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Nonlocal Elasticity Theory as a Continuous Limit of 3D Networks of Pointwise Interacting Masses

Mariya Goncharenko and Eugen Khruslov

Small oscillations of an elastic system of point masses (particles) with a nonlocal interaction are considered. The asymptotic behavior of the system is studied when a number of particles tend to infinity and the distances between them and the forces of interaction tend to zero. The first term of the asymptotic is described by the homogenized system of equations, which is a nonlocal model of oscillations of elastic medium.

Key words: nonlocal elasticity, homogenization, integral model, Eringen model.

Mathematical Subject Classification 2010: 35Q70, 35Q74, 35B27.

1. Introduction

The progress in development of new materials and the modelling of nanostructures caused the emergence of nonlinear elasticity theories (see, for example, [7, 10, 15]). Classical local theory is based on the concept of contact interaction and it can not explain some observed experimental phenomena. Therefore, it is necessary to take into account the long-range interaction between the particles of the material that leads to the nonlocal elasticity theory.

The nonlocal elasticity theory can be traced back to the works of Kröner, who formulated the continuum theory of elastic materials with long-range interaction forces [11, 12]. At present, there are two different approaches considering the nonlocal mechanics of the elastic continuum: the gradient elasticity theory (weak nonlocality) and the integral nonlocal theory (strong nonlocality).

The first approach related to the study of the gradients of strain tensors leads to the models with spatial derivatives of order more than 2 [1,8,9]. The main difficulty in using these models is setting boundary conditions for the corresponding boundary value problems (see [21]).

The second approach was developed almost independently. The nonlocal interaction here is represented in the form of a convolution integral of the strain tensor with a kernel that depends on the distance between the particles of the elastic material. This approach leads to the models described by integro-differential equations [4, 5, 13, 14].

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The correctness of these continuum models of nonlocal elasticity theory depends on the effectiveness of long-range molecular forces (cohesive forces) in the material. Therefore, a natural approach to their justification is the so-called microstructural approach, which studies discrete elastic systems (lattice models). This approach was used mainly in physical works [3, 18, 23–25]. Apparently, [2] was one of the first mathematical works, in which the system of equations of the local elasticity theory was derived using the microstructural approach. The short-range interactions between the particles were considered when only the nearest particles interacted in the system. The asymptotic behavior of the oscillations of this system was studied when the distances between the nearest neighbors and the forces of interaction between them tended to zero. A homogenized system of differential equations describing the leading term of asymptotic was obtained. This system is a continuum model of the local theory of elasticity.

In this work, the method based on the studying of the asymptotic behavior of the system, when the scale of the microstructure tends to zero, is applied. The method used is the basis for the homogenization of partial differential equations [17, 20, 22].

We apply the homogenization to study the asymptotic behavior of the oscillations of an elastic system of point masses (particles) with a nonlocal interaction. It is assumed that the system depends on a small parameter ε. More precisely, the distance between the nearest neighbors is of order $O(\varepsilon)$, and the long-range cohesive forces are of order $O(\varepsilon^6)$. It is proved that the main term of asymptotic is described by a homogenized system of integro-differential equations. The integral term is a convolution of the difference of the displacements of the elastic medium at various space points with some kernel. A similar system of integro-differential equations was considered in [21] as a continuum model of nonlocal elasticity theory and used to calculate steel plates. Note that this system formally differs from the well-known Eringen model, where the convolution of the strain tensor with a kernel is taken. The Eringen model was proposed without justifications and any assumptions about cohesive forces [4, 11–14, 18]. In this paper, we assume that cohesive forces are of special form which provides a continuum nonlocal model reducible to the model of Eringen's type.

2. Statement of the problem

We consider a system M_ε of interacting point masses (we will call them particles) in a fixed bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. It is assumed that the system depends on a small parameter $\varepsilon > 0$. The total number of particles in the system is $O(\varepsilon^{-3})$ and the distances between the nearest particles are of order $O(\varepsilon)$. We denote by x_ε^i $(i = 1, \ldots, N_\varepsilon)$ the positions of the particles in the equilibrium state of the system M_ε, and we denote by $u_\varepsilon^i = u_\varepsilon^i(t)$ the displacements of the particles relative to their equilibrium positions x_ε^i.

The potential energy for a small deviation of the system M_ε from the equilibrium position is determined by the equality

$$H_\varepsilon(u_\varepsilon) = H_0 + \frac{1}{2}\sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}(u_\varepsilon^i - u_\varepsilon^j), (u_\varepsilon^i - u_\varepsilon^j)\rangle, \quad H_0 = \text{const}, \tag{2.1}$$

where $u_\varepsilon = \{u_\varepsilon^1, \ldots, u_\varepsilon^{N_\varepsilon}\}$, the parentheses $\langle , \rangle$ denote the scalar product in $\mathbb{R}^3$, and E_ε^{ij} are symmetric nonnegative matrices of the pair interaction between the i-th and the j-th particles. If the particles interact through the central elastic forces (for example, they are connected by elastic springs), then the matrices E_ε^{ij} satisfy the equalities

$$E_\varepsilon^{ij} u = K_\varepsilon^{ij}\langle u, e_\varepsilon^{ij}\rangle e_\varepsilon^{ij}, \quad \forall u \in \mathbb{R}^3, \tag{2.2}$$

where $e_\varepsilon^{ij} = (x_\varepsilon^i - x_\varepsilon^j)|x_\varepsilon^i - x_\varepsilon^j|^{-1}$ is the unit vector of direction between the i-th and the j-th particles and the coefficient K_ε^{ij} characterizes the intensity of interaction (the stiffness of springs).

The coefficient K_ε^{ij} depends on the distances $|x_\varepsilon^i - x_\varepsilon^j|$ between the particles. Generally speaking, it can be zero if in the corresponding pair the particles do not interact with each other. In this paper, we assume that the coefficient K_ε^{ij} is defined by the formula

$$K_\varepsilon^{ij} = \varepsilon^6\left[K(|x_\varepsilon^i - x_\varepsilon^j|) + \frac{K^{ij}}{|x_\varepsilon^i - x_\varepsilon^j|^5}\varphi\left(\frac{|x_\varepsilon^i - x_\varepsilon^j|}{\varepsilon}\right)\right] A_\varepsilon^{ij}, \tag{2.3}$$

where $K(r)$, $\varphi(r) \in C([0, L])$, $K(r) \geq 0$, $\varphi(r) = 1$ as $r \leq \alpha$ and $\varphi(r) = 0$ as $r \geq \beta$ $(0 < \alpha < \beta < L = \operatorname{diam}\Omega)$; $A_\varepsilon^{ij} = 1$ (for interacting pairs of particles) and $A_\varepsilon^{ij} = 0$ (for noninteracting pairs of particles), $a_0 \leq K^{ij} \leq A_0$.

The formula above simulates a weak interaction (of order $O(\varepsilon^6)$) between not very close particles ($|x_\varepsilon^i - x_\varepsilon^j| > \beta\varepsilon$) and a stronger interaction $O(\varepsilon)$ between close ones ($|x_\varepsilon^i - x_\varepsilon^j| < \alpha\varepsilon$) (see Fig. 2.1). This type of interaction is typical for some intermolecular forces (for example, van der Waals forces). The interaction energy

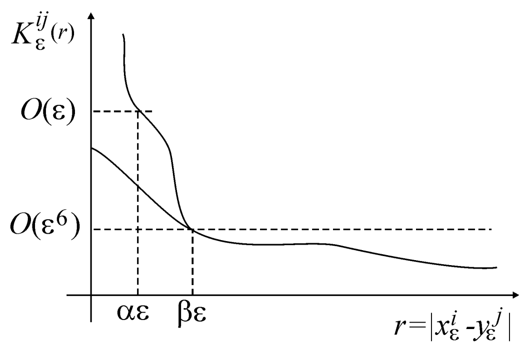

Fig. 2.1

of the system M_ε (2.1)–(2.3) is invariant under rotations and shifts. Therefore, the equilibrium state $(x_\varepsilon^1, \ldots, x_\varepsilon^{N_\varepsilon})$ of the system is not isolated: rotations and shifts are admitted. To exclude this, we fix a part of the particles $M_\varepsilon^0 \subset M_\varepsilon$ on the boundary $\partial\Omega$ (at the corresponding points $x_\varepsilon^i \in \partial\Omega$, $u_\varepsilon^i = 0$). We assume the following conditions hold.

