Szegö-Type Theorems for One-Dimensional Schrödinger Operator with Random Potential (Smooth Case)

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Dedicated to Professor V.A. Marchenko
on the occasion of his 95th birthday

The paper is a continuation of work [15] in which the general setting for analogs of the Szegö theorem for ergodic operators was given and several interesting cases were considered. Here we extend the results of [15] to a wider class of test functions and symbols which determine the Szegö-type asymptotic formula for the one-dimensional Schrödinger operator with ergodic random potential. We show that in this case the subleading term of the formula is given by a Central Limit Theorem in the spectral context, hence the term is asymptotically proportional to $L^{1/2}$, where L is the length of the interval to which the Schrödinger operator is initially restricted. This has to be compared with the classical Szegö formula, where the subleading term is bounded in L, $L \to \infty$. We prove an analog of standard Central Limit Theorem (the convergence of the probability of the corresponding event to the Gaussian Law) as well as an analog of the almost sure Central Limit Theorem (the convergence with probability 1 of the logarithmic means of the indicator of the corresponding event to the Gaussian Law). We illustrate our general results by establishing the asymptotic formula for the entanglement entropy of free disordered fermions for non-zero temperature.

Key words: random operators, asymptotic trace formulas, limit theorems.

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1. Introduction

The Szegö theorem (also known as the strong Szegö theorem) is an interesting asymptotic formula for the restrictions of functions of the Toeplitz operators as the size of the domain of restriction tends to infinity. The theorem has a number of applications and extensions pertinent to analysis, mathematical physics, operator theory, probability theory and statistics and (recently) quantum information theory, see [5, 6, 8, 26, 27]. In this paper we consider an extension of the theorem viewed as an asymptotic trace formula for a certain class of selfadjoint operators. We will start with an outline of the continuous version of the Szegö theorem presenting it in the form which explains our motivation.

Let $k : \mathbb{R} \to \mathbb{R}$ be an even and sufficiently smooth function from $L^1(\mathbb{R})$,

$$\Lambda = [-M, M], \quad |\Lambda| = 2M, \tag{1.1}$$

K and $K_\Lambda := K|_\Lambda$ be a selfadjoint convolution operator in $L^2(\mathbb{R})$ and its restriction to $L^2(\Lambda)$ given by

$$(Ku)(x) = \int_{-\infty}^{\infty} k(x-y)u(y)dy, \qquad x \in \mathbb{R}, \tag{1.2}$$

$$(K_\Lambda u)(x) = \int_{-M}^{M} k(x-y)u(y)dy, \qquad x \in \Lambda.$$

Set $A = \mathbf{1}_{L^2(\mathbb{R})} + K$ and $A_\Lambda = \mathbf{1}_{L^2(\Lambda)} + K_\Lambda$ and consider $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(A_\Lambda)$ is of trace class in $L^2(\Lambda)$. Then we have according to Szegö and subsequent works

$$\mathrm{Tr}_\Lambda \, \varphi(A_\Lambda) = |\Lambda| \int_{-\infty}^{\infty} \varphi(a(t))dt + \mathcal{T} + o(1), \quad |\Lambda| \to \infty, \tag{1.3}$$

where Tr_Λ is the trace in $L^2(\Lambda)$, $a(t) = 1 + \widehat{k}(t)$, $t \in \mathbb{R}$, $\widehat{k}$ is the Fourier transform of k and the subleading term $\mathcal{T}$ is a Λ-independent functional of φ and a. We will call φ and a the test function and the symbol respectively.

Let $P = i\frac{d}{dx}$ be the selfadjoint operator in $L^2(\mathbb{R})$. Then the r.h.s. of is $\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(P))$, i.e., is determined by the triple (φ, a, P), and since a is even and smooth enough, we have $a(x) = b(x^2)$, hence the triple (φ, b, P^2). It was proposed in [15] to consider instead of P^2 the Schrödinger operator $H = P^2 + V$, where the potential $V : \mathbb{R} \to \mathbb{R}$ is an ergodic process. It seems that the replacement is of interest in itself since the ergodicity of the potential guarantees the sufficient regular large Λ behavior of $\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(H))$, hence a well defined asymptotic formulas. Besides, the quantity $\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(H))$ for certain φ, a and V arises in quantum information theory and quantum statistical mechanics, see [10], Remark 2.4 and references therein.

Similar setting is also possible in the discrete case. In fact, it is this case of which was initially studied by Szegö for Toeplitz operators, while the continuous case outlined above was considered later by Akhiezer, Kac and Widom, see, e.g., [6] for a review. We will also consider in this paper the discrete case.

In [15] simple but rather non-trivial discrete cases were studied. There $a(x) = x$ and $\varphi(x)$ is $(x - x_0)^{-1}$ or $\log(x - x_0)$, where x_0 is outside the spectrum of the discrete Schrödinger operator with ergodic potential (random and almost periodic). In particular, it was shown that if the potential in the discrete Schrödinger equation is a collection of independent identically distributed (i.i.d.) random variables, then the leading term on the right of the analog of (1.3) is again of the order $|\Lambda|$ and is not random, but the subleading term is of the order $|\Lambda|^{1/2}$ and is a Gaussian random variable. In fact, a certain Central Limit Theorem for an appropriately normalized quantity $\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(H))$ was established. In this paper we extend this result for those φ and a which, roughly speaking, have the Lipshitz derivative (see condition (2.17) below). Note that similar conditions were used Szegö in his pioneering works, although the conditions were seriously weakened in subsequent works, see [6, 8, 26, 27].

2. Problem and results

Let H be the one-dimensional Schrödinger operator in $l^2(\mathbb{Z})$

$$H = H_0 + V, \tag{2.1}$$

where

$$(H_0 u)_j = -u_{j+1} - u_{j-1}, \quad j \in \mathbb{Z}, \tag{2.2}$$

and

$$(V u)_j = V_j u_j, \quad |V_j| \leq \overline{V} < \infty, \quad j \in \mathbb{Z}, \tag{2.3}$$

is a potential which we assume to be a sequence of independent and identically distributed (i.i.d.) random variables bounded for the sake of technical simplicity.

The spectrum $\sigma(H)$ is a non-random closed set and

$$\sigma(H) \subset K := [-2 - \overline{V}, 2 + \overline{V}], \tag{2.4}$$

see [19].

an Let also $a : \sigma(H) \to \mathbb{R}$ (symbol) and $\varphi : a(\sigma(H)) \to \mathbb{R}$ (test function) be bounded functions. Introduce the integer valued interval (cf. (1.1))

$$\Lambda = [-M, M] \subset \mathbb{Z}, \quad |\Lambda| = 2M + 1, \tag{2.5}$$

and the operator $\chi_\Lambda : l^2(\mathbb{Z}) \to l^2(\Lambda)$ of restriction, i.e., if $x = \{x_j\}_{j \in \mathbb{Z}} \in l^2(\mathbb{Z})$, then $\chi_\Lambda x = x_\Lambda := \{x_j\}_{j \in \Lambda} \in l^2(\Lambda)$. For any operator $A = \{A_{jk}\}_{j,k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ we denote (cf. (1.2))

$$A_\Lambda := \chi_\Lambda A \chi_\Lambda = \{A_{jk}\}_{j,k \in \Lambda} \tag{2.6}$$

its restriction to $l^2(\Lambda)$. Note that the spectra of A and A_Λ are related as follows

$$\sigma(A_\Lambda) \subset \sigma(H). \tag{2.7}$$

Our goal is to study the asymptotic behavior of

$$\mathrm{Tr}_\Lambda \varphi(a_\Lambda(H)) := \sum_{j \in \Lambda} (\varphi(a_\Lambda(H)))_{jj}, \quad |\Lambda| \to \infty, \tag{2.8}$$

where

$$\mathrm{Tr}_\Lambda \cdots = \mathrm{Tr}\, \chi_\Lambda \cdots \chi_\Lambda. \tag{2.9}$$

As was mentioned above, this problem dates back to works of Szegö [12] and has been extensively studied afterwards for the Toeplitz and convolution operators, see, e.g., [6, 8, 26] and references therein. Recall that any sequence

$$\{A_j\}_{j \in \mathbb{Z}}, \quad \overline{A_j} = A_{-j}, \quad \sum_{j \in \mathbb{Z},} |A_j| < \infty \tag{2.10}$$

determines a selfadjoint (discrete convolution) operator in $l^2(\mathbb{Z})$, cf. (1.2)

$$A = \{A_{j-k}\}_{j,k \in \mathbb{Z}}, \quad (A u)_j = \sum_{k \in \mathbb{Z}} A_{j-k} u_k. \tag{2.11}$$

Let

$$a(p) = \sum_{j \in \mathbb{Z}} A_j e^{2\pi i p j}, \quad p \in \mathbb{T} = [0,1),$$

be the Fourier transform of $\{A_j\}_{j \in \mathbb{Z}}$. Then, according to Szegö (see, e.g., [12]), if φ and a are sufficiently regular, then we have the two-term asymptotic formula (cf 1.3)

$$\operatorname{Tr}_\Lambda \varphi(A_\Lambda) = |\Lambda| \int_{\mathbb{T}} \varphi(a(t)) dt + \mathcal{T} + o(1), \quad |\Lambda| \to \infty, \tag{2.12}$$

where the subleading term $\mathcal{T}$ is again a Λ-independent functional of φ and a. Note that the traditional setting for the Szegö theorem uses the Toeplitz operators defined by the semi-infinite matrix $\{A_{j-k}\}_{j,k \in \mathbb{Z}_+}$ and acting in $l^2(\mathbb{Z}_+)$. The restrictions of Toeplitz operators are the upper left blocks $\{A_{j-k}\}_{j,k=0}^{L}$ of $\{A_{j-k}\}_{j,k=0}^{\infty}$. On the other hand, we will use in this paper the convolution operators (2.11) defined by the double infinite matrix $\{A_{j-k}\}_{j,k \in \mathbb{Z}}$, acting in $l^2(\mathbb{Z})$ and having their central $L \times L$, $L = 2M + 1$ blocks as restrictions. The latter setting seems more appropriate for the goal of this paper dealing with ergodic operators where the setting seems more natural. The same setting is widely used in multidimensional analogs of Szegö theorem [6].

