

UDC 519.6

## TWO-STEP METHOD FOR SOLVING NONLINEAR EQUATIONS WITH NONDIFFERENTIABLE OPERATOR

СТЕПАН ШАКНО, ХАЛІНА ЯРМОЛА

**РЕЗЮМЕ.** Запропоновано двокроковий метод для розв'язування нелінійних рівнянь з недиференційовним оператором, побудований на базі двох методів з порядком збіжності  $1 + \sqrt{2}$ . Вивчено локальну та напівлокальну збіжність запропонованого методу та встановлено порядок збіжності. Проведено числове дослідження на тестових задачах та зроблено порівняння отриманих результатів.

**ABSTRACT.** In this paper we propose a two-step method for solving nonlinear equations with a nondifferentiable operator. Its method is based on two methods of order of convergence  $1 + \sqrt{2}$ . We study a local and a semilocal convergence of the proposed method and set an order of convergence. We apply our results to the numerical solution of a nonlinear equation and systems of nonlinear equations.

### 1. INTRODUCTION

We consider the equation

$$H(x) \equiv F(x) + G(x) = 0, \quad (1)$$

where  $F$  and  $G$  are nonlinear operators, defined on a convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ .  $F$  is a Fréchet-differentiable operator,  $G$  is a continuous operator.

There are kinds of methods to find a solution of (1). In [1] Argyros studied the two-point iterative process

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots, \quad (2)$$

where  $A_n = A(x_{n-1}, x_n)$  is a bounded linear operator. There was provided a local and a semilocal convergence analysis for the method (2) and some cases where  $A_n = F'(x_n)$ ,  $A_n = F'(x_n) + G(x_{n-1}; x_n)$  were considered. Here  $G(x; y)$  is a first order divided difference of the operator  $G$  at the points  $x$  and  $y$ . The convergence analysis for the case where  $A_n = F'(x_n)$  was given by Zabrejko and Nguen [11]. In the paper [3] the convergence analysis results for modification of the method (2) for some cases of  $A_n$  were presented. There are studies in which there are considered difference methods, i.e., the secant method, the parametric secant method [5, 6] and the method based on the method of linear interpolation and the secant method [7]. In [4] Chen studied a Broyden-like method for solving (1). In [9] we researched a semilocal convergence of the

---

<sup>†</sup>*Key words.* Nondifferentiable operator, convergence order, local and semilocal convergence.

method (2) for  $A_n = F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})$ . The Newton's method cannot be applied, as differentiability of operator  $H$  is required.

In this work we propose a two-step method which is based on the methods with the order of convergence  $1 + \sqrt{2}$  [8, 10],

$$\begin{aligned} x_{n+1} &= x_n - \left[ F' \left( \frac{x_n + y_n}{2} \right) + G(x_n; y_n) \right]^{-1} (F(x_n) + G(x_n)), \\ y_{n+1} &= x_{n+1} - \left[ F' \left( \frac{x_n + y_n}{2} \right) + G(x_n; y_n) \right]^{-1} (F(x_{n+1}) + G(x_{n+1})), \\ n &= 0, 1, \dots \end{aligned} \quad (3)$$

Although the numbers of evaluations of the function values increases by one at each step for the proposed method (3), the convergence order is higher than for the one-step methods.

## 2. CONVERGENCE ANALYSIS

**Definition 1.** Let  $F$  be a nonlinear operator defined on a subset  $D$  of a linear space  $X$  with values in a linear space  $Y$  and let  $x, y$  be two points of  $D$ . A linear operator from  $X$  into  $Y$ , denoted as  $G(x; y)$ , which satisfies the condition

$$G(x; y)(x - y) = G(x) - G(y).$$

is called a divided difference of  $G$  at the points  $x$  and  $y$ .