I. *The condition of "ε-net" on the boundary $\partial\Omega$.* The set M_ε^0 of the particles assigned to $\partial\Omega$ is an ε-net for $\partial\Omega$. It is clear that the number of the particles is $N_\varepsilon^0 = O(\varepsilon^{-2}) \ll N_\varepsilon$

II. *The triangulation condition.* Let Γ_ε be a graph with vertices at the points x_ε^i and the edges $(x_\varepsilon^i, x_\varepsilon^j)$ $(i,j = 1,\dots,N_\varepsilon,\ i \neq j)$. Assume that for any $\varepsilon > 0$ there exists a subgraph $\Gamma_\varepsilon' \subset \Gamma_\varepsilon$ with the same set of vertices M_ε and edges of length $|x_\varepsilon^i - x_\varepsilon^j| = d^{ij}\varepsilon$ $(0 < d_1 \leq d^{ij} < d_2)$ that corresponds to the interaction coefficients $K_\varepsilon^{ij} = k^{ij}\varepsilon$ $(0 < a \leq k^{ij} \leq A)$. The subgraph Γ_ε' triangulates the domain Ω. The volumes $|P_\varepsilon^\alpha|$ of the corresponding simplexes of the triangulation P_ε^α $(\alpha = 1\dots\hat{N}_\varepsilon)$ satisfy the inequality $|P_\varepsilon^\alpha| > C\varepsilon^3$ $(C > 0)$.

Under these conditions, the equilibrium state $(x_\varepsilon^1,\dots,x_\varepsilon^{N_\varepsilon})$ is isolated. In the small neighborhood of $(x_\varepsilon^1,\dots,x_\varepsilon^{N_\varepsilon})$ the nonstationary oscillations of the system M_ε are described by the following problem:

$$m_\varepsilon^i \ddot{u}_\varepsilon^i = -\nabla_{u_\varepsilon^i} H_\varepsilon(u_\varepsilon^1,\dots,u_\varepsilon^{N_\varepsilon}), \qquad x_\varepsilon^i \in \Omega, t > 0, \tag{2.4}$$

$$u_\varepsilon^i(t) = 0, \qquad x_\varepsilon^i \in \partial\Omega, t > 0, \tag{2.5}$$

$$u_\varepsilon^i(0) = a_\varepsilon^i, \quad \dot{u}_\varepsilon^i(0) = b_\varepsilon^i, \qquad i = 1,\dots,N_\varepsilon, \tag{2.6}$$

where m_ε^i is the mass of the i-th particle, a_ε^i are the given initial displacements of the particles, b_ε^i are the given initial velocities ($a_\varepsilon^i = 0$, $b_\varepsilon^i = 0$ when $x_\varepsilon^i \in \partial\Omega$). There is the unique solution $\{u_\varepsilon\} = \{u_\varepsilon^1,\dots,u_\varepsilon^{N_\varepsilon}\}$ of this problem. The main goal of the paper is to study the asymptotic behavior of the solution as $\varepsilon \to 0$. We obtain a homogenized system of equations. This system describes the leading term of asymptotic. It is a macroscopic model of the oscillation of an elastic medium with nonlocal interaction.

3. Quantitative characteristics of the system of interacting particles and formulation of the main result

We denote by $K_h^x = K(x,h)$ the cubes with centers at the points $x \in \Omega$ and sides of length h with a fixed orientation. It is assumed that $0 < \varepsilon \ll h \ll 1$ and the cube K_h^x contains a large number of the particles of order $O\left(\frac{h^3}{\varepsilon^3}\right)$. Consider the functional of the symmetric tensor $T = \{T_{np}\}_{n,p=1}^3$:

$$H_{K_h^x}(T) = \inf_{v_\varepsilon} \left\{ \frac{1}{2} \sum_{\substack{i,j \\ |x_\varepsilon^i - x_\varepsilon^j| \leq \beta\varepsilon}}{}_{K_h^x} \langle E_\varepsilon^{ij}(v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j)\rangle \right.$$
$$\left. + \sum_i{}_{K_h^x} h^{-2-\gamma} \left| v_\varepsilon^i - \sum_{n,p=1}^3 \psi^{np}(x_\varepsilon^i) T_{np} \right|^2 \right\}. \tag{3.1}$$

The sum $\sum_{K_h^x}$ consists of the particles $x_\varepsilon^i \in K_h^x$ and the inf is taken over the displacements $v_\varepsilon = \{v_\varepsilon^i, i = 1, \dots, N_\varepsilon\}$ of the particles. The vector function $\psi^{np}(x)$ is defined by the equality $\psi^{np}(x) = \frac{1}{2}(x_n e^p + x_p e^n)$, and γ is an arbitrary penalty parameter $0 < \gamma < 2$.

The functional $H_{K_h^x}(T)$ is quadratic and we can rewrite it in the form

$$H_{K_h^x}(T) = \sum_{n,p,q,r=1}^{3} a_{npqr}(x; \varepsilon, h; \gamma) T_{np} T_{qr}, \tag{3.2}$$

where $a_{npqr}(x; \varepsilon, h; \gamma)$ are the components of the symmetric tensor of rank 4 in $\mathbb{R}^3$: $a_{npqr} = a_{qrnp} = a_{pnqr} = \dots$. This tensor is a mesoscopic ($0 < \varepsilon \ll h \ll 1$) characteristic of the concentration of the short-range interaction energy in the neighborhood of the point $x \in \Omega$.

Assume that the limits

$$\lim_{h\to 0} \underline{\lim}_{\varepsilon\to 0} \frac{a_{npqr}(x; \varepsilon, h; \gamma)}{h^3} = \lim_{h\to 0} \overline{\lim}_{\varepsilon\to 0} \frac{a_{npqr}(x; \varepsilon, h; \gamma)}{h^3} = a_{npqr}(x) \tag{3.3}$$

exist.

Remark 3.1. Formally, the limit tensor $\{a_{npqr}(x)\}_{n,p,q,r=1}^3$ must depend on the parameter γ and the orientation of the cubes $K(x,h)$. But the main result and the example in Section 6 show that the limiting tensor $\{a_{npqr}(x)\}_{n,p,q,r=1}^3$ does not depend on the parameter γ and the orientation of the cubes $K(x,h)$.

Let $\rho_\varepsilon(x) \in L_\infty(\Omega)$ be a density of the distribution of the mass of particles and let $\varphi_\varepsilon(x,y) \in L_\infty(\Omega \times \Omega)$ be a function of the distribution of the pairs of particles in $\Omega \times \Omega$ with long-range interaction. We will denote by V_ε^i ($i = 1, \dots, N_\varepsilon$) the Voronoi cells of a set of points $x_\varepsilon^i \in \Omega$,

$$V_\varepsilon^i = \bigcap_{j=1}^{N_\varepsilon} \{x \in \Omega : |x - x_\varepsilon^i| < |x - x_\varepsilon^j|\},$$

$|V_\varepsilon^i|$ is the volume of the cell and $\chi_\varepsilon^i(x)$ is a characteristic function of the cell. We put

$$\rho_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} \frac{m_\varepsilon^i}{|V_\varepsilon^i|} \chi_\varepsilon^i(x), \tag{3.4}$$

$$\varphi_\varepsilon(x,y) = \varepsilon^6 \sum_{\substack{i,j=1 \\ |x_\varepsilon^i - x_\varepsilon^j| \geq \beta\varepsilon}}^{N_\varepsilon} \frac{A_\varepsilon^{ij}}{|V_\varepsilon^i||V_\varepsilon^j|} \chi_\varepsilon^i(x) \chi_\varepsilon^j(y), \tag{3.5}$$

where m_ε^i are the masses of the particles, A_ε^{ij} are the elements of the adjacency matrix $A_\varepsilon = \{A_\varepsilon^{ij}\}_{i,j=1}^{N_\varepsilon}$ of the complete graph Γ_ε for the system M_ε (see (2.3)).

Suppose that for any $i = 1, \dots, N_\varepsilon$,

$$m_\varepsilon^i = m^i \varepsilon^3 \qquad (0 < m_1 \leq m_\varepsilon^i \leq m_2 < \infty). \tag{3.6}$$

By condition II, $|V_\varepsilon^i| = c_\varepsilon^i \varepsilon^3$ $(0 < C_1 \leq c^i \leq C_2 < \infty)$ and the estimates $\|\rho_\varepsilon\|_{L_\infty(\Omega)} < C$, $\|\varphi_\varepsilon\|_{L_\infty(\Omega\times\Omega)} < C$ are valid uniformly with respect to ε. Hence the set of functions $\{\rho_\varepsilon(x), \varepsilon > 0\}$ is *-weakly compact in $L_\infty(\Omega)$ and the set $\{\varphi_\varepsilon(x,y), \varepsilon > 0\}$ is *-weakly compact in $L_\infty(\Omega\times\Omega)$ (see [20, 22]).

We assume that

$$\rho_\varepsilon(x) \rightharpoonup \rho(x) \qquad \text{*-weakly in } L_\infty(\Omega), \tag{3.7}$$

$$\varphi_\varepsilon(x,y) \rightharpoonup \varphi(x,y) \qquad \text{*-weakly in } L_\infty(\Omega\times\Omega), \tag{3.8}$$

as $\varepsilon \to 0$. Here $\rho(x) > 0$ and $\varphi(x,y) \geq 0$ are the functions in $L_\infty(\Omega)$ and $L_\infty(\Omega\times\Omega)$ respectively. It follows from (3.5) that $\varphi(x,y) = \varphi(y,x)$.

