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The Discrete Self-Adjoint Dirac Systems of General Type: Explicit Solutions of Direct and Inverse Problems, Asymptotics of Verblunsky-Type Coefficients and the Stability of Solving of the Inverse Problem

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To V.A. Marchenko with admiration

We consider discrete self-adjoint Dirac systems determined by the potentials (sequences) $\{C_k\}$ such that the matrices C_k are positive definite and j-unitary, where j is a diagonal $m \times m$ matrix which has m_1 entries 1 and m_2 entries -1 $(m_1 + m_2 = m)$ on the main diagonal. We construct systems with the rational Weyl functions and explicitly solve the inverse problem to recover systems from the contractive rational Weyl functions. Moreover, we study the stability of this procedure. The matrices C_k (in the potentials) are the so-called Halmos extensions of the Verblunsky-type coefficients ρ_k. We show that in the case of the contractive rational Weyl functions the coefficients ρ_k tend to zero and the matrices C_k tend to the identity matrix I_m.

Key words: discrete self-adjoint Dirac system, Weyl function, inverse problem, explicit solution, stability of solution of the inverse problem, asymptotics of the potential, Verblunsky-type coefficient.

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1. Introduction

The discrete self-adjoint Dirac systems of general type have the form

$$y_{k+1}(z) = (I_m + \mathrm{i}zjC_k)y_k(z) \quad (k \in \mathbb{N}_0), \tag{1.1}$$

where $\mathbb{N}_0$ stands for the set of non-negative integers, I_m is the $m \times m$ identity matrix, "i" is the imaginary unit ($\mathrm{i}^2 = -1$) and the $m \times m$ matrices $\{C_k\}$ are positive and j-unitary:

$$C_k > 0, \quad C_kjC_k = j, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \ (m_1 + m_2 = m; m_1, m_2 \neq 0). \tag{1.2}$$

First, we will consider (in Section 2) explicit solutions of the direct and inverse problems for system (1.1), (1.2) in terms of the Weyl–Titchmarsh (or simply Weyl) functions. Direct and inverse problems of general type for this system were studied (in terms of the Weyl functions) in [5] and explicit solutions for the case $m_1 = m_2$, in [4]. In Section 2 and Appendix, we complete the results from [5] by adding the properties of the Weyl functions in the lower half-plane and generalize the explicit results from [4] for the case where m_1 does not necessarily equal m_2. We will often shorten our proofs in Section 2 and Appendix and refer to more detailed proofs in [4, 5]. However, a complete procedure of explicitly solving the inverse problem from Section 2 is missing in [4] (and so it is new for $m_1 = m_2$ as well).

The case of explicit solutions of direct and inverse problems corresponds to the *rational Weyl functions*. The results in Section 2 are based on our generalized Bäcklund–Darboux (GBDT) approach, which was initiated by the seminal book [14] by V.A. Marchenko. For various versions of Bäcklund–Darboux transformations and related commutation methods see, for instance, [1, 2, 7, 9, 11, 15, 17, 21] and references therein.

Section 3 is dedicated to the asymptotics of the *potentials* (sequences) $\{C_k\}$ corresponding to the rational Weyl functions. For this purpose, we first derive the asymptotics of the so-called [20] Verblunsky-type coefficients.

Finally, in Section 4, we study the stability of our method of explicit solving of the inverse problem for system (1.1), (1.2), and these results are new even for the cases $m_1 = m_2$ and $m_1 = m_2 = 1$. We note that various important early results on the stability of solutions for inverse problems were obtained by V.A. Marchenko (see, e.g., [13]).

In the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the real axis, $\mathbb{C}$ stands for the complex plane, and $\mathbb{C}_+$ ($\mathbb{C}_-$) stands for the open upper (lower) half-plane. The spectrum of a square matrix A is denoted by $\sigma(A)$.

2. GBDT and direct and inverse problems

1. The fundamental $m \times m$ solution $\{W_k\}$ of (1.1) is normalized by

$$W_0(z) = I_m. \tag{2.1}$$

For the case $z \in \mathbb{C}_+$, the definition of the Weyl function $\varphi(z)$ of Dirac system (1.1), (1.2) was given in [5] in terms of $W_k(z)$. Below we define the Weyl function in $\mathbb{C}_-$, which is somewhat more convenient for our purposes. Clearly, this Weyl function has the properties similar to those in [5, Theorem 3.8].

Definition 2.1. The Weyl function of Dirac system (1.1) (which is given on the semi-axis $0 \leq k < \infty$ and satisfies (1.2)) is an $m_1 \times m_2$ matrix function $\varphi(z)$ in the lower half-plane such that the following inequalities hold:

$$\sum_{k=0}^{\infty} q(z)^k \begin{bmatrix} \varphi(z)^* & I_{m_2} \end{bmatrix} W_k(z)^* C_k W_k(z) \begin{bmatrix} \varphi(z) \\ I_{m_2} \end{bmatrix} < \infty \quad (z \in \mathbb{C}_-), \tag{2.2}$$

$$q(z) := (1 + |z|^2)^{-1}. \tag{2.3}$$

The properties of the Weyl function are described in the theorem below, which is proved in Appendix (using the standard Weyl disk procedure).

Theorem 2.2. *There is a unique Weyl function of the discrete Dirac system* (1.1)*, which is given on the semi-axis* $0 \le k < \infty$ *and satisfies* (1.2)*. This Weyl function* φ *is analytic and contractive (i.e.,* $\varphi^*\varphi \le I_{m_2}$*) on* $\mathbb{C}_-$*.*

In the proof of Theorem 2.2, given in Appendix, we will need the inequalities

$$C_k \ge j, \tag{2.4}$$

which (together with the inequalities $C_k \ge -j$) immediately follow from [5, Proposition 2.2].