Note now that the convolution operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, $d \geq 1$ admit a generalization, known as ergodic (or metrically transitive) operators, see [19]. We recall their definition in the (discrete) case of $l^2(\mathbb{Z})$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T is an ergodic automorphism of the space. A measurable map $A = \{A_{jk}\}_{j,k \in \mathbb{Z}}$ from Ω to bounded operators in $l^2(\mathbb{Z})$ is called ergodic operator if we have with probability 1 for every $t \in \mathbb{Z}$

$$A_{j+t,k+t}(\omega) = A_{jk}(T^t \omega), \quad j,k \in \mathbb{Z}. \tag{2.13}$$

Choosing $\Omega = \{0\}$, we obtain from (2.13) that A is a convolution operator (2.11). Thus, ergodic operators comprise a generalization of convolution operators, while the latter can be viewed as non-random ergodic operators.

It is easy to see that the discrete Schrödinger operator with ergodic potential (2.1)–(2.3) is an ergodic operator. Moreover, if $\sigma(H)$ is the spectrum of H, then $\sigma(H)$ is non-random, for any bounded and measurable $f : \sigma(H) \to \mathbb{R}$ the operator $f(H)$ is also ergodic and if $\{f_{jk}\}_{j,k \in \mathbb{Z}}$ is its matrix, then $\{f_{jj}\}_{j \in \mathbb{Z}}$ is an ergodic sequence [19]. Besides, there exists a non-negative and non-random measure N_H on $\sigma(H)$, $N(\mathbb{R}) = 1$ such that

$$\mathbf{E}\{f_{jj}(H)\} = \mathbf{E}\{f_{00}(H)\} = \int_{\sigma(H)} f(\lambda) N_H(d\lambda). \tag{2.14}$$

The measure N_H is an important spectral characteristic of selfadjoint ergodic operators known as the Integrated Density of States [19]. In particular, we have for any bounded $f : \sigma(H) \to \mathbb{R}$ with probability 1

$$\lim_{|\Lambda| \to \infty \Lambda} |\Lambda|^{-1} \operatorname{Tr}_\Lambda f(H_\Lambda) = \int_{\sigma(H)} f(\lambda) N_H(d\lambda). \tag{2.15}$$

This plays the role of the Law of Large Numbers for $\operatorname{Tr} f(H_\Lambda)$.

Accordingly, it is shown in [15] (see also formula (2.18) below) that the leading term in an analog of (2.12) for an ergodic Schrödinger operator is always

$$|\Lambda| \int_{\sigma(H)} \varphi(a(\lambda)) N_H(d\lambda). \tag{2.16}$$

On the other hand, the order of magnitude and the form of the subleading term depend on the "amount of randomness" of an ergodic potential and on the smoothness of φ and, especially, a, see, e.g., [6, 8, 11, 15, 22, 27] for recent problems and results.

In this paper we consider the discrete Schrödinger operator with random i.i.d. potential, known also as the Anderson model. Thus, our quantity of interest (2.8) as well as the terms of its asymptotic form are random variables in general (except the leading term (2.16), which is not random). Correspondingly, we will prove below two types of asymptotic trace formulas, both having the subleading terms of the order $|\Lambda|^{1/2}$ (cf. (2.12)). The formulas of the first type are valid in the sense of distributions, i.e., are analogs of the classical Central Limit Theorem (see Theorems 2.1 and 2.3), while the formulas of the second type are valid with probability 1, i.e., are analogs of the so called almost sure Central Limit Theorem (see Theorem 2.5).

Theorem 2.1. *Let H be the ergodic Schrödinger operator* (2.1)–(2.3) *with a bounded i.i.d. potential and let $\sigma(H)$ be its spectrum. Consider bounded functions $a : \sigma(H) \to \mathbb{R}$ and $\varphi : a(\sigma(H)) \to \mathbb{R}$ and assume that a, φ and $\gamma := \varphi \circ a : \sigma(H) \to \mathbb{R}$ admit extensions $\widetilde{a}$, $\widetilde{\varphi}$ and $\widetilde{\gamma}$ on the whole axis such that their Fourier transforms $\widehat{a}$, $\widehat{\varphi}$ and $\widehat{\gamma}$ satisfy the conditions*

$$\int_{-\infty}^{\infty} \left(1 + |t|^\theta\right) |\widehat{f}(t)| \, dt < \infty, \quad \theta > 1, \qquad f = a, \varphi, \gamma. \tag{2.17}$$

Denote

$$\Sigma_\Lambda = |\Lambda|^{-1/2} \left(\operatorname{Tr}_\Lambda \varphi(a_\Lambda(H)) - |\Lambda| \int_{\sigma(H)} \gamma(\lambda) N_H(d\lambda) \right) \tag{2.18}$$

and

$$\sigma_\Lambda^2 = \mathbf{E}\left\{\Sigma_\Lambda^2\right\}. \tag{2.19}$$

Then:

(i) *there exists the limit*

$$\lim_{\Lambda \to \infty} \sigma_\Lambda^2 = \sigma^2, \tag{2.20}$$

where

$$\sigma^2 = \sum_{l \in \mathbb{Z}} C_l \tag{2.21}$$

with

$$C_j = \mathbf{E}\{\overset{\circ}{\gamma}_{00}(H)\overset{\circ}{\gamma}_{jj}(H)\}, \quad \overset{\circ}{\gamma}_{jj}(H) = \gamma_{jj}(H) - \mathbf{E}\{\gamma_{jj}(H)\}, \tag{2.22}$$

and also

$$\sigma^2 = \mathbf{E}\left\{\left(\mathbf{E}\left\{A_0|\mathcal{F}_0^\infty\right\} - \mathbf{E}\left\{A_0|\mathcal{F}_1^\infty\right\}\right)^2\right\}$$
$$= \mathbf{E}\left\{\mathbf{Var}\left\{\mathbf{E}\{A_0|\mathcal{F}_0^\infty\}\,|\mathcal{F}_1^\infty\right\}\right\}, \tag{2.23}$$

where

$$A_0 = V_0 \int_0^1 \gamma'_{00}(H|_{V_0 \to uV_0})du \tag{2.24}$$

and $\mathcal{F}_a^b$, $-\infty \le a \le b \le \infty$ *is the σ-algebra generated by* $\{V_j\}_{j=a}^b$;

(ii) *if γ is non constant monotone function on the spectrum of H, then*

$$\sigma^2 > 0 \tag{2.25}$$

and we have

$$\mathbf{P}\left\{\sigma^{-1}\Sigma_{[-M,M]} \in \Delta\right\} = \Phi(\Delta) + o(1), \quad M \to \infty, \tag{2.26}$$

where $\Delta \subset \mathbb{R}$ is an interval and Φ is the standard Gaussian law (of zero mean and unit variance).

Remark 2.2. The theorem is an extension of Theorem 2.1 of [15], where the cases $a(\lambda) = \lambda$ and $\varphi(\lambda) = (\lambda - x_0)^{-1}$ or $\varphi(\lambda) = \log(\lambda - x_0)$, $x_0 \notin \sigma(H)$ were considered. In these cases a, φ and $\gamma = \varphi \circ a$ are real analytic on $\sigma(H)$ (see (2.4)), hence admit real analytic and fast decaying at infinity extensions to the whole line. Besides, γ is monotone on $\sigma(H)$, hence Theorem 2.1 applies.

It is worth also mentioning that conditions (2.17) are not optimal in general. Consider, for instance, the case where $\varphi(\lambda) = \chi_{(-\infty,E]}(\lambda)$, $E \in \sigma(H)$, $a(\lambda) = \lambda$ with $\chi_{(-\infty,E]}$ being the indicator of $(-\infty, E] \subset \mathbb{R}$. Here $\gamma := \varphi \circ a = \chi_{(-\infty,E]}$ and

$$\mathrm{Tr}_\Lambda\, \varphi(a_\Lambda(H)) = \mathrm{Tr}_\Lambda\, \chi_{(-\infty,E]}(H) := \mathcal{N}_\Lambda(E) \tag{2.27}$$

is the number of eigenvalues of H_Λ not exceeding E. It is known that if the potential in H is ergodic, then with probability 1

$$\lim_{|\Lambda|} |\Lambda|^{-1}\mathcal{N}_\Lambda(E) = N(E),$$

where $N(E)$ is defined in (2.14). This plays the role of the Law of Large Numbers for $\mathcal{N}_\Lambda(E)$ [19]. The Central Limit Theorem for $\mathcal{N}_\Lambda(E)$ is also known [25]. Its proof is based on a careful analysis of a Markov chain arising in the frameworks of the so called phase formalism, an efficient tool of spectral analysis of the one dimensional Schrödinger operator [19]. It can be shown that the theorem can also be proved following the scheme of proof of Theorem 2.1, despite that γ is discontinuous in this case. However, one has to use more sophisticated facts on the Schrödinger operator with i.i.d. random potential, in particular the bound

$$\sup_{\varepsilon>0} \mathbf{E}\left\{\left|(H - E - i\varepsilon)_{jk}^{-1}\right|^s\right\} \le Ce^{-c|j-k|} \tag{2.28}$$

valid for some $s \in (0,1)$, $C < \infty$ and $c > 0$ [2] if the probability law of potential possesses certain regularity, e.g., a bounded density. The bound is one of the basic results of the spectral theory of the random Schrödinger operator, implying the pure point character of the spectrum of H and a number of its other important properties. It is worth noting that the monotonicity of γ on the spectrum remains true in this case. Thus, the monotonicity of γ seems a pertinent sufficient condition for the positivity of the limiting variance.

Here, however, is a version of Theorem 2.1, applicable to the case where γ is a certain convex function on $\sigma(H)$.