To each discrete function $u_\varepsilon(x) = \{u_\varepsilon^1, \dots, u_\varepsilon^{N_\varepsilon}\}$ defined at the points x_ε^i: $u_\varepsilon(x_\varepsilon^i) = u_\varepsilon^i$ we will match the vector function $\tilde{u}_\varepsilon(x) \in L_\infty(\Omega)$ by the formula

$$\tilde{u}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} u_\varepsilon^i \chi_\varepsilon^i(x). \tag{3.9}$$

The vector-functions $\tilde{a}_\varepsilon(x) \in L_\infty(\Omega)$, $\tilde{b}_\varepsilon(x) \in L_\infty(\Omega)$ correspond to the initial data $\{a_\varepsilon^1, \dots, a_\varepsilon^{N_\varepsilon}\}$ and $\{b_\varepsilon^1, \dots, b_\varepsilon^{N_\varepsilon}\}$ in (2.4)–(2.6). The vector-function $\tilde{u}(x,t) \in L_\infty(\Omega \times [0,T])$ $\forall T > 0$ corresponds to the solution $\{u_\varepsilon^1(t), \dots, u_\varepsilon^{N_\varepsilon}(t)\}$ of the problem.

We assume that

$$\tilde{a}_\varepsilon(x) \to a(x), \quad \tilde{b}_\varepsilon(x) \to b(x) \quad \text{in } L_2(\Omega), \tag{3.10}$$

as $\varepsilon \to 0$. Here $a(x)$ and $b(x)$ are the vector functions from $\overset{\circ}{W}{}_2^1(\Omega)$. Suppose that the inequality

$$\sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}(a_\varepsilon^i - a_\varepsilon^j), (a_\varepsilon^i - a_\varepsilon^j)\rangle < C \tag{3.11}$$

holds uniformly with respect to ε.

Now we can formulate the main result.

Theorem 3.2. *Let the system of interacting particles M_ε with the interaction energy (2.1)–(2.3) and the masses m_ε^i (3.6) be located in $\bar{\Omega}$ and conditions I and II be fulfilled. Suppose that conditions (3.3), (3.7), (3.8) and (3.10), (3.11) hold as $\varepsilon \to 0$. Then the vector function $\tilde{u}_\varepsilon(x,t)$ constructed by (3.9) using the solution $u_\varepsilon(t) = \{u_\varepsilon^1(t), \dots, u_\varepsilon^{N_\varepsilon}(t)\}$ of problem (2.4)–(2.6) converges in $L_2(\Omega \times [0,T])$ as $\varepsilon \to 0$ to the solution $u(x,t)$ of the initial-boundary value problem:*

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q}\{a_{npqr}(x) e_{np}[u] e^r\} + \int_\Omega G(x,y)(u(x,t) - u(y,t))dy = 0, \qquad x \in \Omega,\ t > 0, \tag{3.12}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{3.13}$$

$$u(x,0) = a(x), \qquad \frac{\partial u}{\partial t}(x,0) = b(x), \qquad x \in \Omega. \tag{3.14}$$

Here $e_{np}[u] = \frac{1}{2}\left(\frac{\partial u_n}{\partial x_p} + \frac{\partial u_p}{\partial x_n}\right)$ *are the components of the elasticity tensor,* e^r *is the unit vector of the* x_r *axis, and the elements of the matrix* $G(x,y)$ *are defined by*

$$G_{kl}(x,y) = \frac{K(|x-y|)\varphi(x,y)}{|x-y|^2}(x_k - y_k)(x_l - y_l). \tag{3.15}$$

The theorem is proved in Sections 4, 5. In remainder of this section we give only the main ideas of the proof. In Section 4, we reduce the initial problem to a stationary problem with a spectral parameter λ ($\operatorname{Re}\lambda > 0$) by the Laplace transform in time. We introduce the variational formulation of the problem for real $\lambda > 0$. Then we study the asymptotic behavior of its solution as $\varepsilon \to 0$ and obtain the homogenized equation. In Section 5, we study analytic properties of solutions of the initial and homogenized stationary problems on λ for $\operatorname{Re}\lambda > 0$ using Vitali's theorem. We prove the convergence of the solutions and, finally, prove the convergence of the solutions of original nonstationary problem (2.3)–(2.6) to the solution of homogenized problem (3.11)–(3.13) with the help of the inverse Laplace transform.

4. Auxiliary propositions

Let us denote by $L^i_\varepsilon(x)$ a continuous function in $\mathbb{R}^3$ that is linear in every simplex $P^\alpha_{k\varepsilon}$ (condition II), and $L^i_\varepsilon(x^j_\varepsilon) = \delta_{ij}$ at x^i_ε. It is clear that $L^i_\varepsilon(x)$ is nonzero only in the simplexes with vertices x^i_ε.

Using this function, we construct a piecewise linear spline $\hat{u}_\varepsilon(x)$ to interpolate the given discrete vector-function $u_\varepsilon = \{u^1_\varepsilon, \ldots, u^{N_\varepsilon}_\varepsilon\}$:

$$\hat{u}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} u^i_\varepsilon L^i_\varepsilon(x), \tag{4.1}$$

where $u^i_\varepsilon = u_\varepsilon(x^i_\varepsilon)$.

Further, we will assume that $u^i_\varepsilon = 0$ for $x^i_\varepsilon \in \partial\Omega$. Then, $\hat{u}_\varepsilon(x) \in \overset{\circ}{W}{}^1_2(\Omega)$ for any $\varepsilon > 0$ if the domain Ω is convex. If Ω is not convex, then $\hat{u}_\varepsilon(x) \in \overset{\circ}{W}{}^1_2(\Omega_\delta)$ for sufficiently small $\varepsilon \le \varepsilon(\delta)$. Here Ω_δ is a domain in $\mathbb{R}^3$ such that $\Omega \Subset \Omega_\delta$ and $\operatorname{dist}(\partial\Omega, \partial\Omega_\delta) = \delta$ ($\forall\delta > 0$). This statement follows from conditions I, II, and the smoothness of $\partial\Omega$.

Lemma 4.1. *Let us construct the vector-functions* $\tilde{u}_\varepsilon(x)$ *and* $\hat{u}_\varepsilon(x)$ *by formulas (3.9) and (4.1) for the same set of vectors* $(u^1_\varepsilon, \ldots, u^{N_\varepsilon}_\varepsilon)$ $(u^i_\varepsilon = 0$ *for* $x^i_\varepsilon \in \partial\Omega)$. *If the inequality*

$$\|\hat{u}_\varepsilon\|_{W^1_2(\Omega)} < C$$

holds uniformly with respect to ε, *then*

$$\|\hat{u}_\varepsilon - \tilde{u}_\varepsilon\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Denote $v_\varepsilon(x) = \hat{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x)$. Let V_ε^i be the Voronoi cell at the point x_ε^i, and P_ε^α be a simplex with vertex at the point x_ε^i (see condition II). By (4.1) with $x \in V_\varepsilon^i \bigcap P_\varepsilon^\alpha$, we get

$$|\nabla v_\varepsilon(x)|^2 = |\nabla \hat{u}_\varepsilon(x)|^2 \equiv |M_\varepsilon^{i\alpha}|^2 = \text{const}. \tag{4.2}$$

Taking into account $v_\varepsilon(x_\varepsilon^i) = 0$, we obtain

$$v_\varepsilon(x) = \int_0^{|x-x_\varepsilon^i|} \frac{\partial v_\varepsilon}{\partial r}(x_\varepsilon^i + r(x - x_\varepsilon^i))dr, \quad x \in V_\varepsilon^i \bigcap P_\varepsilon^\alpha.$$

By this equality, condition II and (4.2), we have

$$|v_\varepsilon(x)|^2 \leq C\varepsilon^2 |M_\varepsilon^{i\alpha}|^2, \quad x \in V_\varepsilon^i \bigcap P_\varepsilon^\alpha,$$

and, consequently,

$$\int_\Omega |v_\varepsilon(x)|^2 dx = \sum_{i,j} \int_{V_\varepsilon^i \bigcap P_\varepsilon^\alpha} |v_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \sum_{i,\alpha} \int_{V_\varepsilon^i \bigcap P_\varepsilon^\alpha} |M_\varepsilon^{i\alpha}|^2 \, dx.$$

Thus, according to (4.2), the inequality

$$\int_\Omega |v_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_\Omega |\nabla \hat{u}_\varepsilon|^2 \, dx$$

holds, which establishes the assertion of the lemma. □

Consider the function $G_{\varepsilon kl}(x, y) \in L_\infty(\Omega \times \Omega)$ $(k, l = 1, 2, 3)$,

$$G_{\varepsilon kl}(x, y) = \varepsilon^6 \sum_{\substack{i,j=1 \\ |x_\varepsilon^i - x_\varepsilon^j| > \beta\varepsilon}}^{N_\varepsilon} \frac{K(|x_\varepsilon^i - x_\varepsilon^j|) e_{\varepsilon k}^{ij} e_{\varepsilon l}^{ij}}{|V_\varepsilon^i||V_\varepsilon^j|} A_\varepsilon^{ij} \chi_\varepsilon^i(x) \chi_\varepsilon^j(y), \tag{4.3}$$

where $e_{\varepsilon k}^{ij}$ are the k-th components of the vectors $e_\varepsilon^{ij} = (x_\varepsilon^i - x_\varepsilon^j)|x_\varepsilon^i - x_\varepsilon^j|^{-1}$.