Another way to prove Theorem 2.2 and the uniqueness of the solution for the inverse problem, which we will need further, is to consider the Dirac systems

$$\widetilde{y}_{k+1}(z) = (I_m + \mathrm{i}z\widetilde{j}\,\widetilde{C}_k)\widetilde{y}_k(z) \quad (k \in \mathbb{N}_0), \tag{2.5}$$

$$\widetilde{j} := -JjJ^* = \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix}, \quad J := \begin{bmatrix} 0 & I_{m_2} \\ I_{m_1} & 0 \end{bmatrix}, \quad \widetilde{C}_k := JC_kJ^*. \tag{2.6}$$

Systems (2.5), (2.6) are dual to systems (1.1), (1.2), and it is immediate from (1.2), (2.6) that the relations

$$J^*J = I_m, \quad \widetilde{C}_k > 0, \quad \widetilde{C}_k\widetilde{j}\,\widetilde{C}_k = \widetilde{j} \tag{2.7}$$

are valid. Hence, systems (2.5) are again self-adjoint Dirac systems. Similarly to $\widetilde{j}$ and $\widetilde{C}_k$, we use "tilde" in other notations (introduced for self-adjoint Dirac systems), when it goes about systems (2.5). For instance, clearly we have $\widetilde{m}_1 = m_2$, $\widetilde{m}_2 = m_1$. It is easy to see that the fundamental solution $\{\widetilde{W}_k(z)\}$ of systems (2.5) is connected with the fundamental solution $\{W_k(z)\}$ of (1.1) by the equality

$$\widetilde{W}_k(z) = W_k(-z). \tag{2.8}$$

Thus, according to (2.2) and (2.8), the function

$$\widetilde{\varphi}(z) = \varphi(-z), \tag{2.9}$$

where φ is the Weyl function of system (1.1), satisfies the inequalities

$$\sum_{k=0}^{\infty} q(z)^k \begin{bmatrix} I_{m_2} & \widetilde{\varphi}(z)^* \end{bmatrix} \widetilde{W}_k(z)^*\widetilde{C}_k\widetilde{W}_k(z) \begin{bmatrix} I_{m_2} \\ \widetilde{\varphi}(z) \end{bmatrix} < \infty \quad (z \in \mathbb{C}_+). \tag{2.10}$$

Therefore, by virtue of [5, Definition 3.6], the matrix function $\widetilde{\varphi}(z)$ is the Weyl function (on $\mathbb{C}_+$) of dual system (2.5). Moreover, we see that there is a one to one correspondence (2.6), (2.9) between systems (1.1) and (2.5) and their Weyl functions (on $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively). Hence, [5, Corollary 4.7] yields the theorem below.

Theorem 2.3. *Dirac system* (1.1)*,* (1.2) *is uniquely recovered from its Weyl function* $\varphi(z)$ $(z \in \mathbb{C}_-)$ *introduced by* (2.2)*.*

2. In order to consider the case of rational Weyl functions, we introduce the generalized Bäcklund–Darboux transformation (GBDT) of discrete Dirac systems. Each GBDT of the initial discrete Dirac system is determined by a triple $\{A, S_0, \Pi_0\}$ of parameter matrices. Here, we take a trivial initial system and choose $n \in \mathbb{N}$ $(n > 0)$, two $n \times n$ parameter matrices A $(\det A \neq 0)$ and $S_0 > 0$, and an $n \times m$ parameter matrix Π_0 such that

$$AS_0 - S_0A^* = \mathrm{i}\Pi_0 j \Pi_0^*. \tag{2.11}$$

Define recursively the sequences $\{\Pi_k\}$ and $\{S_k\}$ $(k > 0)$ by the relations

$$\Pi_{k+1} = \Pi_k + \mathrm{i}A^{-1}\Pi_k j, \tag{2.12}$$

$$S_{k+1} = S_k + A^{-1}S_k(A^*)^{-1} + A^{-1}\Pi_k\Pi_k^*(A^*)^{-1}. \tag{2.13}$$

From (2.11)–(2.13), the validity of the matrix identity

$$AS_r - S_rA^* = \mathrm{i}\Pi_r j \Pi_r^* \quad (r \geq 0), \tag{2.14}$$

follows by induction.

Definition 2.4. The triple $\{A, S_0, \Pi_0\}$, where $\det A \neq 0$, $S_0 > 0$ and (2.11) holds, is called admissible.

In view of (2.13), for the admissible triple we have $S_k > 0$ $(k \geq 0)$. Thus, the sequence

$$C_k := I_m + \Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1} \tag{2.15}$$

is well-defined. We say that the sequence $\{C_k\}$ is *determined* by the admissible triple $\{A, S_0, \Pi_0\}$. We will need also the matrix function w_A, which for each $k \geq 0$ is a so-called transfer matrix function in Lev Sakhnovich's form [18, 21, 22] and is defined by the relation

$$w_A(k, \lambda) := I_m - \mathrm{i}j\Pi_k^* S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k. \tag{2.16}$$

Now, similarly to [4, 9], we obtain the theorem below.

Theorem 2.5. *Let the triple $\{A, S_0, \Pi_0\}$ be admissible and assume that the recursions (2.12) and (2.13) are valid. Then the matrices C_k given by (2.15) (i.e., determined by $\{A, S_0, \Pi_0\}$) are well-defined and satisfy (1.2). Moreover, in this case the fundamental solution $\{W_k\}$ of Dirac system (1.1) admits the representation*

$$W_k(z) = w_A(k, -1/z)\,(I_m + \mathrm{i}zj)^k\, w_A(0, -1/z)^{-1} \quad (k \geq 0), \tag{2.17}$$

where w_A is defined in (2.16).

Proof. Recall that since $S_0 > 0$, relation (2.13) yields by induction that $S_k > 0$, and so the sequence $\{C_k\}$ is well-defined.

Next, formula (2.17) easily follows from the equality

$$w_A(k+1,\lambda)\left(I_m - \frac{\mathrm{i}}{\lambda}j\right) = \left(I_m - \frac{\mathrm{i}}{\lambda}jC_k\right) w_A(k,\lambda) \quad (k \geq 0), \tag{2.18}$$

which is proved quite similarly to the proof of [4, (2.24)] (and so we omit this proof here).

It remains to prove (1.2). The second equality in (1.2), that is, $C_k j C_k = j$, follows from (2.18) and the equalities

$$w_A(k,\lambda) j w_A(k,\overline{\lambda})^* = j, \tag{2.19}$$

which are to be found in [22] (see also [21, (1.84)]). Indeed, we can easily check that

$$\left(I_m - \frac{\mathrm{i}}{\lambda}j\right) j \left(I_m + \frac{\mathrm{i}}{\lambda}j\right) = \left(1 + \frac{1}{\lambda^2}\right) j, \tag{2.20}$$

and formulas (2.18)–(2.20) imply that

$$\left(I_m - \frac{\mathrm{i}}{\lambda}jC_k\right) j \left(I_m + \frac{\mathrm{i}}{\lambda}C_k j\right) = \left(1 + \frac{1}{\lambda^2}\right) j. \tag{2.21}$$

Clearly, the second equality in (1.2) is immediate from (2.21).