Theorem 2.3. *Consider the functions $r_\alpha : [0,1] \to [0,1]$ and $n_F : \mathbb{R} \to [0,1]$ given by*

$$r_\alpha(\lambda) = (1-\alpha)^{-1} \log_2\left(\lambda^\alpha + (1-\lambda)^\alpha\right), \quad \lambda \in [0,1], \ \alpha > 0, \tag{2.29}$$

and

$$n_F(\lambda) = \left(e^{\beta(\lambda - E_F)} + 1\right)^{-1}, \quad \lambda \in \mathbb{R}, \ \beta > 0, \ E_F \in \sigma(H). \tag{2.30}$$

Assume that the random i.i.d. potential in (2.1)–(2.3) has zero mean $\mathbf{E}\{V_0\} = 0$ and that the support of its probability law contains zero. Then the conclusions of Theorem 2.1 remain valid for $\varphi = r_\alpha$ and $a = n_F$, i.e., the random variable Σ_Λ of (2.18) converges in distribution to the Gaussian random variable of zero mean and a certain variance $\sigma^2 > 0$.

Remark 2.4. The quantity $\mathrm{Tr}_\Lambda \, r_\alpha((n_F(H))_\Lambda)$ is known in quantum statistical mechanics and quantum information theory as the Rényi entanglement entropy of free fermions in the thermal state of the inverse temperature $\beta^{-1} > 0$ and the Fermi energy E_F and having H as the one body Hamiltonian, see, e.g., [1, 3, 10]. An important particular case where $\alpha = 1$, hence $h_1(\lambda) = -\lambda \log_2 \lambda - (1-\lambda) \log_2(1-\lambda)$, $\lambda \in [0,1]$, is known as the von Neumann entanglement entropy. One is interested in the large-$|\Lambda|$ asympotic form of the entanglement entropy. In the translation invariant case, i.e., for the case of constant potential in (2.1)–(2.3) one can use the Szegö theorem (see (2.12) and (2.38)) to find a two-term asymptotic formula for the entanglement entropy. In this case the term proportional to $|\Lambda|$ in (2.12) and (2.38), i.e., to the one dimensional analog of the volume of the spatial domain occupied by the system, is known as the volume law, while the second term in (2.12), which is independent of $|\Lambda|$, i.e., proportional to the one dimensional analog $\{-M, M\}$ of the surface area of the domain, is known as the area law [10]. In view of the above theorem we conclude that in the disorder case (random potential in H) the leading term of the entanglement entropy is non-random and is again the volume law while the subleading term is random, proportional to $|\Lambda|^{1/2}$ and describes random fluctuations of the volume law. The $O(1)$ in $|\Lambda|$ term can also be found for some φ and a [15]. It is random and is now the "subsubleading" term of the asymptotic formula. Of particular interest is the zero-temperature case $\beta = \infty$, where $n_F = \chi_{-\infty,E}$ and this term is leading. We refer the reader to recent works [1, 11, 18, 21, 22, 24, 27] for related results and references.

The above results can be viewed as stochastic analogs of the Szegö theorem (see more on the analogy in [15] and below). It is essentially a Central Limit Theorem in its traditional form, i.e., an assertion on the convergence of distribution of an appropriately normalized sums of random variables to the Gaussian random variable. In recent decades there has been a considerable interest to the almost sure versions of classical (distributional) limit theorems. The prototype of such theorems dates back to P. Levy and P. Erdős and is as follows, see, e.g., [4,9] for reviews.

Let $\{X_l\}_{l=1}^{\infty}$ be a sequence of i.i.d. random variables of zero mean and unit variance. Denote $S_m = \sum_{l=1}^{m}$, $Z_m = m^{-1/2}S_m$. Then we have with probability 1

$$\frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \mathbf{1}_\Delta(Z_m) = \Phi(\Delta) + o(1), \quad M \to \infty, \tag{2.31}$$

In other words, the random ("empirical") distribution of Z_m converges with probability 1 to the (non-random) Gaussian distribution.

On the other hand, the classical Central Limit Theorem implies

$$\frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \mathbf{E}\{\mathbf{1}_\Delta(Z_m)\} = \Phi(\Delta) + o(1), \quad M \to \infty, \tag{2.32}$$

i.e., just the convergence of expectations of the random distributions on the l.h.s. of (2.31). Thus, replacing the expectation by the logarithmic average, a sequence of random variables satisfying the CLT can be observed along all its typical realizations.

The situation with the almost sure CLT (2.31) for independent random variables is rather well understood, see, e.g., [4, 9] and references therein, while the case of dependent random variable is more involved and diverse, see, e.g., [7, 13, 16, 23]. As in the case of classical CLT (2.32), the existing results concern mostly the weakly dependent stationary sequences, e.g., strongly mixing sequences. This and the approximation techniques developed [14, Section 18.3], allow us to prove an almost sure version of Theorem 2.1.

Theorem 2.5. *We have with probability 1 under the conditions of Theorem 2.1*

$$\frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \mathbf{1}_\Delta \left(\sigma^{-1} \Sigma_{[-m,m]} \right) = \Phi(\Delta) + o(1), \quad M \to \infty, \tag{2.33}$$

where $\Sigma_{[-m,m]}$ is given by (2.18) with $\Lambda = [-m,m]$, $\Delta \subset \mathbb{R}$ is an interval and Φ is the standard Gaussian law.

Remark 2.6. Given a sequence $\{\xi_m\}_{m \geq 1}$ of random variables and a random variable ξ, write

$$\xi_M \overset{\mathcal{D}}{=} M^{1/2}\xi + o\left(M^{1/2}\right), \quad M \to \infty, \tag{2.34}$$

if we have

$$\mathbf{P}\{\xi_M/M^{1/2} \in \Delta\} = G(\Delta) + o(1), \quad M \to \infty, \qquad (2.35)$$

where G is the probability law of ξ, and write

$$\xi_M \overset{\mathcal{L}}{=} M^{1/2}\xi + o\left(M^{1/2}\right), \quad M \to \infty, \qquad (2.36)$$

if we have with probability 1 (assuming that all $\{\xi_m, \ m \geq 1\}$) are defined on the same probability space)

$$\frac{1}{\log M}\sum_{m=1}^{M} I_\Delta\left(\xi_m/m^{1/2}\right) = G(\Delta) + o(1), \quad M \to \infty. \qquad (2.37)$$

Then, we can formulate Theorems 2.1 and 2.5 in the form similar to that of the Szegö theorem (cf. (2.12)), namely as

$$\mathrm{Tr}_\Lambda\, \varphi(a_\Lambda(H)) \overset{\mathcal{D}}{=} |\Lambda| \int_{\sigma(H)} \gamma(\lambda) N_H(d\lambda)$$
$$+ |\Lambda|^{1/2}\sigma^{-1}\xi + o\left(|\Lambda|^{1/2}\right), \quad |\Lambda| = (2M+1) \to \infty, \qquad (2.38)$$

for Theorem 2.1 and with probability 1 as

$$\mathrm{Tr}_\Lambda\, \varphi(a_\Lambda(H)) \overset{\mathcal{L}}{=} |\Lambda| \int_{\sigma(H)} \gamma(\lambda) N_H(d\lambda)$$
$$+ |\Lambda|^{1/2}\Phi\left(\sigma^{-1}\Delta\right) + o\left(|\Lambda|^{1/2}\right), \ |\Lambda| = (2M+1) \to \infty, \qquad (2.39)$$

for Theorem 2.5, i.e., as two-term "Szegö-like" asymptotic formulas valid in the sense of the $\mathcal{D}$- and the $\mathcal{L}$-convergence, the latter valid with probability 1. An apparent difference between the Szegö formula (2.12) and its stochastic counterparts (2.38) and (2.39) is that the subleading term of the Szegö theorem is independent of $|\Lambda|$ while the subleading term of its stochastic counterparts grows as $|\Lambda|^{1/2}$ although with stochastic oscillations (see below).

We will comment now on the errors bounds in the above asymptotic formulas. We will mostly use known results on the rates of convergence for the both CLT (2.35) and (2.37) with ξ_m being the sum of i.i.d. random variable (see (2.32) and (2.31)), despite that in our (spectral) context the terms of the sum in (2.8) are always dependent even if the "input" potential is a collection of i.i.d. random variables. It seems plausible that the error bounds for the i.i.d. case provide best possible but not too overestimated versions of the error bounds for the case of sufficiently weakly dependent terms. Known results on the sums of weakly dependent random variables support this approach, see, e.g., [4, 7, 9, 13, 16, 23].

Recall first that for the classical Szegö (non-random) case (2.12), i.e., for the Toeplitz and convolution operators, the subleading term is Λ-independent and the error is just $o(1)$ in general. However, if φ and a are infinitely differentiable, one can construct the whole asymptotic series in the powers of $|\Lambda|^{-1}$ [29].

On the other hand, it follows from the standard CLT for bounded i.i.d. random variables (see (2.35)) and the Berry–Esseen bound that we have in (2.35) the error term $O(M^{-1/2})$ instead of $o(1)$, and, hopefully, $O(|\Lambda^{-1/2}|)$ in the $\mathcal{D}$-convergence stochastic analog (2.26) of the Szegő theorem.

As for the "point-wise" case treated in Theorem 2.5, we note first that this is a "frequency"-type result, analogous to the Law of Large Numbers or, more generally, to the ergodic theorem. This is clear from the following observation on the well known Gaussian random processes [4]. Namely, let $W : [0, \infty) \to \mathbb{R}$ be the Wiener process and $U : \mathbb{R} \to \mathbb{R}$ be the Uhlenbeck–Ornstein process. They are related as $U(s) = e^{-s/2}W(e^s)$, $s \in \mathbb{R}$, thus

$$\frac{1}{\log M} \int_1^M \mathbf{1}_\Delta \left(W(t)/t^{1/2} \right) dt = \frac{1}{\log M} \int_0^{\log M} \mathbf{1}_\Delta (U(s))\, ds.$$

Since U is ergodic and its one-point (invariant) distribution is the standard Gaussian, the r.h.s. converges with probability 1 to $\Phi(\Delta)$ as $M \to \infty$ according to the ergodic theorem. We obtained the almost sure Central Limit Theorem for the Wiener process, the continuous time analog of the sequence of i.i.d. Gaussian random variables, see (2.31).