Lemma 4.2. *Let conditions (3.8) hold. Then the function $G_{\varepsilon kl}(x, y)$ converges to the function*

$$G_{kl}(x, y) = \frac{K(|x - y|)\varphi(x, y)}{|x - y|^2}(x_k - y_k)(x_l - y_l) \tag{4.4}$$

**-weakly in $L_\infty(\Omega \times \Omega)$ as $\varepsilon \to 0$.*

Proof. Let $f(x, y)$ be an arbitrary function in $L_1(\Omega \times \Omega)$. By (4.3), we write

$$\int_\Omega \int_\Omega G_{\varepsilon kl}(x, y) f(x, y) \, dx \, dy$$
$$= \int_\Omega \int_\Omega \left(\varepsilon^6 \sum_{\substack{i,j=1 \\ |x_\varepsilon^i - x_\varepsilon^j| \geq \beta\varepsilon}} \frac{A_\varepsilon^{ij}}{|V_\varepsilon^i||V_\varepsilon^j|} \chi_\varepsilon^i(x) \chi_\varepsilon^j(y) \right) R_{kl}(x, y) f(x, y) \, dx \, dy$$

$$+\int_\Omega\int_\Omega \varepsilon^6 \sum_{\substack{i,j=1\\ |x_\varepsilon^i - x_\varepsilon^j|>\delta}} \left(R_{kl}(x_\varepsilon^i, x_\varepsilon^j - R_{kl}(x,y)\right) \frac{A_\varepsilon^{ij}}{|V_\varepsilon^i||V_\varepsilon^j|}\chi_\varepsilon^i(x)\chi_\varepsilon^j(y) f(x,y)\,dx\,dy$$

$$+\int_\Omega\int_\Omega \varepsilon^6 \sum_{\substack{i,j=1\\ \beta\varepsilon<|x_\varepsilon^i - x_\varepsilon^j|\le\delta}} \left(R_{kl}(x_\varepsilon^i, x_\varepsilon^j - R_{kl}(x,y)\right) \frac{A_\varepsilon^{ij}}{|V_\varepsilon^i||V_\varepsilon^j|}\chi_\varepsilon^i(x)\chi_\varepsilon^j(y) f(x,y)\,dx\,dy$$

$$= I_{kl}^{\varepsilon 1} + I_{kl}^{\varepsilon 2}(\delta) + I_{kl}^{\varepsilon 3}(\delta). \tag{4.5}$$

Here

$$R_{kl}(x,y) = K(x,y)\frac{(x_k - y_k)(x_l - y_l)}{|x-y|^2} \tag{4.6}$$

and δ is an arbitrary number $\delta \gg \varepsilon$.

Since $R_{kl}(x,y)f(x,y) \in L_\infty(\Omega\times\Omega)$,

$$\lim_{\varepsilon\to 0} I_{kl}^{\varepsilon 1} = \int_\Omega\int_\Omega G_{kl}(x,y)f(x,y)\,dx\,dy. \tag{4.7}$$

This follows from (3.5), (3.8) and (4.4), (4.6).

As $f(x,y) \in L_1(\Omega\times\Omega)$, the function $R_{kl}(x,y)$ is continuous for $|x-y| > \delta > 0$, and $\operatorname{diam} V_\varepsilon^i \le C\varepsilon$, $|V_\varepsilon^i| \ge C_2\varepsilon^3$ (condition II), we have

$$\lim_{\varepsilon\to 0} I_{kl}^{\varepsilon 2}(\delta) = 0 \tag{4.8}$$

and

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\, I_{kl}^{\varepsilon 2}(\delta) = 0 \tag{4.9}$$

for any fixed $\delta > 0$.

From (4.5)–(4.9) there follows the assertion of the lemma. □

The next lemma is fundamental for studying the compactness of discrete vector-valued functions. The well-known Korn inequality plays the same role for the functions in $\overset{\circ}{W}{}_2^1(\Omega)$ [19].

Lemma 4.3 (discrete analogue of the Korn inequality)**.** *Let conditions* I *and* II *hold. Then*

$$\sum_{i,j}{}' \langle E_\varepsilon^{ij}[u_\varepsilon^i - u_\varepsilon^j], [u_\varepsilon^i - u_\varepsilon^j]\rangle \ge C\|\hat{u}_\varepsilon\|^2_{W_2^1(\Omega)} \ge C_1\Big(\varepsilon\sum_{i,j}{}'|u_\varepsilon^i - u_\varepsilon^j|^2 + \varepsilon^3\sum_i{}'|u_\varepsilon^i|^2\Big)$$

for any discrete function $u_\varepsilon(x)$ defined at the points x_ε^i by $u_\varepsilon(x_\varepsilon^i) = u_\varepsilon^i$ $i = 1,\dots,N_\varepsilon$, and $u_\varepsilon^i = 0$ for $x_\varepsilon^i \in \partial\Omega$. Here E_ε^{ij} are the pair interaction matrices (*see* (2.1)–(2.3)), the sum $\sum_{i,j}{}'$ is taken over all (i,j) of the edges $(x_\varepsilon^j, x_\varepsilon^j)$ of the triangulation subgraph Γ_ε', and C, $C_1 > 0$ are the constants that do not depend on ε.

Lemma 4.3 was proved in [2].

The next lemma establishes the estimates of the solution $\{v_\varepsilon^i\}$ of problem (3.1). The lemma is given without the proof (for more details see [2]).

Lemma 4.4. *Let conditions* (3.2) *hold. Then*

$$\sum_{\substack{i,j\\ |x_\varepsilon^i - x_\varepsilon^j|\le\beta\varepsilon}} {}_{K_h^x}\langle E_\varepsilon^{ij}(v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j)\rangle = O(h^3),$$

$$\sum_{i} {}_{K_h^x}\Big| v_\varepsilon^i - \sum_{n,p}\psi^{np}(x_\varepsilon^i)T_{np}\Big|^2 \le O(h^{5+\gamma}),$$

$$\sum_{|x_\varepsilon^i - x_\varepsilon^j|\le\beta\varepsilon} {}_{K_h^x\setminus K_{h'}^x}\langle E_\varepsilon^{ij}(v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j)\rangle = o(h),$$

$$\sum_{i} {}_{K_h^x\setminus K_{h'}^x}\Big| v_\varepsilon^i - \sum_{n,p}\psi^{np}(x_\varepsilon^i)T_{np}\Big|^2 = o(h^{5+\gamma}),$$

where v_ε^i *is a solution of problem* (3.1)*;* $h' = h - 2h^{1+\gamma/2}$*,* $\gamma > 0$ *and* $\varepsilon \le \hat\varepsilon(h)$*.*

5. Variational formulation of the problem and asymptotic behavior of the solution as $\varepsilon \to 0$

By Laplace transform we convert the function $u_\varepsilon^i(t)$ of a real variable t to the function of a complex variable λ:

$$u_\varepsilon^i(t) \to u_\varepsilon^i(\lambda) = \int_0^\infty u_\varepsilon^i(t)e^{-\lambda t}\,dt, \quad i = 1,\dots,N_\varepsilon,\ \operatorname{Re}\lambda > 0.$$

Applying the Laplace transform to problem (2.4)–(2.6) and taking into account (2.1), we get the stationary problem for $u_\varepsilon(\lambda) = \{u_\varepsilon^1(\lambda),\dots,u_\varepsilon^{N_\varepsilon}(\lambda)\}$ with a spectral parameter λ:

$$\lambda^2 m_\varepsilon^i u_\varepsilon^i(\lambda) + \sum_{j=1}^{N_\varepsilon} E_\varepsilon^{ij}(u_\varepsilon^i(\lambda) - u_\varepsilon^j(\lambda)) = m_\varepsilon^i f_\varepsilon^i(\lambda), \qquad x_\varepsilon^i \in \Omega, \tag{5.1}$$

$$u_\varepsilon^i(\lambda) = 0, \qquad x_\varepsilon^i \in \partial\Omega, \tag{5.2}$$

where $f_\varepsilon^i(\lambda) = \lambda a_\varepsilon^i + b_\varepsilon^i$, $i = 1,\dots,N_\varepsilon$.

This problem has a unique solution for all $\lambda \in \mathbb{C}$, except the finite number of the spectrum points $\lambda = \pm i\mu_{\varepsilon k}$ ($\mu_{\varepsilon k} > 0$, $k = 1,\dots,N_\varepsilon' < N_\varepsilon$). For $\lambda = 0$, the problem describes the equilibrium elastic system under the action of the forces $m_\varepsilon^i f_\varepsilon^i$.