Finally, the first equality in (1.2) is proved in the same way as [4, Proposition 3.1]. □

3. It is convenient to partition Π_0 into the $n \times m_i$ blocks ϑ_i and to partition $w_A(0,\lambda)$ into the four blocks of the same orders as for j in (1.2):

$$\Pi_0 = [\vartheta_1 \quad \vartheta_2], \quad w_A(0,\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}. \tag{2.22}$$

Theorem 2.6. *Let a sequence $\{C_k\}$ and so Dirac system (1.1), (1.2) be determined by some admissible triple $\{A, S_0, \Pi_0\}$. Then the unique Weyl function of this system is given by the formula*

$$\varphi(z) = -\mathrm{i}z\vartheta_1^* S_0^{-1}\left(I_n + zA^\times\right)^{-1}\vartheta_2, \quad A^\times = A + \mathrm{i}\vartheta_2\vartheta_2^* S_0^{-1}. \tag{2.23}$$

Proof. Recall the definition (2.2) of the Weyl function $\varphi(z)$, where $q(z) = (1+|z|^2)^{-1}$. First, let us show that the summation formula

$$\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) = \frac{\mathrm{i}\left(1+|z|^2\right)}{(\overline{z}-z)}\left(q(z)^{r+1} W_{r+1}(z)^* j W_{r+1}(z) - j\right) \tag{2.24}$$

is valid. Indeed, according to (1.1) and (1.2), we have

$$\begin{aligned} W_{k+1}(z)^* j W_{k+1}(z) &= W_k(z)^*\left(I_m - \mathrm{i}\overline{z}C_k j\right) j \left(I_m + \mathrm{i}zjC_k\right) W_k(z) \\ &= q(z)^{-1} W_k(z)^* j W_k(z) + \mathrm{i}(z-\overline{z}) W_k(z)^* C_k W_k(z), \end{aligned}$$

that is,

$$q(z)^k W_k(z)^* C_k W_k(z) = \frac{\mathrm{i}q(z)^{k-1}}{(\overline{z}-z)}\left(q(z)W_{k+1}(z)^* jW_{k+1}(z) - W_k(z)^* jW_k(z)\right), \quad (2.25)$$

and (2.24) is immediate from (2.25).

Next, we will need the inequality

$$w_A\left(k, -\frac{1}{z}\right)^* j w_A\left(k, -\frac{1}{z}\right) \leq j \quad (z \in \mathbb{C}_-), \quad (2.26)$$

which together with (2.19), follows from a more general formula (see, e.g., [21, (1.88)]), of the form

$$w_A(k,\lambda)^* j w_A(k,\lambda) = j - \mathrm{i}(\lambda - \overline{\lambda})\Pi_k^*(A^* - \overline{\lambda}I_n)^{-1}S_k^{-1}(A-\lambda I_n)^{-1}\Pi_k. \quad (2.27)$$

Formulas (2.17) and (2.26) yield (in $\mathbb{C}_-$) the inequality

$$W_{r+1}(z)^* jW_{r+1}(z) \leq \left(w_A(0, -1/z)^{-1}\right)^* (I_m - \mathrm{i}\overline{z}j)^{r+1} j (I_m + \mathrm{i}zj)^{r+1} w_A(0, -1/z)^{-1}. \quad (2.28)$$

Setting

$$\varphi(z) = b(-1/z)d(-1/z)^{-1} \quad (2.29)$$

and taking into account (2.22) and (2.29), we derive

$$(I_m + \mathrm{i}zj)^{r+1} w_A(0, -1/z)^{-1}\begin{bmatrix}\varphi(z)\\ I_{m_2}\end{bmatrix} = (I_m + \mathrm{i}zj)^{r+1}\begin{bmatrix}0\\ I_{m_2}\end{bmatrix} d(-1/z)^{-1} = (1-\mathrm{i}z)^{r+1}\begin{bmatrix}0\\ I_{m_2}\end{bmatrix} d(-1/z)^{-1}. \quad (2.30)$$

It is immediate from (2.28) and (2.30) that

$$\begin{bmatrix}\varphi(z)^* & I_{m_2}\end{bmatrix} W_{r+1}(z)^* jW_{r+1}(z)\begin{bmatrix}\varphi(z)\\ I_{m_2}\end{bmatrix} \leq 0 \quad (z \in \mathbb{C}_-). \quad (2.31)$$

For $\varphi(z)$ given by (2.29), relations (2.24) and (2.31) imply that (2.2) holds, and thus $\varphi(z)$ is the Weyl function. (We did not discuss the singularities of $d(-1/z)$ and $d(-1/z)^{-1}$, but $\varphi(z)$ is analytic in $\mathbb{C}_-$ because it is meromorphic and it is the Weyl function.)

It remains to show that the right-hand sides of (2.23) and (2.29) coincide. By virtue of (2.16) and (2.22), using the inversion formula from the system theory (see, e.g., [21, Appendix B] and references therein), we obtain

$$b(\lambda)d(\lambda)^{-1} = -\mathrm{i}\vartheta_1^* S_0^{-1}(A-\lambda I_n)^{-1}\vartheta_2\left(I_{m_2} + \mathrm{i}\vartheta_2^* S_0^{-1}(A-\lambda I_n)^{-1}\vartheta_2\right)^{-1}$$

$$= -\mathrm{i}\vartheta_1^* S_0^{-1}(A - \lambda I_n)^{-1}\vartheta_2 \left(I_{m_2} - \mathrm{i}\vartheta_2^* S_0^{-1}(A^\times - \lambda I_n)^{-1}\vartheta_2 \right),$$

where $A^\times = A + \mathrm{i}\vartheta_2\vartheta_2^* S_0^{-1}$. Since $\mathrm{i}\vartheta_2\vartheta_2^* S_0^{-1} = A^\times - A = (A^\times - \lambda I_n) - (A - \lambda I_n)$, we essentially simplify the right-hand side in the formula above:

$$b(\lambda)d(\lambda)^{-1} = -\mathrm{i}\vartheta_1^* S_0^{-1}(A^\times - \lambda I_n)^{-1}\vartheta_2. \tag{2.32}$$

Hence, the right-hand sides of (2.23) and (2.29), indeed, coincide. □

4. We note that the Weyl function $\varphi(z)$ in (2.23) is rational and contractive on $\mathbb{C}_-$. Moreover, $\varphi(-1/z)$ is strictly proper rational and contractive. It is well known (see, e.g., [10, 12]) that each strictly proper rational $m_1 \times m_2$ matrix function $\psi(z)$ admits a representation (the so-called *realization*)

$$\psi(z) = \mathcal{C}(zI_n - \mathcal{A})^{-1}\mathcal{B}, \tag{2.33}$$

where $\mathcal{A}$ is an $n \times n$ matrix, $\mathcal{C}$ is an $m_1 \times n$ matrix and $\mathcal{B}$ is an $n \times m_2$ matrix. Further in the text we will assume that (2.33) is a *minimal realization*, that is, the value of n in (2.33) is minimal (among the corresponding values in different realizations of ψ). The following proposition is immediate from [19, Lemma 3.1] (and is based on several theorems from [12], for details, see [19]).

