (2.3) with the initial condition $\Phi(x,0) = \Phi_0(x)$ which satisfies conditions (2.4). Then $\Phi(x,t)$ also satisfies conditions (2.4).*

The proof of this lemma is straightforward and we omit it.

In [8], the matrix equation (2.3) is studied in the class of real-valued solutions $\Phi(x,t)$. By use of the inverse scattering transform it is proved there that equation (2.3) is completely integrable. Using the same approach, one can show that these results are valid for complex valued $\Phi(x,t)$. In the next section, we will list without proof those of the results of [8] which are necessary for our further investigations.

3. The inverse scattering transform method

Matrix equation (2.3) admits the Lax representation

$$\partial_t \mathcal{L} = [\mathcal{L}, \mathcal{A}], \tag{3.1}$$

where $[\cdot,\cdot]$ is the commutator, L and A are the matrix differential operators

$$\begin{aligned} \mathcal{L} &= \mathbb{I}\partial_x^2 + \Phi(x,t)\partial_x, \\ \mathcal{A} &= 4\mathbb{I}\partial_x^3 + 12\Phi(x,t)\partial_x^2 + 6(\partial_x\Phi(x,t) + \Phi^2(x,t))\partial_x, \end{aligned}$$

and $\mathbb{I}$ is the identity 3×3 matrix. Consider the spectral equation

$$(\mathcal{L} + k^2\mathbb{I})U \equiv \partial_x^2 U + 2\Phi(x,t)\partial_x U + k^2 U = 0, \quad x \in \mathbb{R}, \tag{3.2}$$

where t is fixed. In what follows, we assume that $\Phi(x,t)$ tends to 0 sufficiently fast as $x \to \pm\infty$ such that

$$\int_{-\infty}^{\infty} \left(\left(1+x^2\right) |\Phi(x,t)| + (1+|x|) \left| \frac{\partial}{\partial x}\Phi(x,t) \right| \right) dx < \infty, \quad t > 0.$$

In this case, there exists a matrix solution of equation (3.2) which can be represented as

$$U_{\pm}(x,k,t) = e^{\pm ikx}\mathbb{I} \pm ik \int_x^{\pm\infty} A^{\pm}(x,y,t) e^{\pm iky} dy, \tag{3.3}$$

where the matrices $A^+(x,y,t)$ and $A^-(x,y,t)$ belong to the spaces $L_1(x,\infty)$ and $L_1(-\infty,x,t)$ with respect to y. The matrix $\Phi(x,t)$ can be computed as (recall that t is a parameter)

$$\Phi(x,t) = \pm \frac{dA^{\pm}(x,x,t)}{dx} \left(\mathbb{I} \mp A^{\pm}(x,x,t)\right)^{-1}. \tag{3.4}$$

For $\operatorname{Im} k = 0$, the matrix-functions $U_+(x,k,t)$ and $U_+(x,-k,t)$ form a fundamental system of solutions of (3.2). Then

$$U_-(x,k,t) = U_+(x,k,t)C_{11}(k,t) + U_+(x,-k,t)C_{12}(k,t), \tag{3.5}$$

where the matrices $C_{11}(k,t), C_{12}(k,t)$ are independent of x. Notice that the matrix $C_{11}(k,t)$ is continuous on $\mathbb{R}$, and $C_{11}(k,t) = O(|k|^{-1})$ as $|k| \to \infty$. The matrix $C_{12}(k,t)$ is continuous on $\mathbb{R}$ and it can be continued analytically into the upper half-plane $\mathbb{C}^+$ with $C_{12}(k,t) = C + O(|k|^{-1})$ as $|k| \to \infty$, $\operatorname{Im} k \geq 0$, $\det C_{12} \neq 0$.

We assume that $\det C_{12} \neq 0$ for real k. Consider the matrices

$$\begin{aligned} S_{12}(k,t) &= C_{11}(k,t)C_{12}^{-1}(k,t), && \text{as } \operatorname{Im} k = 0, \\ S_{11}(k,t) &= C_{12}^{-1}(k,t), && \text{as } \operatorname{Im} k \geq 0, \end{aligned}$$

which, by (3.3) and (3.5), are the reflection and the transmission matrices, respectively. The matrix $S_{11}(k,t)$ is meromorphic in $\mathbb{C}^+$, its poles do not depend on t and are located at the points k_ν, $\operatorname{Im} k_\nu > 0$, $\nu = 1,2,\dots,N$ such that $\lambda_\nu = k_\nu^2$ are eigenvalues of the operator $\mathcal{L}$.

The poles k_ν can be multiple and in their neighborhoods $G_\nu \subset \mathbb{C}^+$ the decomposition is valid,

$$S_{11}(k,t) = (k-k_\nu)^{-n_\nu} S^\nu_{n_\nu}(t) + \cdots + (k-k_\nu)^{-1} S^\nu_1(t) + S^\nu_0(k,t).$$