In view of this observation (explaining, in particular, the appearance of the logarithmic average in the almost sure Central Limit Theorem) and the Law of Iterated Logarithm we have to have with probability 1 in (2.35) the oscillating error term $O((\log \log \log M / \log M)^{1/2})$ instead of $o(1)$, hence the error term $O((\log \log \log |\Lambda| / \log \Lambda)^{1/2})$ in the $\mathcal{L}$-convergence stochastic analog (2.33) of the Szegő theorem. More precisely, it follows from the invariance principle that with probability 1 we have to have the additional terms $\widetilde{\sigma} W(\log M) + O(\log M^{1/2-\varepsilon})$, $M \to \infty$ in (2.37) and, correspondingly, the terms

$$\widetilde{\sigma} W\left(\log |\Lambda|\right) + O\left(|\log |\Lambda||^{1/2-\varepsilon}\right), \quad \log |\Lambda| \to \infty,$$

with $\widetilde{\sigma} > 0$ and some $\varepsilon > 0$ in (2.33).

We prove in this paper asymptotic formulas for traces of certain random operators related to the restrictions to the expanding intervals $\Lambda = [-M, M] \subset \mathbb{Z}$, $M \to \infty$ of the one dimensional discrete Schrödinger operator H assuming that its potential is a collection of random i.i.d. variables. We do not use, however, a remarkable property of H, the pure point character of its spectrum. This spectral type holds for any bounded i.i.d. potential [2] and can be contrasted with the absolute continuous type of the spectrum of H with constant or periodic potential. Moreover, if the common probability law of the on-site potential is Lipschitzian, we have the bound (2.28). It can be shown that the use of the bound makes the conditions of our results somewhat weaker (it suffices to have $\theta = 1$ in (2.17), certain bounds somewhat stronger ($O(1)$ instead $o(|\Lambda|^{1/2})$ in (3.1), Ce^{-cp} instead C/p^θ in (4.16), etc.) and proofs simpler (Lemmas 4.2 and (4.4) are not necessary). On the other hand, the bound (2.28) holds only under the condition of some regularity of the common probability law of the i.i.d. potential (e.g., the Lipschitz continuity of its probability law). This is why we prefer to

use rather standard spectral tools, somewhat less optimal conditions (2.17) on a and φ and somewhat more involved proofs but to have corresponding results valid for a larger class of random i.i.d. potentials of Theorems 2.1 and 2.5.

It is worth noting, however, that the bound (2.28) is an important necessary tool in the analysis of the large-Λ behavior of $\mathrm{Tr}_\Lambda \varphi(a_\Lambda(H))$ with not too smooth a and φ, e.g., $a = n_F|_{\beta=\infty} = \chi_{[E_F,\infty)}$ with n_F of (2.30) and $\varphi = r_\alpha$, $\alpha \leq 1$ with r_α of (2.29) corresponding to the entanglement entropy of the ground state of free disordered fermions at zero temperature, see [11, 22] and references therein.

3. Proof of results

Proof of Theorem 2.1. It follows from (2.17) and Lemma 4.5 that we have uniformly in potential

$$\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(H)) = \mathrm{Tr}_\Lambda \, \varphi(a(H)) + o\left(|\Lambda|^{1/2}\right), \quad |\Lambda| \to \infty. \tag{3.1}$$

Hence, we obtain in view of (2.14) and the definition (2.9) of Tr_Λ

$$\mathrm{Tr}_\Lambda \, \varphi(a_\Lambda(H)) = |\Lambda| \int_{-\infty}^{\infty} \gamma(\lambda) N(d\lambda) + \overset{\circ}{\gamma}_\Lambda + o\left(|\Lambda|^{1/2}\right),$$

where

$$\overset{\circ}{\gamma}_\Lambda := \gamma_\Lambda - \mathbf{E}\{\gamma_\Lambda\} = \sum_{j \in \Lambda} (\gamma_{jj}(H) - \mathbf{E}\{\gamma_{jj}(H)\}),$$

$$\mathbf{E}\{\gamma_\Lambda(H)\}) = |\Lambda| \int_{-\infty}^{\infty} \gamma(\lambda) N_H(d\lambda). \tag{3.2}$$

The above formulas reduce the proof of the theorem to that of the Central Limit Theorem for $|\Lambda|^{-1/2}\gamma_\Lambda$, i.e., for the sequence $\{\gamma_{jj}(H)\}_{j \in \mathbb{Z}}$. The sequence is ergodic according to (2.13) for $j = k$.

We use in this case a general Central Limit Theorem for stationary weakly dependent sequences given by Proposition 4.1 with $X_j = V_j$, $j \in \mathbb{Z}$ and $Y_0 = \gamma_{00}(H)$. To verify the approximation condition (4.1) of the proposition it is convenient to write $V = (V_<, V_>)$, where $V_< = \{V_j\}_{|j| \leq p}$ and $V_> = \{V_j\}_{|j| > p}$ are independent collections of independent random variables whose probability laws we denote $P_<$ and $P_>$ so that the probability law P of V is symbolically $P = P_< \cdot P_>$. Denoting $\gamma_{00}(H) = g(V_<, V_>)$, we have

$$\mathbf{E}\{|\gamma_{00}(H) - \mathbf{E}\{\gamma_{00}(H)|\mathcal{F}_{-p}^p\}|\}$$
$$= \int \left| g(V_<, V_>) - \int g(V_<, V_>')P(dV_>') \right| P(dV_>)P(dV_<)$$
$$\leq \int \left(\int \left| g(V_<, V_>) - g(V_<, V_>') \right| P(dV_>') \right) P(dV_>)P(dV_<).$$

Applying to the difference in the third line of the above formula Lemma 4.6 with $f = \gamma_{00}$, we find that the expression in the first line of the formula is bounded by C/p^θ. Thus, the series (4.1) is convergent in our case.

This and Proposition (4.1) imply the validity of (2.20)–(2.22). The formula for the limiting variance (2.23)–(2.24) is proved in Lemma 4.7.

Let us prove now the positivity of the limiting variance σ^2 (2.25). According to (2.23)–(2.24), the hypothesis $\sigma^2 = 0$ implies that for an almost every event from $\mathcal{F}_1^\infty$ the expression

$$V_0 \int_0^1 ds\ \mathbf{E}\{\gamma'_{00}(H|_{V_0 \to uV_0})|\mathcal{F}_1^\infty\}\,du \tag{3.3}$$

is independent of $V_0 \in \mathrm{supp}F$. Assume without loss of generality that zero is in support of F. Then the above expression is zero. On the other hand, if our i.i.d. random potential is non-trivial, then there exists a non-zero point $V_0 \neq 0$ in the support. If, in addition, γ' does not change the sign on the spectrum of H and is not zero, then (3.3) cannot be zero, and we have a contradiction.

Now it suffices to use a general argument (see, e.g., Theorem 18.6.1 of [14] or Proposition 3.2.9 of [20]) to finish the proof of Theorem 2.1. □

Proof of Theorem 2.3. We will first use Theorem 2.1. Indeed, according to (2.4) and (2.30), $a = n_F$ is real analytic on the finite interval K of (2.4) and admits a real analytic and fast decaying at infinity extension to the whole axis. Besides, $a(K) = [a_-, a_+]$, $0 < a_- < a_+ < 1$ is also finite, hence, $\varphi = r_\alpha$ of (2.29) is real analytic on $a(K)$ and admits a real analytic and fast decaying at infinity extension to the whole axis. Thus, assertion (i) of Theorem 2.1 is valid in this case.

We cannot, however, use assertion (ii) of Theorem 2.1, since $\gamma = r_\alpha \circ n_F$ is not monotone but convex on K. Here is another argument proving the positivity (2.25) of the limiting variance (2.23)–(2.24).

Assuming that the variance is zero and using the fact that zero is in support of the probability law F of the potential, we obtain from (2.23)–(2.24), as in the proof of Theorem 2.1, that for almost every event from $\mathcal{F}_1^\infty$ we have

$$V_0 \int_0^1 \mathbf{E}\left\{\gamma'_{00}(H_0(u))|\mathcal{F}_1^\infty\right\}du = 0, \quad V_0 \in \mathrm{supp}F,$$

where $H_0(u) := H|_{V_0 \to uV_0}$. Integrating here by parts with respect to u, we get

$$V_0\mathbf{E}\{\gamma'_{00}(H_0(0))|\mathcal{F}_1^\infty\}$$
$$+ V_0 \int_0^1 \mathbf{E}\left\{\frac{\partial}{\partial u}\gamma'_{00}(H_0(u))|\mathcal{F}_1^\infty\right\}(1-u)\,du = 0, \quad V_0 \in \mathrm{supp}F,$$

and since $\mathbf{E}\{V_0\} = 0$ and $\gamma'_{00}(H_0(0))$ is independent of V_0, the expectation with respect to V_0 yields for almost every event from $\mathcal{F}_1^\infty$

$$\int_0^1 (1-u)du \int V_0\mathbf{E}\left\{\frac{\partial}{\partial u}\gamma'_{00}(H(u))|\mathcal{F}_1^\infty\right\}F(dV_0) = 0. \tag{3.4}$$

We will now use the formula

$$\frac{\partial}{\partial u}\gamma'_{00}(H(u)) = V_0 \iint \frac{\gamma'(\lambda_1) - \gamma'(\lambda_2)}{\lambda_1 - \lambda_2}\mu_{H(u)}(d\lambda_1)\mu_{H(u)}(d\lambda_2),$$

where $\mu_{H(u)}(d\lambda) = (\mathcal{E}_{H(u)}(d\lambda))_{00}$, and $\mathcal{E}_{H(u)}$ is the resolution of identity of $H(u)$. Thus, $\mu_0 \geq 0$ and $\mu_0(\mathbb{R}) = 1$. The formula can be obtained by iterating twice the Duhamel formula (4.3).