Proposition 2.7. *Assume that a strictly proper rational $m_1 \times m_2$ matrix function $\psi(z)$ is contractive on $\mathbb{C}_-$ and that (2.33) is its minimal realization. Then there is a unique Hermitian solution X of the Riccati equation*

$$X\mathcal{B}\mathcal{B}^*X - \mathrm{i}(\mathcal{A}^*X - X\mathcal{A}) + \mathcal{C}^*\mathcal{C} = 0 \tag{2.34}$$

such that the relation

$$\sigma(\mathcal{A} - \mathrm{i}\mathcal{B}\mathcal{B}^*X) \subset (\mathbb{C}_+ \cup \mathbb{R}) \tag{2.35}$$

holds. Moreover, this solution X is positive.

Next, we give an explicit procedure of solving the inverse problem to recover Dirac system from its Weyl function.

Theorem 2.8. *Let $\varphi(z)$ be a rational $m_1 \times m_2$ matrix function such that $\psi(z) = \varphi(-1/z)$ is a strictly proper rational matrix function, which is contractive on $\mathbb{R}$ and has no poles on $\mathbb{C}_-$. Assume that (2.33) is a minimal realization of ψ and that $X > 0$ is a solution of (2.34).*

Then $\varphi(z)$ is the Weyl function of the Dirac system (1.1), (1.2), the potential $\{C_k\}$ of which is determined by the admissible triple

$$A = \mathcal{A} - \mathrm{i}\mathcal{B}\mathcal{B}^*X, \;\; S_0 = X^{-1}, \;\; \vartheta_1 = \mathrm{i}X^{-1}\mathcal{C}^*, \;\; \vartheta_2 = \mathcal{B}, \;\; \Pi_0 = [\vartheta_1 \; \vartheta_2]. \tag{2.36}$$

Proof. Since $\psi(z)$ is contractive on $\mathbb{R}$ and has no poles on $\mathbb{C}_-$, it is contractive on $\mathbb{C}_-$. Thus, according to Proposition 2.7, a positive definite solution X of (2.34) exists. In view of (2.36), by choosing $X > 0$, we have $S_0 > 0$. Moreover, relations (2.34) and (2.35) yield the equality

$$\vartheta_2\vartheta_2^* + \mathrm{i}\left(\left(A + \mathrm{i}\vartheta_2\vartheta_2^* S_0^{-1}\right) S_0 - S_0 \left(A + \mathrm{i}\vartheta_2\vartheta_2^* S_0^{-1}\right)^*\right) + \vartheta_1\vartheta_1^* = 0, \tag{2.37}$$

which is equivalent to (2.11). Hence the triple $\{A, S_0, \Pi_0\}$ is admissible.

It remains to show that for the Weyl function $\varphi(z)$ of the Dirac system (determined by this triple), the function $\psi(z) = \varphi(-1/z)$ coincides with $\psi(z)$ admitting realization (2.33). Taking into account Theorem 2.6 and equalities (2.36), we see that $\psi(z)$ determined by our triple has the form

$$\psi(z) = \mathrm{i}\vartheta_1^* S_0^{-1}(zI_n - \mathcal{A})^{-1}\vartheta_2 = \mathcal{C}(zI_n - \mathcal{A})^{-1}\mathcal{B}, \tag{2.38}$$

and the right-hand sides of (2.33) and (2.38), indeed, coincide. □

3. Verblunsky-type coefficients and asymptotics of the potentials

Recall that the matrices C_k from the potential (sequence) $\{C_k\}$ are positive definite and j-unitary (i.e., they satisfy (1.2)). According to [5, Proposition 2.4], it means that they admit the representations

$$C_k = \mathcal{D}_k H_k, \quad \mathcal{D}_k := \mathrm{diag}\left\{\left(I_{m_1} - \rho_k\rho_k^*\right)^{-\frac{1}{2}}, \left(I_{m_2} - \rho_k^*\rho_k\right)^{-\frac{1}{2}}\right\}, \tag{3.1}$$

$$H_k := \begin{bmatrix} I_{m_1} & \rho_k \\ \rho_k^* & I_{m_2} \end{bmatrix} \quad (\rho_k^*\rho_k < I_{m_2}). \tag{3.2}$$

Here, the $m_1 \times m_2$ matrices ρ_k are the so-called Verblunsky-type coefficients, which were studied in detail in [20]. It is well known (see, e.g., [3]) that $\mathcal{D}_k H_k = H_k \mathcal{D}_k$. Clearly, $\rho_k^*\rho_k < I_{m_2}$ yields $\rho_k\rho_k^* < I_{m_1}$ and vice versa.

In this section, we show that

$$\lim_{k\to\infty} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} = I_{m_1}, \tag{3.3}$$

and so $\rho_k \to 0$ and $C_k \to I_m$. More precisely, we prove the following statement.

Theorem 3.1. *Let the triple $\{A, S_0, \Pi_0\}$ be admissible and assume that $-\mathrm{i} \notin \sigma(A)$. Then, for the potential $\{C_k\}$ (of the Dirac system (1.1)) determined by this triple, the asymptotic relations*

$$\lim_{k\to\infty} \rho_k = 0, \quad \lim_{k\to\infty} C_k = I_m \tag{3.4}$$

are valid.