The solution $u_\varepsilon = \{u_\varepsilon^1,\dots,u_\varepsilon^{N_\varepsilon}\}$ of problem (5.1), (5.2) for $\lambda^2 \ge 0$ minimizes the functional

$$\Phi_\varepsilon[v_\varepsilon] = \frac{1}{2}\sum_{i,j=1}^{N_\varepsilon}\langle E_\varepsilon^{ij}[v_\varepsilon^i - v_\varepsilon^j], [v_\varepsilon^i - v_\varepsilon^j]\rangle + \lambda^2\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |v_\varepsilon^i|^2 - 2\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i\langle f_\varepsilon^i, v_\varepsilon^i\rangle \tag{5.3}$$

in the space $\overset{\circ}{J}_\varepsilon$ of discrete vector-functions $v_\varepsilon(x) = \{v_\varepsilon^1,\dots,v_\varepsilon^{N_\varepsilon}\}$ that equal 0 on $\partial\Omega$: $v_\varepsilon^i = 0$ when $x_\varepsilon^i \in \partial\Omega$. Thus, the vector-function $u_\varepsilon = \{u_\varepsilon^1,\dots,u_\varepsilon^{N_\varepsilon}\}$ is the solution of the minimization problem

$$\Phi_\varepsilon[u_\varepsilon] = \min_{v_\varepsilon \in \overset{\circ}{J}_\varepsilon} \Phi_\varepsilon[v_\varepsilon]. \tag{5.4}$$

To describe the asymptotic behavior of u_ε as $\varepsilon \to 0$, we introduce in $\overset{\circ}{W}{}_2^1(\Omega)$ the functional

$$\begin{aligned}\Phi[v] = &\int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr}(x) e_{np}[v] e_{qr}[v]\, dx + \lambda^2 \int_\Omega \rho(x)|v|^2\, dx \\ &+ \frac{1}{2}\int_\Omega\int_\Omega \langle G(x,y)(v(x)-v(y)), (v(x)-v(y)\rangle\, dx\, dy \\ &- 2\int_\Omega \rho(x)\langle f, v\rangle\, dx.\end{aligned} \tag{5.5}$$

Here

$$e_{np}[v] = \frac{1}{2}\left[\frac{\partial v_n}{\partial x_p} + \frac{\partial v_p}{\partial x_n}\right],$$

the tensor $\{a_{npqr}(x)\}_{n,p,q,r=1}^3$ is given by (3.3), the functions $\rho(x)$ and $\varphi(x,y)$ are defined by (3.7) and (3.8), the vector-function $f(x) = \lambda a(x) + b(x)$ is given by (3.10), and the elements of the matrix $G(x,y)$ are defined by (4.4).

Consider the minimization problem

$$\Phi[u] = \min_{w \in \overset{\circ}{W}{}_2^1(\Omega)} \Phi[w]. \tag{5.6}$$

Theorem 5.1. *Let conditions I, II, (2.2), (2.3) hold and let limits (3.3), (3.7), (3.8) exist as $\varepsilon \to 0$. Then the vector-function $\tilde{u}_\varepsilon(x)$ constructed by (3.9) on the solution $u_\varepsilon = \{u_\varepsilon^1, \dots, u_\varepsilon^{N_\varepsilon}\}$ of the minimization problem (5.4) converges in $L_2(\Omega)$ to the solution of the minimization problem (5.6).*

Proof. Taking into account that $\{0\} \in \overset{\circ}{J}_\varepsilon$ and $\Phi_\varepsilon[0] = 0$, we get the inequality

$$\begin{aligned}&\sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}[u_\varepsilon^i - u_\varepsilon^j], [u_\varepsilon^i - u_\varepsilon^j]\rangle + 2\lambda^2 \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |u_\varepsilon^i|^2 \\ &\qquad \le 4\left\{\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |f_\varepsilon^i|^2\right\}^{1/2} \left\{\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |u_\varepsilon^i|^2\right\}^{1/2}.\end{aligned} \tag{5.7}$$

From (3.4), (3.7), (3.10) and condition II it follows that

$$\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |f_\varepsilon^i|^2 \le C(|\lambda|^2 + 1), \tag{5.8}$$

where C does not depend on ε.

We construct the vector-function $\hat{u}_\varepsilon(x)$ by (4.1), where $u_\varepsilon = \{u_\varepsilon^1, \ldots, u_\varepsilon^{N_\varepsilon}\}$ is the solution of (5.4). By inequalities (5.7), (5.8) and Lemma 4.3, we get

$$\|\hat{u}_\varepsilon\|_{W_2^1(\Omega)} \le C. \tag{5.9}$$

The inequality is satisfied uniformly with respect to ε.

Thus, the set of vector-functions $\{\hat{u}_\varepsilon(x), \varepsilon > 0\}$ is a weakly compact set in $\overset{\circ}{W}{}_2^1(\Omega)$. We can extract a subsequence $\{\hat{u}_{\varepsilon_k}(x), \varepsilon_k \to 0\}$ that converges to the vector-function $u(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ weakly in $\overset{\circ}{W}{}_2^1(\Omega)$ and strongly in $L_q(\Omega)$ $(q \le 6)$.

By (3.9), we construct the subsequence $\{\tilde{u}_{\varepsilon_k}(x), \varepsilon_k \to 0\}$ for the set of vector-functions $\{u_{\varepsilon_k}^1, \ldots, u_{\varepsilon_k}^{N_\varepsilon}\}$. According to Lemma 4.1 and (5.9), the subsequence converges to $u(x)$ in $L_q(\Omega)$. Let us prove that $u(x)$ minimizes (5.6). For this purpose we write the functional Φ_ε (5.3) in the form

$$\Phi_\varepsilon[v_\varepsilon] = \Phi_{1\varepsilon}[v_\varepsilon] + \Phi_{2\varepsilon}[v_\varepsilon], \tag{5.10}$$

where

$$\Phi_{1\varepsilon}[v_\varepsilon] = \frac{1}{2} \sum_{\substack{i,j=1 \\ |x_\varepsilon^i - x_\varepsilon^j| \le \beta\varepsilon}}^{N_\varepsilon} \langle E_\varepsilon^{ij}(v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j)\rangle, \tag{5.11}$$

$$\Phi_{2\varepsilon}[v_\varepsilon] = \frac{1}{2} \sum_{\substack{i,j=1 \\ |x_\varepsilon^i - x_\varepsilon^j| \ge \beta\varepsilon}}^{N_\varepsilon} \langle E_\varepsilon^{ij}(v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j)\rangle + \lambda^2 \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |v_\varepsilon^i|^2 - 2 \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i \langle f_\varepsilon^i, v_\varepsilon^i\rangle. \tag{5.12}$$

Strong interactions between nearby particles are included in the functional $\Phi_{1\varepsilon}$. According to (2.2), (2.3), they are of order $O(\varepsilon)$ (see [2]). Recalling that $\hat{u}_\varepsilon \to u$ converges weakly in $\overset{\circ}{W}{}_2^1(\Omega)$, as $\varepsilon = \varepsilon_k \to 0$, and taking into account (3.2), we get the lower bound for $\Phi_{1\varepsilon}$ using the same method as in [2]:

$$\underline{\lim}_{\varepsilon=\varepsilon_k \to 0} \Phi_{1\varepsilon}[\tilde{u}_\varepsilon] \ge \Phi_1[u] = \int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr}(x) e_{np}[u] e_{qr}[u] dx. \tag{5.13}$$

By (2.2), (2.3), (3.4), (4.5), we can write $\Phi_{2\varepsilon}[v_\varepsilon]$ in the form

$$\Phi_{2\varepsilon}[\tilde{u}_\varepsilon] = \int_\Omega \int_\Omega \sum_{k,l=1}^{3} G_{\varepsilon kl}(\tilde{u}_{\varepsilon k}(x) - \tilde{u}_{\varepsilon k}(y))(\tilde{u}_{\varepsilon l}(x) - \tilde{u}_{\varepsilon l}(y))\, dx\, dy + \lambda^2 \int_\Omega \rho_\varepsilon(x) |\tilde{u}_\varepsilon|^2 dx - 2 \int_\Omega \rho_\varepsilon(x) \langle \tilde{f}_\varepsilon, \tilde{u}_\varepsilon\rangle\, dx, \tag{5.14}$$

where

$$\tilde{f}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} f_\varepsilon^i \chi_\varepsilon^i(x) = \sum_{i=1}^{N_\varepsilon} (\lambda a_\varepsilon^i + b_\varepsilon^i) \chi_\varepsilon^i(x),$$

and $\tilde{u}_{\varepsilon k}(x)$ is the k-th component of the vector-function $\tilde{u}_\varepsilon(x)$.

Since $\tilde{u}_\varepsilon \to u$ in $L_2(\Omega)$ as $\varepsilon = \varepsilon_k \to 0$, we have

$$(\tilde{u}_{\varepsilon k}(x) - \tilde{u}_{\varepsilon k}(y))(\tilde{u}_{\varepsilon l}(x) - \tilde{u}_{\varepsilon l}(y)) \to (u_k(x) - u_k(y))(u_l(x) - u_l(y)) \quad \text{in } L_2(\Omega \times \Omega)$$

and by (3.10),

$$\langle \tilde{f}_\varepsilon, \tilde{u}_\varepsilon \rangle \to \langle f, u \rangle \quad \text{in } L_1(\Omega),$$

where $f(x) = \lambda a(x) + b(x)$.