Here n_ν is the order of the pole k_ν, $S^\nu_l(t)$, $l = 1,2,\dots,n_\nu$ are constant matrices with respect to k, $S^\nu_0(k,t)$ is a holomorphic matrix-function in the domain G_ν. One can show that there exist the matrices $R^\nu_1(t),\dots,R^\nu_{n_\nu}(t)$, $R^\nu_{n_\nu}(t) \neq 0$, which satisfy the equalities (we omit the dependence on t in (3.6))

$$\begin{cases} U_-(x,k_\nu)S^\nu_{n_\nu} = U_+(x,k_\nu)R^\nu_{n_\nu}, \\ U_-(x,k_\nu)S^\nu_{n_\nu-1} + U_-^{(1)}(x,k_\nu)S^\nu_{n_\nu} = U_+(x,k_\nu)R^\nu_{n_\nu-1} + U_-^{(1)}(x,k_\nu)R^\nu_{n_\nu}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ U_-(x,k_\nu)S^\nu_1 + U_-^{(1)}(x,k_\nu)S^\nu_2 + \cdots + \dfrac{1}{(n_\nu-1)!}U_-^{(n_\nu-1)}(x,k_\nu)S^\nu_{n_\nu} = \\ \quad = U_+(x,k_\nu)R^\nu_1 + U^{(1)}(x,k_\nu)R^\nu_2 + \cdots + \dfrac{1}{(n_\nu-1)!}U_+^{(n_\nu-1)}(x,k_\nu)R^\nu_{n_\nu}, \end{cases} \tag{3.6}$$

where $U_\pm^{(s)}(x,k_\nu) := \left.\dfrac{\partial^s}{\partial k^s}U_\pm(x,k,t)\right|_{k=k_\nu}$. We call the matrices $R^s_{n_\nu}(t)$ the norming matrices by an analogy with the norming constants in the scalar case. The set

$$\mathcal{S}(t) = \{S_{12}(k,t), k_\nu, n_\nu, R^\nu_1(t), R^\nu_2(t), \dots, R^\nu_{n_\nu}(t),\ (\nu = 1,2,\dots,N)\} \tag{3.7}$$

is called the scattering data for equation (3.2). The aim of the direct scattering problem is to find these data for a given matrix $\Phi(x,t)$. The inverse scattering problem consists in the reconstruction of the matrix $\Phi(x,t)$ from the scattering data of equation (3.2).

The recovery procedure is the following. Introduce the matrix-function

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{12}(k,t)e^{ikx}dk + \sum_{\nu=1}^{N} M_\nu(x,t)e^{ik_\nu x}, \tag{3.8}$$

where

$$M_\nu(x,t) = -i\sum_{j=0}^{n_\nu-1}\frac{(ix)^j}{j!}R^\nu_{j+1}(t).$$

Consider the integral equation of the Fredholm type with respect to $A(x,z,t)$ as a function of z (x and t being the parameters):

$$A(x,z,t) - \int_x^\infty A(x,y,t)F(y+z,t)dy + \int_{x+z}^\infty F(y,t)dy = 0. \tag{3.9}$$

This equation has the unique matrix solution $A(x,z,t)$. The matrix $\Phi(x,t)$ is connected with this solution by the formula

$$\Phi(x,t) = \frac{dA(x,x,t)}{dx}\left(\mathbb{I} - A(x,x,t)\right)^{-1}. \tag{3.10}$$

Let now $\Phi(x,t)$ be the solution of equation (2.3). Then the scattering data evolve with respect to t as follows:

$$S_{12}(k,t) = S_{12}(k,0)e^{i8k^3t}, \quad k_\nu(t) = k_\nu(0), \quad n_\nu(t) = n_\nu(0), \tag{3.11}$$

and the norming matrices $R^\nu_l(t)$, $l = 1,2,\ldots,n_\nu$ satisfy the differential equations

$$\begin{cases} \dfrac{d}{dt}R^\nu_{n_\nu}(t) - 8ik_\nu^3R^\nu_{n_\nu}(t) = 0, \\ \dfrac{d}{dt}R^\nu_{n_\nu-1}(t) - 8ik_\nu^3R^\nu_{n_\nu-1}(t) = 24ik_\nu^2R^\nu_{n_\nu}(t), \\ \dfrac{d}{dt}R^\nu_{n_\nu-2}(t) - 8ik_\nu^3R^\nu_{n_\nu-2}(t) = 24ik_\nu^2R^\nu_{n_\nu-1}(t) + 24ik_\nu R^\nu_{n_\nu}(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d}{dt}R^\nu_{n_\nu-l}(t) - 8ik_\nu^3R^\nu_{n_\nu-l}(t) = 24ik_\nu^2R^\nu_{n_\nu-l+1}(t) + 24ik_\nu R^\nu_{n_\nu-l+2}(t) \\ \qquad\qquad\qquad + 8iR^\nu_{n_\nu-l+3}(t), \quad l = 3,4,\ldots,n_\nu-1. \end{cases} \tag{3.12}$$

This procedure allows us to solve the Cauchy problem for equation (2.3) with the initial matrix $\Phi_0(x)$. First, let us solve the direct scattering problem for equation (3.2) with matrix $\Phi(x,0) = \Phi_0(x)$ and find the scattering data $\mathcal{S}(0)$. Then, let us transform these data with respect to t according to (3.11), (3.12), get $\mathcal{S}(t)$, introduce the function (3.8), solve equation (3.9), and obtain the solution $\Phi(x,t)$ of the Cauchy problem by formula (3.10).

In view of Lemma 2.1 and the equivalence of conditions (2.2) and (2.4), the recovery procedure is also valid for the vector equation (1.3).