Proof. Consider the equality

$$\begin{aligned} &S_{k+1} - \left(I_n + \mathrm{i}A^{-1}\right) S_k \left(I_n - \mathrm{i}(A^*)^{-1}\right) \\ &= S_{k+1} - S_k - A^{-1}S_k(A^*)^{-1} + \mathrm{i}A^{-1}(AS_k - S_kA^*)(A^*)^{-1}. \end{aligned} \tag{3.5}$$

Using (2.13) and (2.14), we rewrite (3.5):

$$S_{k+1} - \left(I_n + \mathrm{i}A^{-1}\right) S_k \left(I_n - \mathrm{i}(A^*)^{-1}\right) = A^{-1}\Pi_k(I_m - j)\Pi_k^*(A^*)^{-1}. \tag{3.6}$$

Now, we partition Π_k and, taking into account (2.12) and (2.22), write it down in the form

$$\Pi_k = \left[\left(I_n + \mathrm{i}A^{-1}\right)^k \vartheta_1 \quad \left(I_n - \mathrm{i}A^{-1}\right)^k \vartheta_2\right]. \tag{3.7}$$

In view of (3.6) and (3.7), setting

$$R_r := \left(I_n + \mathrm{i}A^{-1}\right)^{-r} S_r \left(I_n - \mathrm{i}(A^*)^{-1}\right)^{-r}, \tag{3.8}$$

we have

$$\begin{aligned} R_{k+1} - R_k = {} & 2\left(I_n + \mathrm{i}A^{-1}\right)^{-k-1} A^{-1}\left(I_n - \mathrm{i}A^{-1}\right)^k \vartheta_2 \\ & \times \vartheta_2^* \left(\left(I_n - \mathrm{i}A^{-1}\right)^k\right)^* \left(A^{-1}\right)^* \left(\left(I_n + \mathrm{i}A^{-1}\right)^{-k-1}\right)^* \geq 0. \end{aligned} \tag{3.9}$$

Since $R_0 = S_0 > 0$, relations (3.9) imply that there is a limit

$$\lim_{k\to\infty} R_k^{-1} = \varkappa_R \geq 0. \tag{3.10}$$

On the other hand, from (3.7) and (3.8), we derive

$$\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \Pi_k^* S_k^{-1} \Pi_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} = \vartheta_1^* R_k^{-1} \vartheta_1, \tag{3.11}$$

and so (3.10) yields

$$\lim_{k\to\infty} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \Pi_k^* S_k^{-1} \Pi_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} = \vartheta_1^* \varkappa_R \vartheta_1. \tag{3.12}$$

The definition (2.15) of C_k and the existence of the limit in (3.12) show that (3.3) holds. It is easy to see that the first equality in (3.4) follows from (3.1)–(3.3). Finally, the second equality in (3.4) is immediate from (3.1), (3.2) and the first equality in (3.4). □

Remark 3.2. According to Theorems 2.3, 2.6, 2.8 and Proposition 2.7, given a potential $\{C_k\}$ determined by some admissible triple we can recover another admissible triple $\{A, S_0, \Pi_0\}$, which determines the same sequence $\{C_k\}$ and has additional property $\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})$. Namely, we construct first the Weyl function using the initial triple and the procedure from Theorem 2.6. Next, we recover another admissible triple $\{A, S_0, \Pi_0\}$ such that $\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})$ in the process of solving the inverse problem.

Thus, we assume $\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})$ without loss of generality, and so the condition $-\mathrm{i} \notin \sigma(A)$ in Theorem 3.1 can be omitted.

We note that in the case of $\{C_k\}$ determined by some admissible triple, Verblunsky-type coefficients can be expressed explicitly. Indeed, in view of (3.1) and (3.2), we have

$$\rho_k = \left(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \tag{3.13}$$

Hence, taking into account (2.15) and (3.11), we derive

$$\begin{aligned} \rho_k = & \left(I_{m_1} + \vartheta_1^* R_k^{-1} \vartheta_1 - \vartheta_1^* R_{k+1}^{-1} \vartheta_1 \right)^{-1} \\ & \times \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \left(\Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1} \right) \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \end{aligned} \tag{3.14}$$

4. Stability of the procedure of solving the inverse problem

It is easy to see that the procedure (given in Theorem 2.8) to recover system (1.1), (1.2) consists of two steps. The first step is the construction of $X > 0$ and the second step is the construction of the potential $\{C_k\}$ using this X.

We start with the matrix function $\varphi(z)$ such that $\psi(z) = \varphi(-1/z)$ is a strictly proper rational $m_1 \times m_2$ matrix function which is contractive on $\mathbb{C}_-$. More precisely, we start with a minimal realization (2.33) of ψ (or, equivalently, with the triple $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$) and consider the stability of the recovery of $X > 0$ satisfying additional condition (2.35). The existence and uniqueness of $X > 0$ satisfying (2.35) follow from Proposition 2.7.

Definition 4.1. By $\mathcal{G}_n$, we denote the class of triples $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\}$ which determine minimal realizations $\widetilde{\psi}(z) = \widetilde{\mathcal{C}}\big(zI_n - \widetilde{\mathcal{A}}\big)^{-1}\widetilde{\mathcal{B}}$ of the $m_1 \times m_2$ matrix functions $\widetilde{\psi}(z)$ contractive on $\mathbb{C}_-$.

The recovery of $X > 0$ satisfying (2.34), (2.35) from the minimal realization (2.33) of $\psi(z)$ (where $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$) is called stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\}$, satisfying the conditions

$$\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\} \in \mathcal{G}_n, \quad \|\mathcal{A} - \widetilde{\mathcal{A}}\| + \|\mathcal{B} - \widetilde{\mathcal{B}}\| + \|\mathcal{C} - \widetilde{\mathcal{C}}\| < \delta, \tag{4.1}$$

there is a solution $\widetilde{X} = \widetilde{X}^*$ of the equation

$$\widetilde{X}\widetilde{\mathcal{B}}\widetilde{\mathcal{B}}^*\widetilde{X} - \mathrm{i}\big(\widetilde{\mathcal{A}}^*\widetilde{X} - \widetilde{X}\widetilde{\mathcal{A}}\big) + \widetilde{\mathcal{C}}^*\widetilde{\mathcal{C}} = 0 \tag{4.2}$$

in the neighbourhood $\|X - \widetilde{X}\| < \varepsilon$ of X.

The stability of the recovery of X follows (similarly to the case of the continuous Dirac system) from [19, Theorem 3.3] based on [16, Theorem 4.4]. Namely, applying [19, Theorem 3.3] to the triples $\{-\mathcal{A}, \mathcal{B}, -\mathcal{C}\}$ and $\{-\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, -\widetilde{\mathcal{C}}\}$, we get our next statement.