From the above, by Lemma 4.2, (3.7), and (5.14), we obtain

$$\lim_{\varepsilon=\varepsilon_k \to 0} \Phi_{2\varepsilon}[\tilde{u}_\varepsilon] = \Phi_2[u] = \int_\Omega \int_\Omega \langle G(x, y)(u(x) - u(y)), (u(x) - u(y)) \rangle \, dx \, dy + \lambda^2 \int_\Omega \rho(x) u^2(x) dx - 2 \int_\Omega \rho(x) \langle f(x), u(x) \rangle \, dx. \quad (5.15)$$

On the account of (5.10), (5.14), (5.15), we get the lower bound for $\Phi_\varepsilon[\tilde{u}_\varepsilon]$:

$$\varliminf_{\varepsilon=\varepsilon_k \to 0} \Phi_\varepsilon[\tilde{u}_\varepsilon] \geq \Phi[u], \quad (5.16)$$

where $u(x)$ is a limit in $L_2(\Omega)$ of the vector-functions $\tilde{u}_\varepsilon(x)$, and the functional $\Phi[u] = \Phi_1[u] + \Phi_2[u]$ is defined by (5.5).

In order to get the upper bound, we introduce the test vector-function $w_{\varepsilon k} = (w_{\varepsilon k}^1, \ldots, w_{\varepsilon k}^{N_\varepsilon})$ in $\mathring{J}_\varepsilon$ for problem (5.4). To this end, we cover Ω by the cubes $K_h^\alpha = K(x^\alpha, h)$ with centers at the points x^α and sides of length h. The centers of the cubes form a cubic lattice with a period $h - h^{1+\gamma/2}$ $(0 < \gamma < 2)$. By this covering, we construct a partition of the unity $\varphi_\alpha(x)$. Namely, a set of functions with the following properties: $\varphi_\alpha(x) \in C_0^2(K_h^\alpha)$, $\sum_\alpha \varphi_\alpha(x) = 1$, $\varphi_\alpha(x) = 0$ when $x \notin K_h^\alpha$, $\varphi_\alpha(x) = 1$ when $x \in K_h^\alpha \setminus \bigcup_{\beta \neq \alpha} K_h^\beta$; $|\nabla \varphi_\alpha(x)| \leq C h^{-1-\gamma/2}$.

Let $w(x)$ be an arbitrary vector-function in $C^2(\Omega)$ with a compact support in Ω. Define

$$w_{\varepsilon h}^i = \sum_\alpha \Big\{ w(x^\alpha) + \sum_{n,p=1}^{3} (e_{np}[w(x^\alpha)] v_{\varepsilon h}^{\alpha np}(x_\varepsilon^i) + \omega_{np}[w(x^\alpha)] \varphi^{np}(x_\varepsilon^i - x^\alpha)) \Big\} \varphi_\alpha(x_\varepsilon^i), \quad i = 1, \ldots, N_\varepsilon. \quad (5.17)$$

Here $v_{\varepsilon h}^{\alpha np}(x_\varepsilon^i)$ is a minimizer of the functional (3.1) in the cube K_h^α for $T = T^{np}$ $(T_{ik}^{np} = \delta_{in}\delta_{pk})$, $e_{np}[w]$, $\omega_{np}[w]$ are symmetric and antisymmetric parts of the tensor ∇w, $\varphi^{np}(x) = \frac{1}{2}(x_n e^p - x_p e^n)$.

Using the properties of the discrete vector-functions $v_{\varepsilon h}^{\alpha np}$ (see Lemma 4.4), the properties of the partition of the unity $\{\varphi_\alpha(x)\}$ and (3.2), we get

$$\lim_{h \to 0} \varlimsup_{\varepsilon \to 0} \Phi_{1\varepsilon}[w_{\varepsilon h}] \leq \Phi_1[w] \quad (5.18)$$

in the same way as in [2]. The functional Φ_1 is defined by (5.13).

To estimate $\Phi_{2\varepsilon}[w_{\varepsilon h}]$, we use the following equality for the vector-functions $w(x) \in C_0^2(\Omega)$ for $x \in K_h^\alpha$:

$$w(x) = w(x^\alpha) + \sum_{n,p} e_{np}[w(x^\alpha)]\psi^{np}(x - x^\alpha) + \omega_{np}[w(x^\alpha)]\varphi^{np}(x - x^\alpha) + O(h^2).$$

Substituting this equality in (5.17) and applying Lemma 4.4, we conclude

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \|w_{\varepsilon h} - w\|^2_{L_2(\Omega)} = 0. \tag{5.19}$$

Taking into account convergence (3.7), (3.10), and Lemma 4.2, we get

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \Phi_{2\varepsilon}[w_{\varepsilon h}] = \Phi_2[w], \tag{5.20}$$

where Φ_2 is defined by (5.15).

Thus, by (5.10), (5.18), (5.20), we obtain

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \Phi_\varepsilon[w_{\varepsilon h}] \le \Phi[w].$$

Recalling that u_ε is the minimizer of Φ_ε in $\overset{\circ}{J}_\varepsilon$ for sufficiently small h ($\varepsilon < \hat{\varepsilon}(h)$) and $w_{\varepsilon h} \in \overset{\circ}{J}_\varepsilon$, we can write

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \Phi_\varepsilon[u_\varepsilon] \le \Phi[w]. \tag{5.21}$$

Combining (5.16) and (5.21), we obtain

$$\Phi[u] \le \Phi[w], \quad w \in C_0^2(\Omega).$$

The inequality is valid for any vector-function $w \in \overset{\circ}{W}{}_1^2(\Omega)$ due to the continuity of the functional $\Phi[w]$ in $\overset{\circ}{W}{}_1^2(\Omega)$. Thus, by the subsequence $\varepsilon = \varepsilon_k \to 0$, the limit $u(x)$ of the vector-functions $\tilde{u}_\varepsilon(x)$ is a solution of minimizing problem (5.6). Hence, $u(x)$ is a weak solution of the boundary value problem:

$$\begin{aligned} \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} \{a_{npqr}(x) e_{np}[u] e^r\} + \lambda^2 \rho(x) u \\ + \int_\Omega \langle G(x,y)(u(x) - u(y))\rangle\, dy = \lambda a(x) + b(x), \quad & x \in \Omega, \quad (5.22) \\ u(x) = 0, \quad & x \in \partial\Omega. \quad (5.23) \end{aligned}$$

Since $\lambda \ge 0$, the function $\rho(x)$ and the matrix-function $G(x,y)$ are nonnegative, and the tensor $\{a_{npqr}(x)\}^3_{n,p,q,r=1}$ is positive definite, the problem has a unique solution. Thus, Theorem 5.1 is proved. □

6. The convergence of solutions of problem (5.1), (5.2) to the solution of problem (5.22), (5.23) for complex λ

1. Consider problem (5.1), (5.2) for complex λ in the semiaxis $\mathrm{Re}\lambda > 0$. Denote by L_ε a Hilbert space of finite sets of N_ε 3-component complex vectors defined in $x_\varepsilon^i \in \bar{\Omega}$: $u_\varepsilon = \{u_\varepsilon^1, \dots, u_\varepsilon^{N_\varepsilon}\}$. If $x_\varepsilon^i \in \partial\Omega$, then $u_\varepsilon^i = 0$. Define a scalar product in L_ε:

$$(u_\varepsilon, w_\varepsilon)_\varepsilon = \sum_{i=1}^{N_\varepsilon} \langle u_\varepsilon^i, \bar{w}_\varepsilon^i \rangle m_\varepsilon^i,$$

where m_ε^i is a mass of the point x_ε^i. By the parentheses $\langle \cdot, \cdot \rangle$, we denote the scalar product in $\mathbb{R}^3$. The bar denotes the complex conjugation. The corresponding norm is denoted by $\|u_\varepsilon\|_\varepsilon = (u_\varepsilon, \bar{u}_\varepsilon)_\varepsilon^{1/2}$.