We do not intend to study this Cauchy problem in details in our paper. We are aimed to construct some exact solutions of equation (1.3) starting from the scattering data. To this end, we have to describe additional conditions on them such that the inverse scattering transform procedure would lead to a matrix (3.10) satisfying (2.4). In a very general form, additional conditions imposed on the scattering data can be formulated as follows:

I. *The poles k_ν of the transmission matrix $S_{11}(k,t)$ are located symmetrically with respect to the imaginary axis and the norming matrices (3.7) satisfy the conditions $R_p^\nu(t) = R_p^\mu(t)$, where $k_\nu = -\overline{k_\mu}$ and $p = 1, \dots, n_\nu$ with $n_\nu = n_\mu$.*

II. *The matrix function $F(x,t)$ in (3.8) can be represented as*

$$F(x,t) = f_0(x,t)\mathbb{I} + i[f_1(x,t)\sigma_1 + f_2(x,t)\sigma_2 + f_3(x,t)\sigma_3], \tag{3.13}$$

where the functions $f_l(x,t)$ are real and σ_l are the Pauli matrices.

III. *Integral equation (3.9) has the solution of the form*

$$A(x,y,t) = a_0(x,y,t)\mathbb{I} + i[a_1(x,y,t)\sigma_1 + a_2(x,y,t)\sigma_2 + a_3(x,y,t)\sigma_3],$$

with the real-valued functions $a_l(x,y,t)$, $l = 0, 1, \dots, 3$ satisfying the equality

$$\sum_{l=0}^{3} a_l^2(x,x,t) = 2a_0(x,x,t). \tag{3.14}$$

Then, according (2.2), (2.1) and (3.10), the $\mathbf{u}(x,t)$ of equation (1.3) are determined by the formulas:

$$\begin{aligned}
\mathbf{u}_1 &= a_2' a_3 - a_3' a_2 - a_1'(1 - a_0) - a_1 a_0', \\
\mathbf{u}_2 &= a_3' a_1 - a_1' a_3 - a_2'(1 - a_0) - a_2 a_0', \\
\mathbf{u}_3 &= a_1' a_2 - a_2' a_1 - a_3'(1 - a_0) - a_3 a_0',
\end{aligned}$$

where $a_l' := \dfrac{\partial a_l}{\partial x}(x,x,t)$, $l = 1, 2, 3$.

Further, put $S_{12}(k,t) \equiv 0$ and choose such k_ν, n_ν and R_l^ν that satisfy **I**, **II** with $f_0(x,t) = 0$. Then the kernel of the integral equation (3.9) is degenerated, and this equation is reduced to a linear algebraic system of equations which can be solved exactly.

In the next section, this approach is used for constructing exact solutions of the vector modified Korteweg–de Vries equation (1.3).

4. Analytical solutions of the vector mKdV equation

4.1. Single soliton solution. We begin with the simplest case. Assume that the transmission matrix $S_{11}(k,t)$ has a simple pole at the point $k_1 = i\mu$, $\mu > 0$, $n_1 = 1$. Choose the matrix $R_1^1(0)$ of the form

$$R_1^1(0) = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 = \begin{pmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma, \end{pmatrix}, \tag{4.1}$$

where α, β, γ are real constants. Then conditions **I** and **II** are satisfied. Solving equation (3.9) with the kernel $F(x,t) = -ie^{-\mu x} R_1^1(t)$, we get the following solution:

$$A(x,y,t) = i\frac{R_1^1(0)}{\mu} e^{-\mu(x+y-8\mu^2)t} \left(1 + i\frac{R_1^1(0)}{2\mu} e^{-\mu(2x-8\mu^2)t}\right)^{-1}.$$

Denote $\mathbf{v} = (\alpha, \beta, \gamma)$, $\|\mathbf{v}\| = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$. Taking into account (3.10), we represent the solution of the matrix equation (2.3) as

$$\Phi(x,t) = -i\frac{2\mu}{\operatorname{ch}\left(2\mu(x - 4\mu^2 t) - \ln\|\mathbf{v}\| + \ln 2\mu\right)} \frac{R_1^1(0)}{\|\mathbf{v}\|}. \tag{4.2}$$

Denote now $a = 2\mu$, $c = a^2$, $\varphi = \dfrac{1}{a}\ln\left(\dfrac{a}{\|\mathbf{v}\|}\right)$, $\mathbf{e} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

According to (2.2), (2.1), (4.1) and (4.2), the vector function

$$\mathbf{u}(x,t) = \frac{a}{\operatorname{ch}(a(x - ct) - \varphi)}\mathbf{e} \tag{4.3}$$

is a single-soliton solution of the vector mKdV equation (1.3). This soliton has an amplitude a, speed c, phase φ, and it is directed along the vector $\mathbf{e}$. This direction does not depend on spatial and time variables. Note that the scalar function in front of the vector $\mathbf{e}$ in (4.3) is the well-known single-soliton solution to the scalar mKdV equation (see [10]).

4.2. Solutions of the soliton-antisoliton type. Consider now the case of a double pole $k_1 = i\mu$, $\mu > 0$, $n_1 = 2$. Choose the matrices $R_1^1(0)$, $R_2^1(0)$ as

$$R_1^1(0) = \alpha_1\sigma_1 + \beta_1\sigma_2 + \gamma_1\sigma_3, \quad R_2^1(0) = i(\alpha_2\sigma_1 + \beta_2\sigma_2 + \gamma_2\sigma_3),$$

where $\alpha_l, \beta_l, \gamma_l$, $l = 1, 2$ are real constants. Denote

$$\mathbf{v}_l = (\alpha_l, \beta_l, \gamma_l), \quad l = 1, 2. \tag{4.4}$$

Since the kernel of the integral equation (3.9) has the form

$$F(x,t) = -i(R_1^1(t) + ixR_2^1(t))e^{-\mu x},$$

then conditions **I**, **II** are satisfied. In this case, the described above approach leads to a solution $\mathbf{u}(x,t)$ to (1.3) of the type soliton-antisoliton. This solution is sometimes called doublet.