Proposition 4.2. *The recovery of $X > 0$, satisfying (2.34), (2.35), from the minimal realization (2.33) (with $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$) is stable.*

Remark 4.3. Note that (according to [16, Theorem 4.4]) we may consider a wider than $\mathcal{G}_n$ class of perturbed triples $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\}$, that is, such perturbed triples that (4.2) has a Hermitian solution $\widetilde{X} = \widetilde{X}^*$.

Recall that given the triple $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ and $X > 0$, we construct the matrices A, S_k, $R_k, \ldots$ For the matrices constructed in a similar way in the case of the triple $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\}$ and of $\widetilde{X} > 0$ satisfying

$$\widetilde{X}\widetilde{\mathcal{B}}\widetilde{\mathcal{B}}^*\widetilde{X} - \mathrm{i}\big(\widetilde{\mathcal{A}}^*\widetilde{X} - \widetilde{X}\widetilde{\mathcal{A}}\big) + \widetilde{\mathcal{C}}^*\widetilde{\mathcal{C}} = 0, \tag{4.3}$$

we use notations with "tilde": $\widetilde{A}$, $\widetilde{S}_k$, $\widetilde{R}_k, \ldots$

The stability of the second step of solving the inverse problem one can prove under the additional condition $\varkappa_R = 0$ or, equivalently,

$$\lim_{k\to\infty} R_k = +\infty, \tag{4.4}$$

which means that all the eigenvalues of R_k tend to infinity. Unlike the skew-self-adjoint case [6], equality (4.4) is not fulfilled automatically.

A sufficient condition of stability can also be expressed in terms of matrices Q_r introduced by the relations

$$Q_r := \left(I_n - \mathrm{i}A^{-1}\right)^{-r} S_r \left(I_n + \mathrm{i}(A^*)^{-1}\right)^{-r}. \tag{4.5}$$

Clearly, we assume in (4.5) that $\mathrm{i} \notin \sigma(A)$. Similarly to equality (3.6), from (2.13) and (2.14), we have

$$S_{k+1} - \left(I_n - \mathrm{i}A^{-1}\right) S_k \left(I_n + \mathrm{i}(A^*)^{-1}\right) = A^{-1}\Pi_k(I_m + j)\Pi_k^*(A^*)^{-1}. \tag{4.6}$$

Hence, taking into account (3.7) (in analogy with relation (3.9) for R_r), we derive

$$\begin{aligned} Q_{k+1} - Q_k = {} & 2\left(I_n - \mathrm{i}A^{-1}\right)^{-k-1} A^{-1}\left(I_n + \mathrm{i}A^{-1}\right)^k \vartheta_1 \\ & \times \vartheta_1^* \left(\left(I_n + \mathrm{i}A^{-1}\right)^k\right)^* \left(A^{-1}\right)^* \left(\left(I_n - \mathrm{i}A^{-1}\right)^{-k-1}\right)^* \geq 0. \end{aligned} \tag{4.7}$$

Since $Q_0 = S_0 > 0$, relations (4.7) imply that there is a limit

$$\lim_{k\to\infty} Q_k^{-1} = \varkappa_Q \geq 0. \tag{4.8}$$

Moreover, (3.7) and (4.5) yield

$$\lim_{k\to\infty} \begin{bmatrix} 0 & I_{m_1} \end{bmatrix} \Pi_k^* S_k^{-1} \Pi_k \begin{bmatrix} 0 & I_{m_1} \end{bmatrix} = \vartheta_2^* \varkappa_Q \vartheta_2. \tag{4.9}$$

Formula (4.9) implies that

$$\lim_{k\to\infty} \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} C_k \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} = I_{m_2}, \tag{4.10}$$

which gives another way of proving Theorem 3.1. The cases where (4.4) or the equality

$$\lim_{k\to\infty} Q_k = +\infty \tag{4.11}$$

holds are considered in the stability theorem below. (Recall that the sequence $\{R_k\}$ is given by (3.8) or, equivalently, by (3.9) together with (2.36) and $R_0 = S_0$.) In Proposition 4.5 at the end of this section we present a wide class where (4.11) is valid.

Theorem 4.4. *Consider the procedure (from Theorem 2.8) of the unique recovery of the potential $\{C_k\}$ of the discrete self-adjoint Dirac system (1.1), (1.2) from a minimal realization (2.33), where $\psi(z) = \varphi(-1/z)$ and $\varphi(z)$ is the Weyl function of the system (1.1), (1.2). Assume that X in this procedure is chosen such that (2.35) holds (which is always possible). Assume also that either the sequence $\{R_k\}$ satisfies (4.4) or $\mathrm{i} \notin \sigma(A)$ and the sequence $\{Q_k\}$ satisfies (4.11).*

Then this procedure of the recovery of the potential $\{C_k\}$ is stable in the class of triples from $\mathcal{G}_n$.

Proof. The recovery of $X > 0$ satisfying (2.34), (2.35) is possible according to Proposition 2.7 and stable according to Proposition 4.2.

Now, in order to show that the recovery of $\{C_k\}$ is stable under condition (4.4), we choose some small $\widehat{\varepsilon} > 0$ and such a large $N > 0$ and a small neighbourhood of $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ that $\|R_k^{-1}\| < \widehat{\varepsilon}$ and $\|\widetilde{R}_k^{-1}\| < 2\widehat{\varepsilon}$ for $X > 0$ satisfying (2.34), (2.35), for $k > N$, and for the matrices $\widetilde{X} > 0$ satisfying (4.3) (where the triples $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\} \in \mathcal{G}_n$ belong to the mentioned above neighbourhood of $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ and $\widetilde{X}$ are those solutions of (4.3) which belong to the neighbourhood of X). Here, we use the fact that the sequence $\{\widetilde{R}_k\}$ is monotonically increasing and if $\widetilde{R}_{r_0}$ is sufficiently large, then $\widetilde{R}_r$ $(r > r_0)$ is sufficiently large as well.

In view of (2.15) and (3.11), we see that for sufficiently small $\widehat{\varepsilon}$ the matrices

$$\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \widetilde{C}_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \tag{4.12}$$

are sufficiently close to I_{m_1}. This, in turn, means that (in view of (3.1) and (3.2)) the matrices ρ_k, $\widetilde{\rho}_k$ are sufficiently small, and thus C_k and $\widetilde{C}_k$ are sufficiently close to I_m. Therefore, for any $\varepsilon > 0$, we may choose $\widehat{\varepsilon}$ such that

$$\|C_k - \widetilde{C}_k\| < \varepsilon \quad \text{for all} \quad k > N(\hat{\varepsilon}).$$

Moreover, for any $\varepsilon > 0$, we may choose a neighbourhood of X and of $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ such that for $\{\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}\}$ from this neighbourhood the inequalities

$$\|C_k - \widetilde{C}_k\| < \varepsilon \quad (0 \le k \le N(\hat{\varepsilon}))$$

are valid as well. Thus, the recovery of $\{C_k\}$ is stable, indeed.