Plugging the r.h.s. of the formula in (3.4) and recalling that γ is strictly convex on the spectrum, hence $(\gamma(\lambda_1) - \gamma(\lambda_2))(\lambda_1 - \lambda)^{-1} < 0$, we conclude that the r.h.s. of (3.4) is not zero. This implies the positivity of the variance. $\square$

Proof of Theorem 2.5. As in the proof of Theorem 2.1 we will start with passing from $\mathrm{Tr}_\Lambda \varphi(a_\Lambda(H))$ to $\mathrm{Tr}_\Lambda \varphi(a(H)) = \mathrm{Tr}_\Lambda \gamma(H)$ with the error $o(|\Lambda|^{1/2})$ by using (2.17) and Lemma 4.5 (see (3.1)), thereby reducing the proof of the theorem to the proof of the almost sure CLT for $|\Lambda|^{-1/2}\gamma_\Lambda$ (see 3.2) i.e., for the same ergodic sequence $\{\gamma_{jj}(H)\}_{j\in\mathbb{Z}}$ as in Theorem 2.1.

Our further proof is essentially based on that in [23] of the almost sure CLT for ergodic strongly mixing sequences (see (3.12)) and on the procedure of approximation of general ergodic sequences by strongly mixing sequences (see (4.1)) given in [14], Section 18.3. In particular, according to Proposition 4.8 (see Theorem 1 in [23]), it suffices to prove the bound

$$\mathbf{Var}\left\{\frac{1}{\log M}\sum_{m=1}^{M}\frac{1}{m}f(Z_m)\right\} = O(1/(\log M)^\varepsilon), \quad M \to \infty, \qquad (3.5)$$

for any bounded Lipschitzian f (see (4.39)),

$$Z_m = \mu_m^{-1/2}\Sigma_{[-m,m]}, \quad \mu_m = 2m + 1, \qquad (3.6)$$

and some $\varepsilon > 0$.

To this end we denote

$$\overset{\circ}{\gamma}_{jj}(H) := \gamma_{jj}(H) - \mathbf{E}\{\gamma_{jj}(H)\} = Y_j, \quad j \in \mathbb{Z}, \qquad (3.7)$$

and introduce for every positive integer s the ergodic sequences $\{\xi_j^{(s)}\}_{j\in\mathbb{Z}}$ and $\{\eta_j^{(s)}\}_{j\in\mathbb{Z}}$ with

$$\xi_j^{(s)} = \mathbf{E}\left\{Y_j | \mathcal{F}_{j-s}^{j+s}\right\}, \quad \eta_j^{(s)} = Y_j - \xi_j^{(s)}. \qquad (3.8)$$

Denote also

$$F_M = \frac{1}{\log M}\sum_{m=1}^{M}\frac{1}{m}f(Z_m)$$

and

$$F_M^{(s)} = \frac{1}{\log M}\sum_{m=1}^{M}\frac{1}{m}f\left(Z_m^{(s)}\right), \quad Z_m^{(s)} = Z_m|_{Y_j \to \xi_j^{(s)}}. \qquad (3.9)$$

We have then from the elementary inequality $\mathbf{Var}\{\xi\} \leq 2\mathbf{Var}\{\eta\} + 2\mathbf{Var}\{\xi - \eta\}$ and (4.39):

$$\mathbf{Var}\{F_M\} \leq 2\mathbf{Var}\left\{F_M^{(s)}\right\} + 2\mathbf{Var}\left\{F_M - F_M^{(s)}\right\}$$

$$\leq 2\mathbf{Var}\left\{F_M^{(s)}\right\} + \frac{2C_1^2}{\log M}\sum_{m=1}^{M}\frac{1}{m}\mathbf{Var}\left\{R_m^{(s)}\right\}, \qquad (3.10)$$

where C_1 is defined in (4.39) and

$$R_m^{(s)} := Z_m - Z_m^{(s)} = \mu_m^{-1/2}\sum_{|j|\leq m}\eta_j^{(s)}. \qquad (3.11)$$

Recall now that an ergodic sequence is said to be strongly mixing if

$$\alpha_k := \sup_{\substack{A\in\mathcal{F}_{-\infty}^n \\ B\in\mathcal{F}_{k+n}^\infty}}|P(AB) - P(A)P(B)| \to 0, \qquad (3.12)$$

as $k \to \infty$ through positive values and α_k is called the mixing coefficient.

Since the random potential is a sequence of i.i.d. random variables, the sequence $\{\xi_j^{(s)}\}_{j\in\mathbb{Z}}$ of (3.8) is strongly mixing and its mixing coefficient is (see (3.12))

$$\alpha_k^{(s)} = \begin{cases}\leq 1, & k\leq 2s, \\ 0, & k > 2s.\end{cases} \qquad (3.13)$$

We are going to bound the first term on the right of (3.10) by using Lemma 1 of [23] on the almost sure CLT for strongly mixing sequences and we will deal with the second term on the right of (3.10) by using the sufficiently good approximation of $\{Y_j\}_{j\in\mathbb{Z}}$ of (3.7) by $\{\xi_j^{(s)}\}_{j\in\mathbb{Z}}$ of (3.8) as $s \to \infty$ following from Lemma 4.6. Note that similar argument has been already used in the proof of Theorem 2.1, see (4.1) in Proposition 4.1 and Theorem 18.6.3 in [14]. This is obtained in Lemmas 4.9 and 4.10 below for $M \to \infty$ and $s \to \infty$. They allow us to continue (3.10) as

$$\mathbf{Var}\{F_M\} = O\left(\log s/\log M\right) + O(1/s^{\theta-1}),$$

where $\theta > 1$ (see (2.17)). Choosing here $s = (\log M)^{1-\varepsilon}$, $\varepsilon \in (0,1)$, we obtain (3.5), hence, the theorem. $\qquad\square$

4. Auxiliary results

We start with a general Central Limit Theorem for ergodic sequences of random variables, see [14], Theorems 18.6.1–18.6.3, more precisely. with its version involving i.i.d. random variables.

Proposition 4.1. *Let $\{X_j\}_{j\in\mathbb{Z}}$ be i.i.d. random variables, $\mathcal{F}_a^b$ be the σ-algebra generated by $\{X_j\}_{j=a}^b$, Y_0 be a function measurable with respect to $\mathcal{F} = \mathcal{F}_{-\infty}^\infty$. Denote T the standard shift automorphism $(X_{j+1}(\omega) = X_j(T\omega))$ and set $Y_j(\omega) = Y_0(T^j\omega)$. Assume that*

(i) *Y_0 is bounded;*

(ii)
$$\sum_{p=1}^{\infty} \mathbf{E}\left\{\left|Y_0 - \mathbf{E}\left\{Y_0 \mid \mathcal{F}_{-p}^p\right\}\right|\right\} < \infty. \tag{4.1}$$

Then

(a) $\sigma^2 := \sum_{k=0}^{\infty} \mathbf{Cov}\{Y_0, Y_k\} < \infty;$

(b) *if $\sigma^2 > 0$, then*

$$|\Lambda|^{-1/2} \sum_{|j| \le M} Y_j \tag{4.2}$$

converges in distribution to the Gaussian random variable of zero mean and variance σ^2.

The proof of the proposition is based on the proof of the CLT for strongly mixing ergodic sequences (see (3.12)) and on the approximation of more general ergodic sequences by strongly mixing sequences provided by condition (4.1).

We will also need several facts on the one-dimensional discrete Schrödinger operator with bounded potential.

We recall first the Duhamel formula for the difference of two one-parametric groups $U_1(t) = e^{itA_1}$ and $U_1(t) = e^{itA_1}$ corresponding to two bounded operators A_1 and A_2 :

$$U_2(t) - U_1(t) = i \int_0^{|t|} U_2(t-s)(A_2 - A_1)U_1(s)\, ds, \quad t \in \mathbb{R}. \tag{4.3}$$

Lemma 4.2. *Let $H = H_0 + V$ be the one-dimensional discrete Schrödinger operator with real-valued bounded potential, $U(t) = e^{itH}$ be the corresponding unitary group and $\{U_{jk}(t)\}_{j,k\in\mathbb{Z}}$ be the matrix of $U(t)$. Then we have for any $t \in \mathbb{R}$ and $\delta > 0$*

$$|U_{jk}(t)| \le e^{-\delta|j-k|+s(\delta)|t|}, \quad s(\delta) = 2\sinh\delta. \tag{4.4}$$

Proof. Introduce the diagonal operator $D = \{D_{jk}\}_{j,k\in\mathbb{Z}}$, with $D_{jk} = e^{\rho j}\delta_{jk}$, $\rho \in \mathbb{R}$ and consider

$$DU(t)D^{-1} = e^{itDHD^{-1}} = e^{itH+itQ},$$

where

$$Q = DHD^{-1} - H = DH_0D^{-1} - H_0.$$

Since H_0 is the operator of second finite difference with the symbol $-2\cos p$, $p \in \mathbb{T}$, the symbol of Q is

$$-2\cos(p + i\rho) + 2\cos p = -2\cos p(\cosh\rho - 1) + 2i\sin p \sinh\rho.$$

Hence $Q = Q_1 + iQ_2$, where Q_1 and Q_2 are selfadjoint operators and

$$\|Q_2\| \le 2\sinh|\rho|.$$

Now, denoting $A_2 = H + Q_1 + iQ_2$ and $A_1 = H + Q_1$, iterating the Duhamel formula (4.3) and using $\|e^{itA_1}\| = 1$, we obtain

$$\|e^{itH_1 - tQ_2}\| \le e^{|t|\|Q_2\|} = e^{|t|2\sinh|\rho|}.$$

This and the relation

$$\left(DU(t)D^{-1}\right)_{jk} = e^{\rho j}U_{jk}(t)e^{-\rho k}$$

imply (4.4). $\qquad\square$

Remark 4.3. Bound (4.4) is an analog of the Combes–Thomas bound for the resolvent $\{((H - z)^{-1})_{jk}\}_{j,k\in\mathbb{Z}}$ of H and the above proof uses an essentially same argument as that in the proof of the bound, see, e.g., [2].