Consider first the special case $R_1^1(0) = 0$. Put $a = 2\mu$, $c = a^2$ and $\phi = \ln \frac{c}{\|\mathbf{v}_2\|}$. After the change of variables

$$x \to x + \frac{1 + 3\varphi}{2a}, \quad t \to t + \frac{1 + \varphi}{3ac},$$

this doublet can be represented as

$$\mathbf{u}(x,t) = 2a \frac{a \operatorname{sh}(a(x-ct))(3ct - x) + \operatorname{ch}(a(x-ct))}{\operatorname{ch}^2(a(x-ct)) + a^2(3ct - x)^2} \mathbf{e}_2, \tag{4.5}$$

where $\mathbf{e}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$ (cf. (4.4)).

For sufficiently large time, the doublet splits into a sum of oppositely directed solitons with the amplitudes a (the soliton and antisoliton). The distance between them grows logarithmically with respect to time

$$\mathbf{u}(x,t) = \mathbf{e}_2 \begin{cases} \frac{a}{\operatorname{ch}(a(x-ct)-\ln(4a^3 t))} - \frac{a}{\operatorname{ch}(a[x-ct]+\ln(4a^3 t))}, & t \to \infty, \\ -\frac{a}{\operatorname{ch}(a(x-ct)-\ln(4a^3|t|))} + \frac{a}{\operatorname{ch}(a(x-ct)+\ln(4a^3|t|))}, & t \to -\infty. \end{cases}$$

When the time increases, the soliton catches up with the antisoliton and then overtakes it. At their closest approach and interaction, there occurs the resulting impulse

$$\mathbf{u}(x,0) = 2a \frac{\operatorname{ch}(ax) - ax \operatorname{sh}(ax)}{\operatorname{ch}^2(ax) + (ax)^2} \mathbf{e}_2.$$

The motion phases of the doublet are shown on Fig. 4.1.

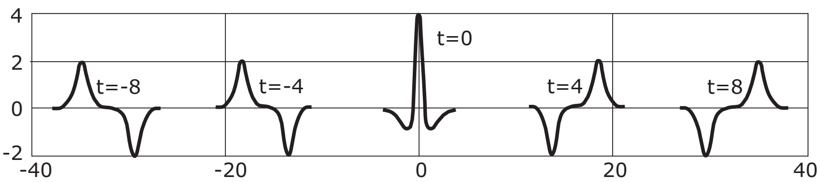

Fig. 4.1: The motion phases of the doublet ($R_1^1(0) = 0$, $\mu = 1$).

The constructed vector solution (4.5) is one-component, because it is directed, for any $x \in \mathbb{R}$ and $t \in \mathbb{R}$, along the constant vector $\mathbf{e}_2$. The scalar function in front of $\mathbf{e}_2$ in (4.5) satisfies the scalar mKdV equation (1.2), for which a doublet-type solution is well known [2, 16].

Consider now a more general case. Suppose that $R_1^1(0) \neq 0$, and this matrix is not proportional to $R_2^1(0)$. Then the doublet-type solution loses the property to be one-component, and the vector equation (1.3) can not be reduced to the scalar mKdV equation. Indeed, suppose that the unit vectors $\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ and $\mathbf{e}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$ are orthogonal. Denote

$$\delta = \frac{\|\mathbf{v}_1\|}{\|\mathbf{v}_2\|}, \quad \rho = \frac{a\delta}{2}. \tag{4.6}$$

Then in the shifted coordinate system

$$x \to x + \frac{1+3\varphi}{2a} - \frac{\delta}{2}, \qquad t \to t + \frac{1+\varphi}{3ac} - \frac{\delta}{4c},$$

this doublet has the form

$$\mathbf{u}(x,t) = u_1(x,t)\mathbf{e}_1 + u_2(x,t)\mathbf{e}_2 + u_3(x,t)(\mathbf{e}_1 \times \mathbf{e}_2), \tag{4.7}$$

where

$$\begin{aligned} u_1(x,t) &= 2a\rho\Bigg\{\frac{-\operatorname{ch}(a(x-ct))}{\operatorname{ch}^2(a(x-ct)) + a^2(3ct-x)^2 + \rho^2} \\ &\qquad + 4\frac{\operatorname{ch}(a(x-ct)) - a(3ct-x)\operatorname{sh}(a(x-ct))}{\left(\operatorname{ch}^2(a(x-ct)) + a^2(3ct-x)^2 + \rho^2\right)^2}\Bigg\}, \\ u_2(x,t) &= 2a\Bigg\{\frac{\operatorname{ch}(a(x-ct)) + a(3ct-x)\operatorname{sh}(a(x-ct))}{\operatorname{ch}^2(a(x-ct)) + a^2(3ct-x)^2 + \rho^2} \\ &\qquad - \frac{\rho^2\operatorname{ch}(a(x-ct))}{\left(\operatorname{ch}^2(a(x-ct)) + a^2(3ct-x)^2 + \rho^2\right)^2}\Bigg\}, \\ u_3(x,t) &= -4a\rho\frac{2a(3ct-x) + \operatorname{ch}(2a(x-ct)}{\left(\operatorname{ch}^2(a(x-ct)) + a^2(3ct-x)^2 + \rho^2\right)^2}. \end{aligned}$$

Asymptotic analysis of these formulas as $t \to \pm\infty$ shows that doublet (4.7) splits into two oppositely directed solitons with the amplitude a oriented along the vector $\mathbf{e}_2$. The solitons approach as $|t| \to 0$ and interact with each other. For $t = 0$, they produce the resulting three-component impulse depending on δ (cf. (4.6)), as is shown on Fig. 4.2.