The stability of the recovery of $\{C_k\}$ under condition (4.11) is proved in a similar way. □

Now, consider the case where A is similar to a diagonal matrix D (A is diagonalisable):

$$A = UDU^{-1}. \tag{4.13}$$

Relations (2.35), (2.36) and (4.13) yield $\sigma(D) \in (\mathbb{C}_+ \cup \mathbb{R})$ or, equivalently,

$$\mathrm{i}(D^* - D) \geq 0. \tag{4.14}$$

Proposition 4.5. *Let the sequence $\{Q_k\}$ be given by (4.5), where A and $\{S_k\}$ are constructed with the use of the procedure from Theorem 4.4, A is diagonalizable (i.e., representation (4.13) holds) and $\mathrm{i} \notin \sigma(A)$. Then (4.11) is valid.*

Proof. According to (4.7), we have

$$Q_{k+n} - Q_k = 2(A - \mathrm{i}I_n)^{-n-k}(A + \mathrm{i}I_n)^k F (A^* - \mathrm{i}I_n)^k (A^* + \mathrm{i}I_n)^{-n-k}, \tag{4.15}$$

$$F := \sum_{\ell=1}^{n} (A - \mathrm{i}I_n)^{n-\ell}(A + \mathrm{i}I_n)^{\ell-1}\vartheta_1\vartheta_1^*(A^* - \mathrm{i}I_n)^{\ell-1}(A^* + \mathrm{i}I_n)^{n-\ell}, \tag{4.16}$$

where F does not depend on k. Let us show that F is strictly positive, that is, $F > 0$. Indeed, it is easy to see (more details are given in the similar part of the proof of [6, Proposition 4.10]) that

$$\mathrm{Span} \bigcup_{\ell=1}^{n} (A - \mathrm{i}I_n)^{n-\ell}(A + \mathrm{i}I_n)^{\ell-1}\vartheta_1 = \mathrm{Span} \bigcup_{\ell=1}^{n} A^{\ell-1}\vartheta_1,$$

and we have only to prove that the pair $\{A, \vartheta_1\}$ is controllable.

Since realization (2.33) is minimal, the pair $\{\mathcal{A}^*, \mathcal{C}^*\}$ is controllable. In view of (2.36), the controllability of the pair $\{X^{-1}\mathcal{A}^*X, \vartheta_1\}$ follows from the controllability of $\{\mathcal{A}^*, \mathcal{C}^*\}$. Hence, the equality

$$X^{-1}\mathcal{A}^*X = A - \mathrm{i}\vartheta_1\vartheta_1^*X \tag{4.17}$$

(which we derive below) implies that the pair $\{A, \vartheta_1\}$ is controllable as well.

Finally, using (2.36), we rewrite (2.11) in the form

$$AX^{-1} - X^{-1}A^* = \mathrm{i}(\vartheta_1\vartheta_1^* - \vartheta_2\vartheta_2^*).$$

This yields in turn that $X^{-1}A^*X = A + \mathrm{i}\mathcal{B}\mathcal{B}^*X - \mathrm{i}\vartheta_1\vartheta_1^*X$. Applying now the first equality from (2.36), we obtain (4.17). Thus $\{A, \vartheta_1\}$ is controllable and the inequality $F > 0$ is proved.

Next, we show that

$$(D - \mathrm{i}I_n)^{-1}(D + \mathrm{i}I_n)\left((D - \mathrm{i}I_n)^{-1}(D + \mathrm{i}I_n)\right)^* \geq I_n. \tag{4.18}$$

Inequality (4.18) is equivalent to the inequality

$$(D + \mathrm{i}I_n)(D^* - \mathrm{i}I_n) \geq (D - \mathrm{i}I_n)(D^* + \mathrm{i}I_n),$$

which follows from (4.14).

Now, formula (4.15), representation (4.13) and inequalities $F > 0$ and (4.18) imply that

$$Q_{k+n} - Q_k \geq \varepsilon I_n \tag{4.19}$$

for some $\varepsilon > 0$, which does not depend on k. The asymptotics (4.11) is immediate from (4.19). □

Appendix. Proof of Theorem 2.2

Proof. It is easy to see that

$$(I_m + \mathrm{i}\overline{z}jC_k)^* j\,(I_m + \mathrm{i}zjC_k) = \left(1+z^2\right) j, \tag{A.20}$$

and so both $(I_m + \mathrm{i}zjC_k)$ and $W_r(z) = \prod_{k=0}^{r-1}(I_m + \mathrm{i}zjC_k)$ are invertible for $z \neq \pm\mathrm{i}$. Now let us consider the sets $\mathcal{N}_r$ of the linear fractional transformations

$$\varphi_r(z,\mathcal{P}) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_r(z)^{-1}\mathcal{P}(z)\left(\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} W_r(z)^{-1}\mathcal{P}(z)\right)^{-1}, \tag{A.21}$$

where $\mathcal{P}(z)$ are nonsingular $m \times m_2$ matrix functions with property-j. That is, $\mathcal{P}(z)$ are meromorphic on $\mathbb{C}_-$ matrix functions such that the inequalities

$$\mathcal{P}(z)^*\mathcal{P}(z) > 0, \quad \mathcal{P}(z)^* j\mathcal{P}(z) \leq 0 \tag{A.22}$$

hold for all the points in $\mathbb{C}_-$ (excluding, possibly, discrete sets of points). The sets $\mathcal{N}_r$ are well-defined because the inequality

$$\det\left(\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} W_r(z)^{-1}\mathcal{P}(z)\right) \neq 0 \tag{A.23}$$

follows from (A.22). Indeed, since relations (1.2) and (2.4) yield

$$(I_m + \mathrm{i}zjC_k)^* j(I_m + \mathrm{i}zjC_k) = \left(1+|z|^2\right) j + \mathrm{i}(z-\overline{z})C_k \geq \widetilde{q}(z)j, \tag{A.24}$$

$$\widetilde{q}(z) := 1 + |z|^2 + \mathrm{i}(z-\overline{z}) > 0, \tag{A.25}$$

we have

$$W_r(z)^* jW_r(z) \geq \widetilde{q}(z)^r j, \quad \text{i.e.,} \quad \left(W_r(z)^{-1}\right)^* jW_r(z)^{-1} \leq \widetilde{q}(z)^{-r} j. \tag{A.26}$$

Thus, the inequalities

$$\mathcal{P}(z)^*\left(W_r(z)^{-1}\right)^* jW_r(z)^{-1}\mathcal{P}(z) \leq 0, \quad \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} j \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} < 0 \tag{A.27}$$

are valid, and (A.23) is immediate from [21, Proposition 1.43].