Consider in L_ε a linear operator $A_\varepsilon : L_\varepsilon \to L_\varepsilon$:

$$(A_\varepsilon u_\varepsilon)_i = \begin{cases} \frac{1}{m_\varepsilon^i} \sum_{i=1}^{N_\varepsilon} E_\varepsilon^{ij} (u_\varepsilon^i - u_\varepsilon^j), & x_\varepsilon^i \in \Omega \\ 0, & x_\varepsilon^i \in \partial\Omega \end{cases}. \tag{6.1}$$

From (2.2), (2.3), (3.5), it follows that A_ε is a bounded selfadjoint operator in L_ε. By Lemma 4.3, A_ε is a positive definite operator (uniformly with respect to ε):

$$(A_\varepsilon u_\varepsilon, u_\varepsilon)_\varepsilon = \sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}(u_\varepsilon^i - u_\varepsilon^j), u_\varepsilon^i - u_\varepsilon^j \rangle \geq \alpha \|u_\varepsilon\|_\varepsilon^2, \quad (\alpha > 0). \tag{6.2}$$

Let us write problem (5.1), (5.2) in the operator form in L_ε:

$$A_\varepsilon u_\varepsilon + \lambda^2 u_\varepsilon = \lambda a_\varepsilon + b_\varepsilon. \tag{6.3}$$

By the indicated properties of the operator A_ε, its resolvent is a meromorphic operator function of the parameter $\tau = \lambda^2$ with poles on the negative semiaxis $\tau < 0$. Hence the solution $u_\varepsilon = u_\varepsilon(\lambda)$ of (6.3) is a holomorphic function of λ in the half-plane $\mathrm{Re}\lambda > 0$. Multiplying (6.3) on $\bar{u}_\varepsilon$ and separating the imaginary and real parts, taking into account (3.5) and (3.10), we obtain the estimate for u_ε in the half-plane $\mathrm{Re}\lambda > \sigma$ ($\forall \sigma > 0$), which is uniform with respect to ε: $\|u_\varepsilon\|_\varepsilon \leq C$ ($C = C(\sigma) < \infty$). This implies that the vector-function $\tilde{u}_\varepsilon = u_\varepsilon(x, \lambda)$ defined by (3.9) is a holomorphic function in $\mathrm{Re}\lambda > \sigma$ ($\forall \sigma > 0$). Moreover, $\tilde{u}_\varepsilon$ is bounded in the norm of $L_2(\Omega)$ uniformly with respect to ε:

$$\|\tilde{u}_\varepsilon\|_{L_2(\Omega)} \leq C < \infty. \tag{6.4}$$

2. We now turn to problem (5.22), (5.23). Denote by $L_2(\Omega, \rho)$ a Hilbert space of the complex-valued vector-functions in $L_2(\Omega)$ with a weight $\rho(x) > 0$. We define the scalar product in $L_2(\Omega, \rho)$ by

$$(u, w)_\rho = \int_\Omega u(x) \overline{w(x)} \rho(x) \, dx.$$

Consider a sesquilinear form defined on the set of vector-valued functions $C_0(\Omega)$ that is dense in $L_2(\Omega, \rho)$,

$$\hat{A}(u, w) = \frac{1}{\rho} \int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr} e_{np}[u] e_{qr}[\bar{w}]\, dx + \frac{1}{2\rho} \int_\Omega \langle G(x,y)[u(x) - u(y)], [\bar{w}(x) - \bar{w}(y)] \rangle\, dx\, dy.$$

Due to the properties of the elasticity tensor $\{a_{npqr}\}_{n,p,q,r=1}^{3}$ and the long-range matrix $G(x, y)$, the form generates a self-adjoint operator A in $L_2(\Omega, \rho)$ [22]. The equality

$$(Au, u)_\rho = \int_\Omega \sum_{n,p,q,r=1}^{3} a_{npqr}(x) |e_{np}[u]|^2\, dx + \frac{1}{2} \int_\Omega \langle G(x,y)[u(x) - u(y)], [\bar{u}(x) - \bar{u}(y)] \rangle\, dx\, dy$$

is valid. From Korn's inequality it follows that

$$(Au, u)_\rho \geq C \|u\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} \quad (C > 0). \tag{6.5}$$

This inequality implies that the operator A is positive definite and has a completely continuous inverse operator. Now we can write problem (5.22), (5.23) in the operator form

$$Au + \lambda^2 u = \lambda a + b. \tag{6.6}$$

The properties of the operator A imply that equation (6.6) has a solution $u(x)$ for complex λ ($\mathrm{Re}\lambda > 0$). This solution is a holomorphic function of λ satisfying the inequality

$$\|u\|_\rho < C.$$

3. By Theorem 3.2, the vector-function $\tilde{u}_\varepsilon(x, \lambda)$ converges in $L_2(\Omega)$ for $\lambda > 0$ to the solution $u(x, \lambda)$ of problem (5.22), (5.23) (or equation (6.6)) as $\varepsilon \to 0$. Moreover, the set of vector-functions $\{\tilde{u}_\varepsilon, \varepsilon > 0\}$ is bounded by the norm in $L_2(\Omega)$, uniformly with respect to ε in the half-plane $\mathrm{Re}\lambda > \sigma$ ($\forall \sigma > 0$). Therefore, using Vitali's theorem and taking into account that $u(x, \lambda)$ is holomorphic, we get the following assertion.

Theorem 6.1. *Let conditions I, II, (3.2), (3.6), (3.9) be fulfilled. Let us construct the function $\tilde{u}_\varepsilon(x, \lambda)$ by (3.9) on the solution of problem (5.1), (5.2). Then the vector-function $\tilde{u}_\varepsilon(x, \lambda)$ converges in $L_2(\Omega)$ to the solution $u(x, \lambda)$ of equation (6.6) (or problem (5.22), (5.23)) uniformly with respect to complex λ from the half-plane $Re\lambda > \sigma$ ($\forall \sigma > 0$).*

7. The end of the proof of the main theorem

By definition (6.1) of the operator A_ε, problem (2.4)–(2.6) in L_ε has the form

$$\ddot{u}_\varepsilon + A_\varepsilon u_\varepsilon = 0,$$
$$u_\varepsilon(0) = a_\varepsilon, \quad \dot{u}_\varepsilon(0) = b_\varepsilon.$$

From this on account of (6.1), we have

$$\|\dot{u}_\varepsilon\|_\varepsilon^2 + \sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}(u_\varepsilon^i - u_\varepsilon^j), (u_\varepsilon^i - u_\varepsilon^j)\rangle = \|b_\varepsilon\|_\varepsilon^2 + \sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}(a_\varepsilon^i - a_\varepsilon^j), (a_\varepsilon^i - a_\varepsilon^j)\rangle.$$

The equality above with discrete Korn's inequality, the properties of E_ε^{ij} and m_ε^i, and (3.9), (3.10), implies the inequality

$$\int_{\Omega_T} \left\{ \left(\frac{\partial \hat{u}_\varepsilon}{\partial t}\right)^2 + |\nabla \hat{u}_\varepsilon|^2 \right\} dx\, dt \leq CT \quad (\forall T > 0),$$

where $\hat{u}_\varepsilon = \hat{u}_\varepsilon(x,t)$ is a spline vector-function, defined by (4.1), and C does not depend on ε.

Thus the set of vector-functions $\{\hat{u}_\varepsilon, \varepsilon \to 0\}$ is bounded in $W_2^1(\Omega_T)$ uniformly with respect to ε. We can extract a subsequence $\{\hat{u}_\varepsilon, \varepsilon = \varepsilon_k \to 0\}$ which converges weakly in $W_2^1(\Omega_T)$ to a function $u(x,t) \in W_2^1(\Omega_T)$ (and by the embedding theorem, converges strongly in $L_q(\Omega \times (0,T))$ $(q \leq 4)$ and for almost all $t \in (0,T]$ converges strongly in $L_2(\Omega)$). By the above and Lemma 4.1, we conclude that the piecewise-constant vector-functions $\tilde{u}_\varepsilon(x,t)$, defined by (3.9), converge in $L_4(\Omega)$ and $L_2(\Omega)$ to $u(x,t)$ for almost all $t \in [0,T]$ as $\varepsilon = \varepsilon_k \to 0$.

Let us prove that the function $u(x,t)$ is a solution of problem (3.11)–(3.13). By the definition of the operator A, this problem can be written in the operator form

$$\ddot{u} + Au = 0, \tag{7.1}$$
$$u(0) = a, \quad \dot{u} = b. \tag{7.2}$$

The solution $u_\varepsilon(x,t)$ of problem (2.4)–(2.6) is an inverse Laplace transform of the solution $u_\varepsilon(x,\lambda)$ of problem (5.1), (5.2):

$$u_\varepsilon(x,t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} u_\varepsilon(x,\lambda)\, d\lambda, \quad \sigma > 0.$$

Thus we have

$$\tilde{u}_\varepsilon(x,t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} \tilde{u}_\varepsilon(x,\lambda)\, d\lambda, \tag{7.3}$$

where $\tilde{u}(x,t)$ and $\tilde{u}(x,\lambda)$ are defined by (3.9). Multiply the equality above by $\psi(x)\varphi(t)$, where $\psi(x) \in L_2(\Omega)$, $\varphi(t) \in C_0^2(0,T]$, and integrate over Ω_T. Changing the integration order and integrating on t by parts, we obtain

$$\int_{\Omega_T} \tilde{u}_\varepsilon(x,t)\psi(x)\varphi(t)\, dx\, dt = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda^2}\left(\int_{\Omega_T} \tilde{u}_\varepsilon(x,\lambda)\psi(x)\frac{\partial^2 \varphi}{\partial t^2}\, dx\, dt\right) d\lambda.$$

Note that due to (6.4), the integrals on λ in the right-hand side converge absolutely.