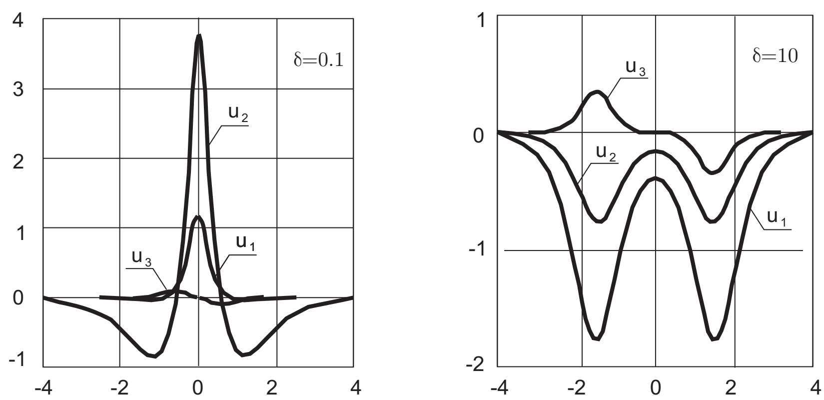

Fig. 4.2: The components of the resulting impulse of the doublet ($\mu = 1$) at $\delta = 0.1$ and $\delta = 10$.

Similarly, one can consider the case of a one pole $k_1 = i\mu$, $\mu > 0$, with a higher multiplicity $n_1 \geq 3$. Analytical formulas for this case are quite cumbersome. For this reason, for the case $n_1 = 3$ we perform numerical computations based on the inverse scattering transform given by (3.7)–(3.10). We choose the norming matrices as $R_l^1(0) = (\mathrm{i})^{l-1}(\alpha_l\sigma_1 + \beta_l\sigma_2 + \gamma_l\sigma_3)$, where $\alpha_l, \beta_l, \gamma_l \in \mathbb{R}$, $l = 1, 2, 3$. One can check that in this case conditions **I**, **II** are also satisfied.

Numerical computations show that in this case the solution of (1.3) is of the type soliton-antisoliton-soliton, and one can call it a triplet. For large times, the triplet splits into three mutually oppositely directed solitons of the same amplitude. The distance between them grows logarithmically with respect to time as $t \to \pm\infty$. When $|t| \to 0$, these solitons merge into the resulting impulse. In the general case, when the vectors $\mathbf{v}_l = (\alpha_l, \beta_l, \gamma_l)$, $l = 1, 2, 3$ are not collinear, this impulse has three components depending on x and t. If the vectors $\mathbf{v}_l$ are all collinear, then the triplet is one-component. This case is illustrated by Fig. 4.3.

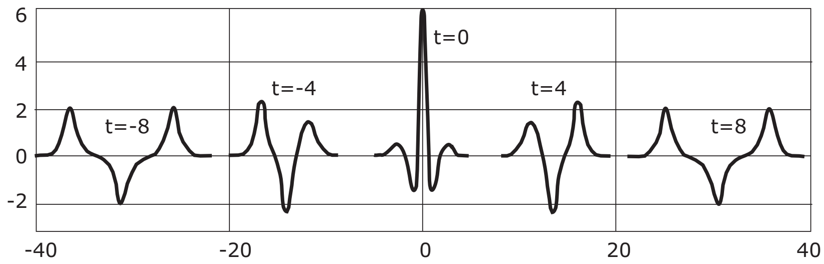

Fig. 4.3: The motion phases of the triplet ($R_1^{(1)}(0) = R_2^{(1)}(0) = 0$, $\mu = 1$).

4.3. The breather-type solutions. Consider now the case of two simple poles $k_{1,2} = \pm\lambda + i\mu$, $\lambda, \mu > 0$. Let $R_1^{(2)} = R_2^{(2)} = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Then

$$F(x, 0) = -2i(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)e^{-\mu x}cos(\lambda x),$$

and one can check that conditions **I**, **II** are satisfied. In this case, we obtain the multisoliton solution of (2.2) of the breather type. Put $\mathbf{v} = (\alpha, \beta, \gamma)$,

$$c_\mu = 4(\mu^2 - 3\lambda^2),\ c_\lambda = 4(3\mu^2 - \lambda^2),\ \psi = \arctan\frac{\lambda}{\mu},\ \varphi = \ln\left(\frac{4\mu^2}{\|\mathbf{v}\|}\frac{|\mu^2 - \lambda^2|}{\mu^2 + \lambda^2}\right).$$

Then in the shifted coordinate system

$$x \to x + \frac{1}{2}\left(\frac{\psi}{\lambda}c_\mu - \frac{\varphi}{\mu}c_\lambda\right)\frac{1}{c_\mu - c_\lambda}, \quad t \to t + \frac{1}{2}\left(\frac{\psi}{\lambda} - \frac{\varphi}{\mu}\right)\frac{1}{c_\mu - c_\lambda},$$

the breather-type solution can be represented as (see Fig. 4.4)

$$\mathbf{u}(x,t) = \frac{\lambda\cos(2\lambda(x - c_\lambda t))\,\mathrm{ch}(2\mu(x - c_\mu t))}{\frac{\lambda}{4\mu}\,\mathrm{ch}^2(2\mu(x - c_\mu t)) + \frac{\mu}{4\lambda}\sin^2(2\lambda(x - c_\lambda t))}$$
$$+ \frac{+\mu\sin(2\lambda(x - c_\lambda t))\,\mathrm{sh}(2\mu(x - c_\mu t))}{\frac{\lambda}{4\mu}\,\mathrm{ch}^2(2\mu(x - c_\mu t)) + \frac{\mu}{4\lambda}\sin^2(2\lambda(x - c_\lambda t))}\mathbf{e},$$