Lemma 4.4. *Let $H = H_0 + V$ be the one-dimensional discrete Schrödinger operator with real-valued potential and $a : \mathbb{R} \to \mathbb{R}$ admits the Fourier transform $\widehat{a}$ and*

$$\int_{-\infty}^{\infty} \left(1 + |t|^\theta\right) |\widehat{a}(t)|\, dt < \infty, \quad \theta > 0. \tag{4.5}$$

If $A = a(H) = \{F_{jk}\}_{j,k\in\mathbb{Z}}$, then we have

$$|A_{jk}| \le C/|j - k|^\theta, \quad C < \infty, \ j \ne k. \tag{4.6}$$

Proof. It follows from the spectral theorem

$$A = a(H) = \int_{-\infty}^{\infty} \widehat{a}(t)U(t)\, dt, \tag{4.7}$$

hence, we have for any $T > 0$

$$\begin{aligned}
A_{jk} &= \int_{-\infty}^{\infty} \widehat{a}(t)U_{jk}(t)\, dt \\
&= \int_{|t|\le T} \widehat{a}(t)U(t)\, dt + \int_{|t|\ge T} \widehat{a}(t)U(t)\, dt = I_1 + I_2.
\end{aligned}$$

We have further

$$|I_1| \le e^{-\delta|j-k| + s(\delta)T} \int_{-\infty}^{\infty} |\widehat{a}(t)|\, dt$$

by using Lemma 4.2 and

$$|I_2| \le \frac{1}{T^\theta} \int_{-\infty}^{\infty} \left(1 + |t|^\theta\right) |\widehat{a}(t)|\, dt$$

by condition (4.5) of the lemma.

Now, choosing $T = \frac{\delta}{s}|j - k| - \theta \log|j - k|$, we obtain (4.6). $\qquad\square$

Lemma 4.5. *Let* $A = \{A_{jk}\}_{j,k\in\mathbb{Z}}$ *be bounded selfadjoint operator in* $l^2(\mathbb{Z})$ *such that*

$$|A_{jk}| \le C/|j-k|^\theta, \quad C < \infty, \quad \theta > 1, \tag{4.8}$$

and $A_\Lambda = \chi_\Lambda A \chi_\Lambda = \{A_{jk}\}_{j,k\in\Lambda}$ *be its restriction to* Λ. *Then for any* $f : \mathbb{R} \to \mathbb{C}$ *admitting the Fourier transform* $\widehat{f}$ *such that*

$$\int_{-\infty}^\infty (1+|t|)|\widehat{f}(t)|\, dt < \infty \tag{4.9}$$

we have uniformly in V *satisfying* (2.3)

$$\left| \operatorname{Tr} \chi_\Lambda f(A_\Lambda)\chi_\Lambda - \operatorname{Tr} \chi_\Lambda f(A)\chi_\Lambda \right| = o(L^{1/2}), \quad L \to \infty. \tag{4.10}$$

Proof. Consider $A_\Lambda \oplus A_{\overline{\Lambda}}, \quad \overline{\Lambda} = \mathbb{Z} \setminus \Lambda$ and

$$A - A_\Lambda \oplus A_{\overline{\Lambda}} = \begin{pmatrix} 0 & \chi_\Lambda A \chi_{\overline{\Lambda}} \\ \chi_{\overline{\Lambda}} A \chi_\Lambda & 0 \end{pmatrix}.$$

Thus, writing an analog of (4.7) for A instead of H and using the Duhamel formula (4.3), we obtain

$$
\begin{aligned}
& f(A) - f(A_\Lambda \oplus A_{\mathbb{Z}\setminus\Lambda}) \\
&\quad = \int_{-\infty}^\infty \widehat{f}(t)dt \int_0^{|t|} U(t-s)(\chi_{\overline{\Lambda}} A \chi_\Lambda + \chi_\Lambda A \chi_{\overline{\Lambda}})U_\Lambda(s) \oplus U_{\overline{\Lambda}}(s)\, ds, \quad (4.11)
\end{aligned}
$$

and

$$
\begin{aligned}
& \operatorname{Tr} \chi_\Lambda f(A)\chi_\Lambda - \operatorname{Tr} \chi_\Lambda f(A_\Lambda)\chi_\Lambda \\
&\qquad = \int_{-\infty}^\infty \widehat{f}(t)dt \int_0^t \operatorname{Tr}\chi_\Lambda U_\Lambda(s)\chi_\Lambda U(t-s)\chi_{\overline{\Lambda}} A \chi_\Lambda\, ds. \quad (4.12)
\end{aligned}
$$

Denoting

$$B = \chi_\Lambda U_\Lambda(s)\chi_\Lambda U(t-s) : l^2(\mathbb{Z}) \to l^2(\Lambda) \tag{4.13}$$

we can write the integrand J in (4.12) as

$$J = \sum_{\substack{j\in\Lambda \\ k\in\overline{\Lambda}}} A_{kj} B_{jk}, \tag{4.14}$$

hence

$$|J| \le \sum_{j\in\Lambda} \left(\sum_{k\in\overline{\Lambda}} |A_{kj}|^2 \sum_{k\in\overline{\Lambda}} |B_{jk}|^2 \right)^{1/2} \le \sum_{j\in\Lambda} \left(\sum_{k\in\overline{\Lambda}} |A_{kj}|^2 \sum_{k\in\mathbb{Z}} |B_{jk}|^2 \right)^{1/2}.$$

We have in view of (4.13)

$$\sum_{k \in \mathbb{Z}} |B_{jk}|^2 = (BB^*)_{jj}$$

$$= (U_\Lambda(s)\chi_\Lambda U(t-s)U^*(t-s)\chi_\Lambda U^*_\Lambda(s))_{jj} = (\chi_\Lambda U_\Lambda(s)U^*_\Lambda(s))_{jj} = 1$$

since $U(t-s)$ is unitary in $l^2(\mathbb{Z})$ and $U_\Lambda(s)$ is unitary in $l^2(\Lambda)$. Thus, we have in view of (4.8)

$$|J| \leq \sum_{j \in \Lambda} \left(\sum_{k \in \overline{\Lambda}} |A_{kj}|^2 \right)^{1/2} \leq C \sum_{j \in \Lambda} \left(\sum_{k \in \overline{\Lambda}} |k-j|^{-2\theta}| \right)^{1/2} = o(L^{1/2})$$

and (4.11) follows. Note that for $\theta > 3/2$ the r.h.s. of the above bound is $O(1)$. □

Similar result was obtained in [17] by another method.

Lemma 4.6. *Let H_1 and H_2 be the one dimensional discrete Schrödinger operators with bounded potentials V_1 and V_2 coinciding within the integer valued interval $[-p, p]$. Consider $f : \mathbb{R} \to \mathbb{C}$ whose Fourier transform $\widehat{f}$ is such that*

$$\int_{-\infty}^{\infty} \left(1 + |t|^\theta\right) |\widehat{f}(t)|\, dt < \infty, \quad \theta > 1. \tag{4.15}$$

Then we have

$$|f_{00}(H_1) - f_{00}(H_2)| \leq C/p^\theta, \tag{4.16}$$

where C is independent of V_1 and V_2.

Proof. We denote

$$V_1 = \{V'\}_{|j|>s} \cup \{V_j\}_{|j| \leq s}, \quad V_2 = \{V''_j\}_{|j|>s} \cup \{V_j\}_{|j| \leq s},$$

$U^{(1)}(t) = e^{itH_1}$ and $U^{(2)}(t) = e^{itH_2}$ and use (4.7) and the spectral theorem to write for any $T > 0$

$$|f_{00}(H_2) - f_{00}(H_1)| \leq \int_{|t| \leq T} |\widehat{f}(t)| |U^{(2)}_{00}(t) - U^{(1)}_{00}(t)|\, dt$$

$$+ \int_{|t| \geq T} |\widehat{f}(t)| |U^{(2)}_{00}(t) - U^{(1)}_{00}(t)| =: I_1 + I_2. \tag{4.17}$$

We have then by the Duhamel formula (4.3) and (2.3)

$$I_1 \leq \int_{|t| \leq T} |\widehat{f}(t)| dt \int_0^{|t|} \sum_{|j|>s} |U^{(2)}_{0j}(t-t')(V''_j - V'_j)U^{(1)}_{j0}(t')|\, dt'$$

$$\leq 2\overline{V} \int_{|t| \leq T} |\widehat{f}(t)| dt \int_0^{|t|} \sum_{|j|>s} |U^{(2)}_{0j}(t-t')| |U^{(1)}_{j0}(t')|\, dt'.$$

We will use now Lemma 4.2 implying

$$I_1 \leq 2\overline{V}e^{-2\delta p + s(\delta)T}\int_{-\infty}^{\infty}(1+|t|)\widehat{f}(t)|\,dt. \tag{4.18}$$

To estimate I_2 of (4.17), we write

$$I_2 \leq 2\int_{|t|\geq T}|\widehat{f}(t)|dt \leq \frac{2}{T^\theta}\int_{-\infty}^{\infty}\left(1+|t|^\theta\right)|\widehat{f}(t)|\,dt. \tag{4.19}$$

Choosing now in (4.18) and (4.19) $T = 2\delta p/s(\delta) - \theta\log p$, we obtain (4.16). $\square$