Let us pass to the limit in the equation above as $\varepsilon = \varepsilon_k \to 0$. We should take into account that $\tilde{u}_{\varepsilon_k}(x,t)$ converges to $u(x,t)$ in $L_2(\Omega)$ and $\tilde{u}_{\varepsilon_k}(x,\lambda)$ converges to the solution $u(x,\lambda)$ of equation (6.6) in $L_2(\Omega)$ uniformly on the compacts Λ in the half-plane Re$\lambda > 0$ (see Theorem 6.1). Hence we get

$$\int_{\Omega_T} u(x,t)\psi(x)\varphi(t)\,dx\,dt = \int_{\Omega_T} \left\{ \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} u(x,\lambda)e^{\lambda t} d\lambda \right\} \psi(x)\varphi(t)\,dx\,dt.$$

Since the linear combination of the functions $\psi(x)\varphi(t)$ form a dense set in $L_2(\Omega_T)$, then $u(x,t)$ is a solution of problem (7.1), (7.2). By the properties of the operator A, this problem has the unique solution. Thus $\tilde{u}_\varepsilon(x,t)$ converges to $u(x,t)$ in $L_2(\Omega_T)$ as $\varepsilon \to 0$. Theorem 3.2 is proved.

8. Periodic structure

We now consider the concrete case when the conditions of Theorem 3.2 are satisfied and the elastic tensor $\{a_{npqr}(x)\}$ and the matrix-function $G(x,y)$ are computed explicitly.

Suppose that the points x_ε^i of the equilibrium state of the system are located periodically. They form a cubic lattice with a period ε. Each point x_ε^i interacts with the tops of the cube x_ε^j. The points x_ε^i, x_ε^j belong to the same cube. For clarity, elastic springs are used to simulate the interaction. The stiffness of the springs (the elasticity coefficient in Hooke's law) directed along the edges of the cubes is $k_1\varepsilon^2$, directed along the diagonals of the faces of the cubes is $k_2\varepsilon$, and directed along the diagonals of the cubes is $k_3\varepsilon^2$ (Fig. 8.1). Thus we have modeled a strong short-range interaction. The corresponding coefficients of interaction

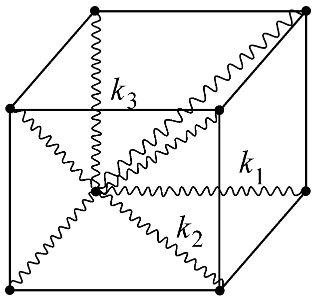

Fig. 8.1

(2.2) K_ε^{ij} are of order $O(\varepsilon)$, $K_\varepsilon^{ij} = k_1\varepsilon,\, 4k_2\varepsilon,\, 9k_3\varepsilon$.

Let us assume that there exists a long-range interaction. Each point x_ε^i interacts with the points x_ε^j of the cubic sublattice $\{x_\varepsilon^j\}^{(i)}$ with the period $N\varepsilon$ ($\exists N \in \mathbb{Z}$, $N \geq 2$), which is a weak interaction, and

$$K_\varepsilon^{ij} = \varepsilon^6 K|x_\varepsilon^i - x_\varepsilon^j|,$$

where $K(r)$ is a nonnegative function (see (2.3)).

The system of the points x_ε^i satisfies condition II. The corresponding interaction is described by (2.2), where $\alpha = \sqrt{3}$, $\beta = 2$ $K_{ij} = k_1, k_2, k_3$; $A_{ij} = 1$ only for $|x_\varepsilon^i - x_\varepsilon^j| = \varepsilon, \sqrt{2}\varepsilon, \sqrt{3}\varepsilon$ and for $|x_\varepsilon^i - x_\varepsilon^j| = \sqrt{k^2+l^2+m^2}N\varepsilon$ $(k, l, m = 1, 2, 3, \dots)$.

By (3.3), the limit dense $\varphi(x, y)$ is equal to $\frac{1}{N^3}$. Therefore, by (4.4),

$$G_{kl}(x,y) = \frac{K(|x-y|)(x_k-y_k)(x_l-y_l)}{N^3|x-y|^2}. \tag{8.1}$$

The components of the elasticity tensor for this system were calculated in [2] and determined by the formulas:

$$a_{nnnn} = k_1 + 2\frac{k_2}{\sqrt{2}} + \frac{4k_3}{3\sqrt{3}}, \quad a_{nnpp} = a_{npnp} = \frac{k_2}{\sqrt{2}} + \frac{4k_3}{3\sqrt{3}} \quad (n \neq p)$$

and $a_{npqr} = 0$ for other cases.

Remark 8.1. If we take $k_1 = \frac{k_2}{\sqrt{2}} + \frac{8k_3}{3\sqrt{32}}$, then the components of the limiting elasticity tensor satisfy the condition $a_{nnnn} = 2a_{npnp} + a_{nnpp}$, and the limit model of the elastic system is isotropic. Equation (3.12) has the form

$$\frac{\partial^2 u}{\partial t^2} - a\Delta u + 2a\nabla \mathrm{div} u + \int_\Omega G(x,y)(u(x)-u(y))\,dy = 0,$$

where $a = a_{nnpp} = a_{npnp}$, and the elements of the matrix $G(x, y)$ are defined by (8.1).

9. The model of Eringen's type

As it was mentioned above, we obtain the homogenized system of integro-differential equations (3.12) that formally differs from the Eringen continuum model of nonlocal elasticity. Now we will show that this system can be written as a model of Eringen's type. Using the method from [11], we introduce the tensor $\{t_{ikjl}\} = \{t_{kijl}(x,y)\}_{i,j,k,l=1}^3$, $x, y \in \Omega$ of rank 4, defined by the formulas,

$$t_{ikjl}(x,y) = \frac{\partial^2}{\partial x_j \partial y_l} \int_\Omega \int_\Omega \frac{G_{ik}(\xi,\eta)}{4\pi^2|x-\xi||y-\eta|}\,d\xi\,d\eta. \tag{9.1}$$

Here $G_{ik}(\xi, \eta)$ are components of the tensor $G(x, y)$ of rank 2 defined by (3.15). It follows from (9.1) and the symmetry of $G(x, y)$ that $t_{ikjl}(x, y)$ satisfy the equations

$$\sum_{j,l=1}^{3} \frac{\partial^2}{\partial x_j \partial y_l} t_{ikjl} = -G_{ik}, \quad x, y \in \Omega \times \Omega, \tag{9.2}$$

and have the symmetries $t_{ikjl}(x, y) = t_{kijl}(x, y) = t_{iklj}(x, y) = t_{jlik}(x, y)$.

Integrating by parts and taking into account (9.2), we obtain for each $u(x) \in \overset{\circ}{W}{}^1_2(\Omega)$:

$$\int_\Omega \int_\Omega \langle G(x,y)(u(x)-u(y)),(u(x)-u(y))\rangle \, dx\, dy$$
$$= \int_\Omega \int_\Omega \sum_{i,j,k,l=1}^{3} t_{ikjl}(x,y) \left[\frac{\partial u_i(y)}{\partial y_l} \frac{\partial u_k(x)}{\partial x_j} + \frac{\partial u_k(y)}{\partial y_l} \frac{\partial u_i(x)}{\partial x_j} \right] dx\, dy. \quad (9.3)$$

Following [11], we introduce a tensor of rank 4 with the components:

$$c_{ijkl}(x,y) = t_{ikjl}(x,y) + t_{jkil}(x,y) - t_{jilk}(x,y). \quad (9.4)$$

The tensor has the symmetries $c_{ijkl}(x,y) = c_{jikl}(x,y) = c_{ijlk}(x,y) = c_{klij}(x,y)$. Taking the above into account, we obtain from (9.3) and (9.4),

$$\int_\Omega \int_\Omega \langle G(x,y)(u(x)-u(y)),(u(x)-u(y))\rangle \, dx\, dy$$
$$\int_\Omega \int_\Omega c_{ijkl}\varepsilon_{ij}[u(x)]\varepsilon_{kl}[u(y)]\, dx\, dy, \quad (9.5)$$

where $\varepsilon_{ij}[u(x)] = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ are the components of the strain tensor.

Now we introduce a nonlocal stress tensor of Eringen's type with the components:

$$\sigma_{ij}[u(x)] = \sum_{i,j,k,l=1}^{3} a_{ijkl}(x)\varepsilon_{kl}[u(x)] + \int_\Omega \sum_{i,j,k,l=1}^{3} c_{ijkl}\varepsilon_{kl}[u(y)]\, dy.$$

Then, according to Theorem 3.2 and equalities (9.3), (9.4), we can rewrite homogenized equation (3.12) as follows:

$$\rho(x)\frac{\partial^2 u_i(x)}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial \sigma_{ij}[u]}{\partial x_j} = 0, x \in \Omega, \quad t > 0, \ i = 1,2,3.$$

Finally, we have obtained the equation which corresponds to the nonlocal model of elasticity theory of Eringen's type.

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Mariya Goncharenko,
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: marusya61@yahoo.co.uk

Eugen Khruslov,
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: khruslov@ilt.kharkov.ua

Нелокальна модель теорії пружності як неперервна межа 3D сітки точкових взаємодіючих мас

Mariya Goncharenko and Eugen Khruslov

Розглядаються малі коливання пружної системи точкових мас (часток) з нелокальною взаємодією. Вивчається асимптотичне поводження системи, коли кількість часток прямує до нескінченності, а відстані між ними та сили взаємодії прямують до нуля. Перший член асимптотики описується усередненою системою рівнянь, що є нелокальною моделлю коливань пружного середовища.

Ключові слова: нелокальна еластичність, усереднення, інтегральна модель, модель Ерінгена.