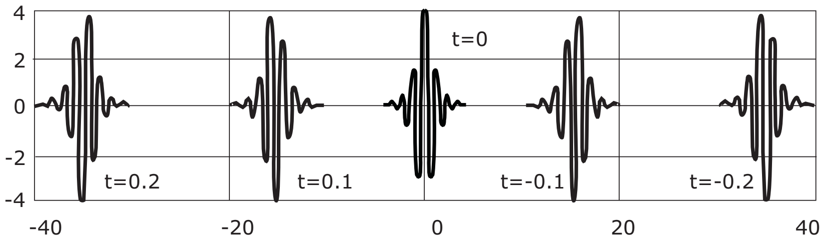

Fig. 4.4: The motion phases of the breather for $\mu = 1,\ \lambda = 4$.

where $\mathbf{e} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

When $\lambda \to 0$, the solution degenerates into a doublet, and when $\mu \ll \lambda$, one gets the soliton with an internal motion

$$\mathbf{u}(x,t) = 4\mu \frac{\sin(2\lambda(x - c_\lambda t))}{\mathrm{ch}(2\mu(x - c_\mu t))}\mathbf{e}, \quad c_\mu = 3c_\lambda < 0. \tag{4.8}$$

This solution is one-component since $\mathbf{e}$ is a constant vector. Therefore solution (4.8) coincides with a breather-type solution for the scalar mKdV equation.

However, if the poles $k_{1,2} = \pm\lambda + i\mu$ are multiple, the solution loses its property to be one-component, and the vector equation (1.3) can not be reduced to a scalar equation. Analytical formulas for the solution in this case are quite cumbersome and thus we compute the solution of (1.3) numerically. Indeed, assume that the pole $k_1 = \lambda + i\mu$ has multiplicity 2, and the respective norming matrices have a form $R_l^1(0) = \alpha_l\sigma_1 + \beta_l\sigma_2 + \gamma_l\sigma_3$, where $\alpha_l, \beta_l, \gamma_l \in \mathbb{R}$, $l = 1, 2$. According to condition **I**, the second pole $k_2 = -\lambda + i\mu$ is also double with the same norming matrices. Condition **II** is also fulfilled due to the structure of the matrices. The simulation shows that for $t \to \pm\infty$ the solution splits into two one-component breathers, and the distance between them increases logarithmically with time. When they approach each other, the resulting impulse has a three-component character if the vectors $\mathbf{v}_l = (\alpha_l, \beta_l, \gamma_l)$, $l = 1, 2$ are not collinear. If the vectors $\mathbf{v}_l$ are collinear, then the vector solution is one-component and directed for all x, t along the same vector. The graph of the scalar function in front of this vector is shown on Fig. 4.5.

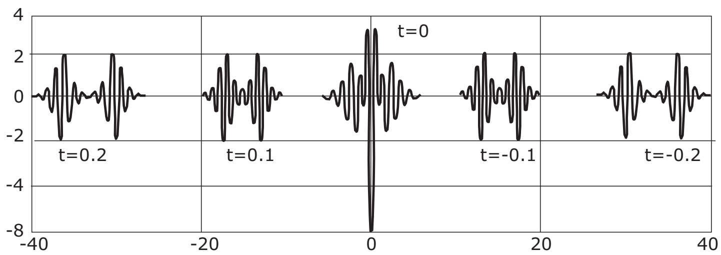

Fig. 4.5: The motion phases of the linked breathers with $\mu = 1,\ \lambda = 4$.

4.4. Interacting solitons. Consider now the case of two simple purely imaginary poles of the transmission matrix $S_{11}(k, 0)$: $k_j = i\mu_j,\ \mu_1 > \mu_2 > 0$. As usual, we choose the norming matrices as $R_1^l(0) = \alpha_l\sigma_1 + \beta_l\sigma_2 + \gamma_l\sigma_3,\ l = 1, 2,$ where $\alpha_l, \beta_l, \gamma_l \in \mathbb{R}$. It is clear that conditions **I**, **II** are fulfilled. Assume first that the vectors $\mathbf{v}_l = (\alpha_l, \beta_l, \gamma_l)$ are collinear, that is,

$$\mathbf{e} := \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \varepsilon\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \quad \varepsilon = \pm 1. \tag{4.9}$$

Put

$$a_j = 2\mu_j,\ c_j = a_j^2,\ \varphi_j = \ln\left(\frac{a_j}{\|\mathbf{v}_j\|}\right),\ y_j := a_j(x - c_j t), \quad j = 1, 2. \tag{4.10}$$