Lemma 4.7. *Consider a bounded $\gamma : \mathbb{R} \to \mathbb{R}$ admitting the Fourier transform $\widehat{\gamma}$ such that*

$$\int_{-\infty}^{\infty}(1+|t|)|\widehat{\gamma}(t)|\,dt < \infty, \tag{4.20}$$

and set $\gamma(H) = \{\gamma_{jk}(H)\}_{j,k\in\mathbb{Z}}$, where H is the one-dimensional Schrödinger operator (2.1)–(2.3) with random i.i.d. potential. Let γ_Λ be defined in (3.2) and

$$\sigma_\Lambda^2 = |\Lambda|^{-1}\mathbf{Var}\{\gamma_\Lambda\}. \tag{4.21}$$

Then there exists the limit

$$\sigma^2 := \lim_{\Lambda\to\infty}\sigma_\Lambda^2 = \mathbf{E}\left\{\left(\mathcal{M}^{(0)}\right)^2\right\}, \tag{4.22}$$

where

$$\begin{aligned}\mathcal{M}^{(0)} = \;&\mathbf{E}\left\{A_0(V_0, \{V_j\}_{j\neq 0}) \mid \mathcal{F}_0^\infty\right\}\\ &- \int_{-\infty}^{\infty}\mathbf{E}\{A_0(V_0', \{V_j\}_{j\neq 0}) \mid \mathcal{F}_0^\infty\}F(dV_0')\end{aligned} \tag{4.23}$$

and

$$A_0(V_0, \{V_j\}_{j\neq 0}) = V_0\int_0^1(\gamma'(H)|_{V_0\to uV_0})_{00}\,du. \tag{4.24}$$

Proof. It is convenient to consider

$$\tau_\Lambda := \mathrm{Tr}_\Lambda\gamma(H_\Lambda) \tag{4.25}$$

instead of γ_Λ of (3.2). It follows from Lemma 4.5 that

$$\sigma_\Lambda^2 = |\Lambda|^{-1}\mathbf{Var}\{\tau_\Lambda\} + o(1), \quad |\Lambda| \to \infty. \tag{4.26}$$

To deal with $\mathbf{Var}\{\tau_\Lambda\}$ we will use a simple version of the martingale techniques (see, e.g., [20], Proposition 18.1.1), according to which if $\{X_j\}_{j=-M}^M$ are the i.i.d. random variables, $\Phi : \mathbb{R}^{2M+1} \to \mathbb{R}$ is bounded and $\Phi = \Phi(X_{-M}, X_{-M+1}, \ldots, X_M)$, then

$$\mathbf{Var}\{\Phi\} := \left\{|\Phi - \mathbf{E}\{\Phi\}|^2\right\}$$

$$= \sum_{|m| \leq M} \mathbf{E}\left\{ |\Phi^{(m)} - \Phi^{(m+1)}|^2 \right\}, \tag{4.27}$$

where

$$\Phi^{(m)} = \mathbf{E}\left\{ \Phi | \mathcal{F}_m^M \right\}, \quad \Phi^{(-M)} = \Phi, \quad \Phi^{(M+1)} = \mathbf{E}\{\Phi\}. \tag{4.28}$$

We choose in (4.27), (4.28) $X_j = V_j$, $|j| \leq M$ and $\Phi = \tau_\Lambda$ (see (4.25)) and we obtain

$$|\Lambda|^{-1} \mathbf{Var}\{\tau_\Lambda\} = |\Lambda|^{-1} \sum_{m=-M}^{M} \mathbf{E}\left\{ |\mathcal{M}_\Lambda^{(m)}|^2 \right\}, \tag{4.29}$$

$$\mathcal{M}_\Lambda^{(m)} = \tau_\Lambda^{(m)} - \tau_\Lambda^{(m+1)},$$

where (see (4.28))

$$\tau_\Lambda^{(m)} = \mathbf{E}\left\{ \tau_\Lambda | \mathcal{F}_m^M \right\}, \quad \tau_\Lambda^{(-M)} = \tau_\Lambda, \quad \tau_\Lambda^{(M+1)} = \mathbf{E}\{\tau_\Lambda\}. \tag{4.30}$$

By using the formula

$$\tau_\Lambda - \tau_\Lambda|_{V_m=0} = \int_0^1 du \frac{\partial}{\partial u} \mathrm{Tr}_\Lambda \ \gamma(H_\Lambda|_{V_m \to uV_m})$$

$$= V_m \int_0^1 du (\gamma'(H_\Lambda|_{V_m \to uV_m}))_{mm} =: A_\Lambda(V_m, \{V\}_{j\neq m}), \tag{4.31}$$

we can write

$$\mathcal{M}_\Lambda^{(m)} = \mathbf{E}\left\{ A_\Lambda(V_m, \{V\}_{j\neq m}) | \mathcal{F}_m^M \right\}$$

$$- \int_{-\infty}^{\infty} A_\Lambda(V_m', \{V\}_{j\neq m}) | \mathcal{F}_m^M \} F(dV_m'). \tag{4.32}$$

Let us show now that

$$\lim_{\Lambda \to \infty} |\Lambda|^{-1} \sum_{m \in \Lambda} \mathbf{E}\left\{ |\mathcal{M}_\Lambda^{(m)}|^2 \right\} = \mathbf{E}\left\{ |\mathcal{M}^{(0)}|^2 \right\}, \tag{4.33}$$

where for any $m \in \mathbb{Z}$

$$\mathcal{M}^{(m)} = \mathbf{E}\{ A(V_m, \{V\}_{j\neq m}) | \mathcal{F}_m^\infty \}$$

$$- \int_{-\infty}^{\infty} \mathbf{E}\{ A(V_m', \{V\}_{j\neq m}) | \mathcal{F}_m^\infty \} F(dV_m'). \tag{4.34}$$

Note first that since $A_\Lambda(V_m, \{V\}_{j\neq m})$ does not depend on $\{V_j\}_{|j|>M}$, we can replace $\mathcal{F}_m^M$ by $\mathcal{F}_m^\infty$ Next, it is easy to see that $\mathcal{M}_\Lambda^{(m)}$ is bounded in Λ and V, thus the proof of (4.33) reduces to the proof of validity with probability 1 of the relation

$$\lim_{\substack{\Lambda \to \infty, \\ \mathrm{dist}(m, \{M, -M\}) \to \infty}} \mathcal{M}_\Lambda^{(m)} = \mathcal{M}^{(m)}. \tag{4.35}$$

Note that $\mathcal{M}^{(m)}$ of (4.34) differs from its prelimit form $\mathcal{M}_\Lambda^{(m)}$ of (4.32) by the replacement of H_Λ by H in the r.h.s. of (4.31).

Indeed, if (4.35) is valid, then we can replace $\mathcal{M}_\Lambda^{(m)}$ by $\mathcal{M}^{(m)}$ in the l.h.s. of (4.33) and then take into account that V is a collection of i.i.d. random variables, hence $\mathbf{E}\{|\mathcal{M}^{(m)}|^2\} = \mathbf{E}\{|\mathcal{M}^{(0)}|^2\}$ for any $m \in \mathbb{Z}$.

To prove (4.35) we will use a version of formula (4.11) with $f = \gamma'$ implying for $m \in \Lambda$

$$(\gamma'(H) - \gamma'(H_\Lambda))_{mm} = i \int_{-\infty}^{\infty} t\widehat{\gamma}(t)dt \int_0^{|t|} (U_\Lambda(t-s)\chi_\Lambda H\chi_{\overline{\Lambda}}U(s))_{mm}\, ds. \quad (4.36)$$

Taking into account that the non-zero entries of $\chi_\Lambda H\chi_{\overline{\Lambda}}$ are $-\delta_{j,M}\delta_{k,M+1}$ and $-\delta_{j,-M}\delta_{k,-(M+1)}$, $|j| \le M, |k| > M$, we obtain

$$|(U_\Lambda(t-s)\chi_\Lambda H\chi_{\overline{\Lambda}}U(s))_{mm}| \le |U_{M+1,m}(s)| + |U_{-(M+1),m}(s)|. \quad (4.37)$$

We write now the integral over t in (4.36) as the sum of the integral I_1 over $|t| \le T$ and that I_2 over $|t| \ge T$ for some T, cf. the proofs of Lemmas 4.4 and 4.6. We have by Lemma 4.2 and (4.37)

$$\begin{aligned} |I_1| \le &2e^{-\delta d + sT} \int_{|t| \le T} |t|^2 |\widehat{\gamma}(t)|\, dt \\ \le &2e^{-\delta d + sT} T \int_{|t| \le T} |t| |\widehat{\gamma}(t)|\, dt, \quad d = \mathrm{dist}(m, \{M, -M\}) \end{aligned}$$

and by (4.37) and the unitarity of $U(s)$

$$|I_2| \le 2 \int_{|t| \ge T} |t| |\widehat{\gamma}(t)|\, dt.$$

Now, choosing $sT = \delta d/2$ and taking into account (4.20), we obtain (4.35), hence, the assertion of the lemma. $\qquad\square$

Proposition 4.8. *Let $\{X_j\}_{j \in \mathbb{Z}}$ be a sequence of random variables on the same probability space with $\mathbf{E}\{X_l\} = 0$, $\mathbf{E}\{X_l^2\} < \infty$. Put (cf. (2.18), (2.19))*

$$S_m = \sum_{|j| \le m} X_l, \quad Z_m = \mu_m^{-1/2} S_m, \quad \mu_m = 2m+1, \quad \sigma_m^2 = \mathbf{E}\left\{Z_m^2\right\}, \quad (4.38)$$

and assume:

(i) $Z_m \overset{\mathcal{D}}{\to} \xi_\sigma$, $m \to \infty$, *where ξ is the Gaussian random variable of zero mean and variance $\sigma^2 > 0$;*

(ii) *for every bounded Lipschitz f:*

$$|f(x)| \le C, \quad |f(x) - f(y)| \le C_1|x - y| \quad (4.39)$$

there exists $\varepsilon > 0$, such that

$$\mathbf{Var}\left\{\frac{1}{\log M} \sum_{m=1}^M \frac{1}{m} f(Z_m)\right\} = O(1/(\log M)^\varepsilon), \quad M \to \infty.$$

Then $\{X_j\}_{j\in\mathbb{Z}}$ satisfies the almost sure Central Limit Theorem, i.e., we have with probability 1

$$\lim_{M\to\infty} \frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \mathbf{1}_{\Delta}(Z_m) = \Phi\left(\sigma^{-1}\Delta\right),$$

where Δ is an interval and Φ is the standard Gaussian law.