In view of (A.21), we have

$$\varphi_{r+1}(z,\mathcal{P}) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_r(z)^{-1}\widetilde{\mathcal{P}}(z)\left(\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} W_r(z)^{-1}\widetilde{\mathcal{P}}(z)\right)^{-1}, \tag{A.28}$$

where

$$\widetilde{\mathcal{P}}(z) = (I_m + \mathrm{i}zjC_r)^{-1}\mathcal{P}(z). \tag{A.29}$$

Relations (A.24), (A.25) and (A.29) imply that

$$\widetilde{\mathcal{P}}(z)^* j\widetilde{\mathcal{P}}(z) \leq 0. \tag{A.30}$$

Compare (A.21), (A.22) with (A.28), (A.30) to see that the sets (Weyl disks) $\mathcal{N}_r$ are embedded:

$$\mathcal{N}_{r+1} \subseteq \mathcal{N}_r. \tag{A.31}$$

Clearly, formulas (A.28)–(A.30) remain valid when we put there $r = 0$. For that case, we partition $\widetilde{\mathcal{P}}$ and (in view of (2.1)) rewrite (A.28) in the form

$$\varphi_1(z, \mathcal{P}) = \widetilde{\mathcal{P}}_1(z)\widetilde{\mathcal{P}}_2(z)^{-1}, \quad \widetilde{\mathcal{P}} =: \begin{bmatrix} \widetilde{\mathcal{P}}_1 \\ \widetilde{\mathcal{P}}_2 \end{bmatrix}, \tag{A.32}$$

where (according to (A.23) with $r = 1$) we have $\det \widetilde{\mathcal{P}}_2(z) \neq 0$. It follows from (A.30) and (A.32) that the functions from $\mathcal{N}_1$ are contractive. Hence, (A.31) implies that all the functions $\varphi_r(z, \mathcal{P})$ given by (A.21) are analytic and contractive in $\mathbb{C}_-$.

Next, using Montel's theorem and arguments from Step 1 in the proof of [5, Theorem 3.8], one can easily show that there is an analytic and contractive in $\mathbb{C}_-$ matrix function $\varphi_\infty(z)$ such that

$$\varphi_\infty \in \bigcap_{r\geq 1} \mathcal{N}_r. \tag{A.33}$$

(We note the functions $\begin{bmatrix} I_{m_1} \\ \varphi \end{bmatrix}$ in the proof of [5, Theorem 3.8] should be substituted by $\begin{bmatrix} \varphi \\ I_{m_2} \end{bmatrix}$ for our case of the Weyl functions in $\mathbb{C}_-$.) Taking into account (A.21) and (A.33), we write the representations

$$\begin{bmatrix} \varphi_\infty(z) \\ I_{m_2} \end{bmatrix} = W_{r+1}(z)\mathcal{P}(z, r+1) \quad (r \geq 0), \tag{A.34}$$

where $\mathcal{P}(z, r + 1)$ are nonsingular with property-j. Using the summation formula (2.24) and representation (A.34), we derive

$$\begin{bmatrix} \varphi_\infty^* & I_{m_2} \end{bmatrix} \sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) \begin{bmatrix} \varphi_\infty \\ I_{m_2} \end{bmatrix} \leq \frac{\mathrm{i}\left(1 + |z|^2\right)}{(\overline{z} - z)} I_{m_2}. \tag{A.35}$$

Compare (A.35) with Definition 2.1 of the Weyl function in order to see that φ_∞ is a Weyl function of (1.1), (1.2). Moreover, this Weyl function is analytic and contractive in $\mathbb{C}_-$. It remains to show that the Weyl function is unique.

First notice that (2.25) yields

$$q(z)W_{k+1}(z)^* jW_{k+1}(z) \geq W_k(z)^* jW_k(z) \quad (k \geq 0). \tag{A.36}$$

Thus, we have $q(z)^{k+1}W_{k+1}(z)^* jW_{k+1}(z) \geq j$, and so (2.4) implies that

$$\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \geq (r + 1)I_{m_1}. \tag{A.37}$$

Therefore, there is an m_1-dimensional subspace of vectors $g \in \mathbb{C}^m$ such that

$$\sum_{k=0}^{\infty} g^* q(z)^k W_k(z)^* C_k W_k(z) g = \infty. \tag{A.38}$$

Further proof of the uniqueness of the values, which the Weyl function may take at any fixed $z \in \mathbb{C}_-$, is easy and coincides with the arguments used in [5, Theorem 3.8]. □

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Дискретні самоспряжені системи Дірака загального типу: явні розв'язки прямої і оберненої задач, асимптотики коефіцієнтів типу Верблюнського та стійкість розв'язання оберненої задачі

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Розглянуто дискретні самоспряжені системи Дірака, визначені потенціалами (послідовностями) $\{C_k\}$ так, що матриці C_k є позитивно-визначеними та j-унітарними, де j — це діагональна матриця розміру $m \times m$, що має на головній діагоналі m_1 та m_2 елементів, які дорівнюють відповідно 1 та -1 $(m_1 + m_2 = m)$. У роботі побудовано системи з раціональними функціями Вейля та точно розв'язано обернену задачу відновлення системи за стискальними раціональними функціями Вейля. Крім цього, у роботі досліджується стійкість цієї процедури. Матриці C_k (з потенціалів) — це так звані розширення Халмоша коефіцієнтів ρ_k типу Верблюнського. У роботі доведено, що у випадку стискальної раціональної функції Вейля коефіцієнти ρ_k прямують до нуля, а матриці C_k прямують до одиничної матриці I_m.

Ключові слова: дискретна самоспряжена система Дірака, функція Вейля, обернена задача, явний розв'язок, стійкість розв'язання оберненої задачі, асимптотики потенціалу, коефіцієнт типу Верблюнського.