Then the change of variables

$$x \to x - \frac{1}{c_1 - c_2}\left(\frac{\varphi_2}{a_2}c_1 - \frac{\varphi_1}{a_1}c_2\right), \quad t \to t + \frac{1}{c_1 - c_2}\left(\frac{\varphi_2}{a_2} - \frac{\varphi_1}{a_1}\right), \tag{4.11}$$

leads to the following solution of (1.3):

$$\mathbf{u}(x,t) = \frac{2(a_1^2 - a_2^2)(a_1 \operatorname{ch}(y_2) + \varepsilon a_2 \operatorname{ch}(y_1))}{4\varepsilon a_1 a_2 + (a_1 + a_2)^2 \operatorname{ch}(y_2 - y_1) + (a_1 - a_2)^2 \operatorname{ch}(y_2 + y_1)}\mathbf{e}. \tag{4.12}$$

One can see that for large $|t|$ this solution splits into two noninteracting solitons

$$\mathbf{u}(x,t) = \mathbf{e}\begin{cases} \frac{a_1}{\operatorname{ch}(y_1 - \psi)} + \frac{\varepsilon a_2}{\operatorname{ch}(y_2 + \psi)}, & t \to -\infty, \\ \frac{a_1}{\operatorname{ch}(y_1 + \psi)} + \frac{\varepsilon a_2}{\operatorname{ch}(y_2 - \psi)}, & t \to \infty, \end{cases} \tag{4.13}$$

where $\psi = \ln\frac{a_1 + a_2}{a_1 - a_2}, \quad a_1 > a_2$.

When these solitons approach each other, they begin to interact strongly with the resulting impulse

$$\mathbf{u}(x,0) = \frac{2(a_1^2 - a_2^2)(a_1 \operatorname{ch}(a_2 x) + \varepsilon a_2 \operatorname{ch}(a_1 x))\mathbf{e}}{4\varepsilon a_1 a_2 + (a_1 + a_2)^2 \operatorname{ch}(a_2 x - a_1 x) + (a_1 - a_2)^2 \operatorname{ch}(a_2 x + a_1 x)}, \tag{4.14}$$

whose graph is shown on Fig. 4.6.

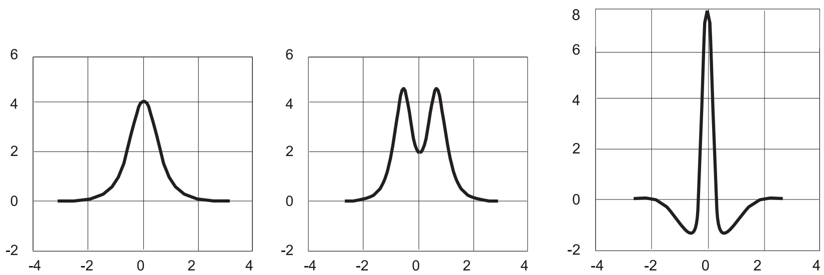

Fig. 4.6: The resulting impulse of the collinear solitons: the merge-split type ($\mu_1 = 3, \mu_2 = 1$), the bounce-exchange type ($\mu_1 = 3, \mu_2 = 2$) for unipolar solitons, and the absorb-emit type ($\mu_1 = 3, \mu_2 = 1$) for heteropolar ones.

The solution of the scalar mKdV equation (1.2) corresponds to the solution (4.12) as it is one-component and directed along the vector $\mathbf{e}$ independently of

x, t. This scalar solution was constructed in [2]. Notice that in [2] an interesting interpretation of the resulting impulse graph is given. The graph is treated as a characteristics of the interaction type for solitons. Namely, the resulting impulse with two humps corresponds to the bounce-exchange interaction of unipolar solitons. This type of interaction occurs when the amplitudes do not differ from each other significantly: $\frac{3-\sqrt{5}}{2}a_1 < a_2 < a_1$. In this case, the solitons do not stick together, but the faster soliton gives its power to the slower soliton, then the faster soliton decelerates, and the other one accelerates. The impulse with one hump corresponds to the merge-split type of the unipolar solitons interaction with a larger difference of amplitudes $a_2 < \frac{3-\sqrt{5}}{2}a_1$. Here the faster soliton absorbs the slower solitons during interaction and then restores. The impulse with three extrema corresponds to the interaction of the heteropolar solitons when the so-called absorbance-emit mode appears (see [2]).

Consider the case when the vectors $\mathbf{v}_l = (\alpha_l, \beta_l, \gamma_l)$, $l = 1, 2$, are not collinear, moreover, they are orthogonal. Then the solution of the vector mKdV equation (1.3) is essentially a three-component solution. Indeed, denote $\mathbf{e}_l = \frac{\mathbf{v}_l}{\|\mathbf{v}_l\|}$. Then in the shifted coordinate system (4.11) the solution of (1.3) is given by the formula

$$\mathbf{u}(x,t) = u_1(x,t)\mathbf{e}_1 + u_2(x,t)\mathbf{e}_2 + u_3(x,t)(\mathbf{e}_1 \times \mathbf{e}_2),$$

where

$$\begin{aligned} u_1(x,t) &= (a_1^2 - a_2^2)a_1 \\ &\quad\times \frac{(a_1-a_2)^2\operatorname{ch}(2y_2+y_1) + (a_1+a_2)^2\operatorname{ch}(2y_2-y_1) + 2(5a_2^2-3a_1^2)\operatorname{ch}(y_1)}{((a_1-a_2)^2\operatorname{ch}(y_2+y_1) + (a_1+a_2)^2\operatorname{ch}(y_2-y_1))^2}, \\ u_2(x,t) &= (a_1^2 - a_2^2)a_2 \\ &\quad\times \frac{(a_1-a_2)^2\operatorname{ch}(y_2+2y_1) + (a_1+a_2)^2\operatorname{ch}(y_2-2y_1) + 2(5a_1^2-3a_2^2)\operatorname{ch}(y_2)}{((a_1-a_2)^2\operatorname{ch}(y_2+y_1) + (a_1+a_2)^2\operatorname{ch}(y_2-y_1))^2}, \\ u_3(x,t) &= 4(a_1^2 - a_2^2)^2\frac{(a_1-a_2)\operatorname{sh}(y_1+y_2) + (a_1+a_2)\operatorname{sh}(y_1-y_2)}{((a_1-a_2)^2\operatorname{ch}(y_2+y_1) + (a_1+a_2)^2\operatorname{ch}(y_2-y_1))^2}. \end{aligned}$$