The proposition is a version of Theorem 1 of [23], where the case of semi-infinite stationary sequences $\{X_l\}_{l=1}^{\infty}$ was considered. For another criterion of the validity of the almost sure CLT see [13].

Lemma 4.9. *Let $\left\{\xi_j^{(s)}\right\}_{j\in\mathbb{Z}}$ be defined in* (3.8), *$Z_m^{(s)}$ be defined in* (3.6) *and* (3.9) *and*

$$(\sigma_m^{(s)})^2 = \mathbf{E}\left\{(Z_m^{(s)})^2\right\}.$$

Then we have:

(i) *for every $m = 1, 2, \ldots$*

$$|\sigma_m^{(s)} - \sigma_m| \leq C/s^{(\theta-1)},$$

where $\sigma_m > 0$ is given in (4.38), *C is independent of m and s and $\theta > 1$ is given in* (2.17);

(ii) *for any $\delta > 0$ there exists $m_0 > 0$ and $s_0 > 0$ such that*

$$|\sigma_m^{(s)} - \sigma| \leq \sigma\delta$$

if $m > m_0$ and $s > s_0$ and $\sigma > 0$ is given in Theorem 2.1;

(iii) *for every $m = 1, 2, \ldots$*

$$\mathbf{E}\left\{(R_m^{(s)})^2\right\} \leq C/s^{\theta-1},$$

where C is independent of m and s and $\theta > 1$ is given in (2.17).

Proof. The lemma is a version of the obvious fact $\lim_{s\to\infty} \xi_m^{(s)} = Y_m$ valid with probability 1 for every m and following from (3.8).

(i) Since $\{Y_j\}_{j\in\mathbb{Z}}$ and $\{\xi_j^{(s)}\}_{j\in\mathbb{Z}}$ are ergodic sequences, we can write

$$\sigma_m^2 - (\sigma_m^{(s)})^2 = \sum_{|l|\leq 2s} (1 - |l|/\mu_m)\left(C_l - C_l^{(s)}\right)$$

$$+ \sum_{2s<|l|\leq 2m} (1 - |l|/\mu_m)C_l, \tag{4.40}$$

where $C_l = \mathbf{E}\{Y_0 Y_l\}$ and $C_l^{(s)} = \mathbf{E}\{\xi_0^{(s)}\xi_l^{(s)}\}$ are the correlation functions of the corresponding sequences (see (2.22)) and we took into account (3.13) implying

that $C_l^{(s)} = 0$, $|l| > 2s$ (and that the second term on the right is present only if $m > 2s$). Since $Y_j = \xi_l^{(s)} + \eta_l^{(s)}$, we have

$$C_l - C_l^{(s)} = \mathbf{E}\left\{\xi_0^{(s)}\eta_l^{(s)}\right\} + \mathbf{E}\left\{\xi_j^{(s)}\eta_0^{(s)}\right\} + \mathbf{E}\left\{\eta_0^{(s)}\eta_l^{(s)}\right\}.$$

Since γ is bounded, it follows from (3.7), (3.8) that

$$\left|\mathbf{E}\left\{\xi_0^{(s)}\eta_l^{(s)}\right\}\right| \le C\psi_s, \quad \left|\mathbf{E}\left\{\xi_j^{(s)}\eta_0^{(s)}\right\}\right| \le C\psi_s, \quad \left|\mathbf{E}\left\{\eta_0^{(s)}\eta_l^{(s)}\right\}\right| \le C\psi_s, \quad (4.41)$$

where

$$\psi_s = \mathbf{E}\left\{\left|\eta_0^{(s)}\right|\right\} \tag{4.42}$$

and by (3.8) and Lemma 4.6

$$\psi_s = O\left(1/s^\theta\right), \quad \theta > 1. \tag{4.43}$$

This and (4.40) imply uniformly in m

$$\left|\sigma_m^2 - (\sigma_m^{(s)})^2\right| \le \sum_{|l| \le 2s} \left|C_l - C_l^{(s)}\right| + \sum_{2s < |l| \le 2m} |C_l|$$

$$= O(s\psi_s) + O\left(1/s^{(\theta-1)}\right) = O\left(1/s^{(\theta-1)}\right), \quad s \to \infty. \tag{4.44}$$

(ii) σ_m of (4.38) is strictly positive for every m and according to Theorem 2.1 and Lemma 4.7

$$\lim_{m \to \infty} \sigma_m^2 = \sigma^2 > 0.$$

This and (4.44) imply the assertion.

(iii) The ergodicity of $\{\eta_j^{(s)}\}_{j \in \mathbb{Z}}$ implies (cf. (4.40))

$$\mathbf{Var}\left\{R_m^{(s)}\right\} = \sum_{|l| \le 2m}(1 - |l|/\mu_m)\,\mathbf{E}\left\{\eta_0^{(s)}\eta_l^{(s)}\right\} = \sum_{|l| \le 6s} + \sum_{6s < |l| \le 2m}. \tag{4.45}$$

It follows then from the proof of Proposition 4.1 (see [14], Theorem 18.6.3) and (3.13) that

$$\mathbf{E}\left\{\eta_0^{(s)}\eta_l^{(s)}\right\} \le C\psi_{[l/3]}, \quad |l| > 6s. \tag{4.46}$$

We will use now (4.41)–(4.43) in the first sum on the r.h.s. of (4.45) and (4.43) and (4.46) in the second sum (cf. (4.44)) to get the bound

$$\mathbf{Var}\left\{R_m^{(s)}\right\} \le C\left(s\psi_s + \sum_{|l| > 6s}\psi_{[l/3]}\right) \le C/s^{\theta-1},$$

proving the assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Lemma 4.10. *Let* $\left\{\xi_j^{(s)}\right\}_{j \in \mathbb{Z}}$ *and* $F_M^{(s)}$ *be defined in* (3.8) *and* (3.9) *respectively. Then we have:*

$$\mathbf{Var}\left\{F_m^{(s)}\right\} = O(\log s / \log M), \quad s \to \infty, \ M \to \infty.$$

Proof. Repeating almost literally the proof of Lemma 1 in [23] (where the case of semi-infinite strongly mixing sequences $\{X_l\}_{l=1}^\infty$ was considered), we obtain

$$\mathbf{Var}\left\{F_m^{(s)}\right\} \le \frac{C'}{\log M^2} + \frac{C''}{\log M} \sum_{m=1}^{M} \frac{\alpha_m^{(s)}}{m}$$

$$+ \frac{C'''}{(\log M)^2} \sum_{m=1}^{M-1} \left(2m^{-1/2}\sigma_{2m}^{(s)} + \mathbf{E}\left\{m^{-1}\left|\xi_0^{(s)}\right|\right\}\right) \sum_{l=m+1}^{M} \frac{1}{l^{3/2}}, \quad (4.47)$$

where C', C'', C''' depend only on C in (4.39) and $\alpha_m^{(s)}$ is the mixing coefficient (3.12) of $\left\{\xi_l^{(s)}\right\}_{l\in\mathbb{Z}}$ given by (3.13). In view of (3.13) the second term is bounded by

$$\frac{C''}{\log M} \sum_{m=1}^{2s} \frac{1}{m} = O\left(\frac{\log s}{\log M}\right) \quad (4.48)$$

as $M \to \infty$ and $s \to \infty$.

Consider now the third term of the r.h.s. of (4.47). It follows from (3.8) and our assumption on the boundedness of Y_0 that the contribution of $\mathbf{E}\{|\xi_0^{(s)}|\}$ is $O(1/(\log M)^2)$. Next, given an M-independent M_0 of Lemma 4.9 (ii), we write

$$\sum_{m=1}^{M-1} 2\sigma_{2m}^{(s)} \sum_{l=m+1}^{M} \frac{1}{l^{3/2}} = \sum_{m=1}^{M_0} \sum_{l=m+1}^{M} + \sum_{m=M_0+1}^{M-1} \sum_{l=m+1}^{M}.$$

The first double sum on the right is bounded in M in view of Lemma 4.9 (i) and the fact that σ_m, $m = 1, 2, \ldots, M_0$ are bounded (e.g., $\sigma_m \le \mathbf{E}^{1/2}\{Y_0^2\}$). The second double sum is in view of Lemma 4.9 (ii)

$$O\left(\sum_{m=M_0+1}^{M-1} \frac{1}{m^{1/2}} \sum_{l=m+1}^{M} \frac{1}{l^{3/2}}\right) = O\left(\log M\right).$$

Hence, the third term on the right of (4.47) is $O\left(1/\log M\right)$. This and (4.48) imply that the r.h.s. of (4.47) is $O(\log s/(\log M))$. $\qquad\square$

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Теореми типу Сеге для одновимірного оператора Шредінгера з випадковим потенціалом (гладкий випадок)

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Ця стаття є продовженням роботи [15], де було поставлено задачу про аналог теореми Сеге для ергодичних операторів загального вигляду та розглянуто декілька цікавих випадків. У даній статті ми розповсюджуємо результати [15] на ширший клас тестових функцій та символів, що задають формули типу Сеге для одновимірного оператора Шредінгера з випадковим потенціалом. Ми доводимо, що в цьому випадку член, що по порядку є наступним після головного у формулі Сеге, відповідає центральній граничній теоремі у спектральному контексті, тобто є пропорційним $L^{1/2}$, де L є довжиною інтервалу, на якому ми розглядаємо оператор Шредінгера. Цей результат слід порівняти з класичною формулою Сеге, де відповідний член є обмеженим за L, коли $L \to \infty$. Ми доводимо аналог стандартної центральної граничної теореми (тобто збіжність ймовірності відповідних подій до гауссівського закону), а також аналог майже напевно центральної граничної теореми (тобто збіжність з ймовірністю 1 логарифмічного середнього індикатора відповідної події до гауссівського закону). Як ілюстрацію нашого загального методу

ми надаємо асимптотичну формулу для "заплутаної" ентропії вільних ферміонів при ненульовій температурі.

Ключові слова: випадкові оператори, асимптотичні формули сліду, граничні теореми.