As $|t| \to \infty$, this solution splits into two noninteracting solitons oriented in the orthogonal directions $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{e}_1\frac{a_1}{\operatorname{ch}(y_1-\psi)} + \mathbf{e}_2\frac{a_2}{\operatorname{ch}(y_2+\psi)}, & t \to -\infty, \\ \mathbf{e}_1\frac{a_1}{\operatorname{ch}(y_1+\psi)} + \mathbf{e}_2\frac{a_2}{\operatorname{ch}(y_2-\psi)}, & t \to \infty, \end{cases} \tag{4.15}$$

where y_j and ψ are defined by (4.10) and (4.13), respectively.

When $|t| \to 0$, these solitons interact and form a three-component resulting impulse as it is shown on Fig. 4.7.

If the vectors $\mathbf{v}_j$ are not orthogonal, the analytical formulas are quite complicated. Therefore the analysis of the resulting three-component impulse was given numerically. We found that the module of its graph can have one, two or three maximums, depending on the ratio of the amplitudes of the solitons and

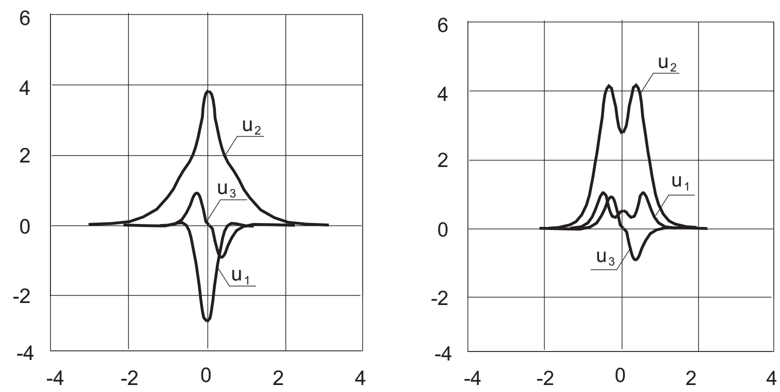

Fig. 4.7: The components of the resulting impulse ($\mu_1 = 3, \mu_2 = 1$ and $\mu_1 = 3, \mu_2 = 2$).

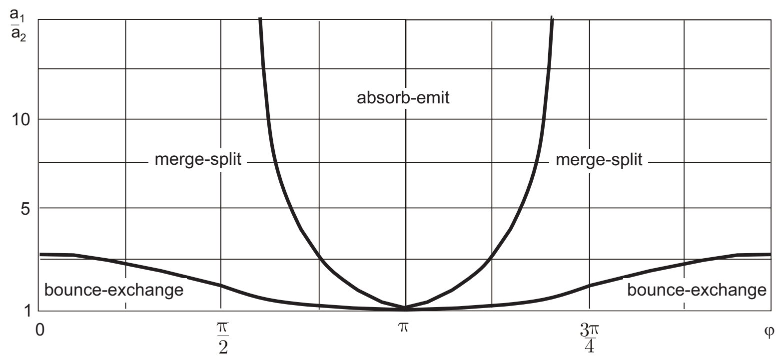

Fig. 4.8: Types of two-soliton interactions depending on the amplitudes and directions (φ is the angle between the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$).

their directions. Following [2], it is naturally to call this cases as merge-split, bounce-exchange and absorbance-emit interactions (Fig. 4.8).

The examples considered above show that one can construct different types of solutions for the vector mKdV equation by choosing appropriate singularities of the transmission matrix $S_{11}(k,0)$. If we choose the transmission matrix with simple purely imaginary poles, then we get a multi-soliton solution. If the poles are multiple, then we get the duplet, triplet etc. solutions. These solutions are connected solitons with the growing with respect to time distance between them. The choice of the transmission matrix with the simple complex pole symmetric with respect to the imaginary axis leads to the solutions of the breather type, i.e., to the solitons with the internal motion. If the poles are multiple, then we get a solution of the type of the connected breathers. The distance between them grows logarithmically with respect to time.

Moreover if the norming matrices $R_p^{\nu}(0)$ are generated by the non-collinear vectors, then all these solutions are essentially three-component.

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Нелінійна динаміка солітонів для модифікованого векторного рівняння Кортевега–де Фріза

V. Fenchenko and E. Khruslov

Розглянуто векторне узагальнення модифікованого рівняння Кортевега–де Фріза та розроблено обернене перетворення розсіювання для розв'язання цього рівняння. Побудовано солітони та брізерні розв'язки рівняння і досліджено процеси їхньої взаємодії. Показано, що поряд з однокомпонентними солітонними розв'язками існують розв'язки, що мають істотно трикомпонентну структуру.

Ключові слова: векторне mKdV, обернене перетворення розсіювання, солітон, зіткнення.