**Theorem 1.** Let  $F$  and  $G$  be nonlinear operators, defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ .  $F$  is a twice Fréchet-differentiable operator,  $G$  is a continuous operator. Let us suppose that equation (1) has a solution  $x^* \in D$ ,  $G$  has a first order divided difference in  $D$  and there exist  $[A(x, y)]^{-1} = \left[ F' \left( \frac{x + y}{2} \right) + G(x; y) \right]^{-1}$  for all  $x \neq y$  and  $\|[A(x, y)]^{-1}\| \leq B$ . Let in  $D$  the following conditions fulfill

$$\|F'(x) - F'(y)\| \leq 2p_1\|x - y\|, \quad (4)$$

$$\|F''(x) - F''(y)\| \leq p_2\|x - y\|^\alpha, \quad \alpha \in (0, 1], \quad (5)$$

$$\|G(x; y) - G(u; v)\| \leq q_1(\|x - u\| + \|y - v\|). \quad (6)$$

Suppose that  $U = \{x : \|x - x^*\| < r_*\} \subset D$ , where  $r_*$  is the smallest positive zero of equations

$$q(r) = 1, \quad (7)$$

$$3B(p_1 + q_1)r q(r) = 1,$$

$$q(r) = B \left[ (p_1 + q_1)r + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r^{1+\alpha} \right].$$

Then the sequences  $\{x_n\}_{n \geq 0}$ ,  $\{y_n\}_{n \geq 0}$  generated by the iterative process (3) are well defined for all  $x_0, y_0 \in U$ , remain in  $U$  and converge to the solution  $x^*$ . Moreover, the following inequalities hold for all  $n \geq 0$

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq B \left[ (p_1 + q_1)\|y_n - x^*\| + \right. \\ &\quad \left. + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} \|x_n - x^*\|^{1+\alpha} \right] \|x_n - x^*\|, \end{aligned} \quad (8)$$

$$\begin{aligned} \|y_{n+1} - x^*\| \leq & B(p_1 + q_1) \left[ \|y_n - x^*\| + \right. \\ & \left. + \|x_n - x^*\| + \|x_{n+1} - x^*\| \right] \|x_{n+1} - x^*\|. \end{aligned} \quad (9)$$

*Proof.* Since the following equality holds for all  $x, h \in D$  [10]

$$F(x+h) = F(x) + F'(x)h + \int_0^1 (1-t)F''(x+th)h^2 dt,$$

then

$$\begin{aligned} & F(x_n) - F(x^*) - F'\left(\frac{x_n + x^*}{2}\right)(x_n - x^*) = \\ & = F(x_n) - F\left(\frac{x_n + x^*}{2}\right) - F'\left(\frac{x_n + x^*}{2}\right)\frac{x_n - x^*}{2} - \\ & - \left[ F(x^*) - F\left(\frac{x_n + x^*}{2}\right) - F'\left(\frac{x_n + x^*}{2}\right)\frac{x^* - x_n}{2} \right] = \\ & = \int_0^1 (1-t)F''\left(\frac{x_n + x^*}{2} + t\frac{x_n - x^*}{2}\right)\frac{x_n - x^*}{2}\frac{x_n - x^*}{2} dt - \\ & - \int_0^1 (1-t)F''\left(\frac{x_n + x^*}{2} + t\frac{x^* - x_n}{2}\right)\frac{x_n - x^*}{2}\frac{x_n - x^*}{2} dt. \end{aligned} \quad (10)$$

Using the condition (5) and the equality (10), we obtain

$$\begin{aligned} & \left\| F(x_n) - F(x^*) - F'\left(\frac{x_n + x^*}{2}\right)(x_n - x^*) \right\| \leq \\ & \leq \frac{p_2 \|x_n - x^*\|^{2+\alpha}}{4} \int_0^1 (1-t)t^\alpha dt = \frac{p_2 \|x_n - x^*\|^{2+\alpha}}{4(\alpha+1)(\alpha+2)}. \end{aligned} \quad (11)$$

Let us choose  $x_0 \in U$  and show that the sequences given in (3) are well defined. We denote  $A_n = F'\left(\frac{x_n + y_n}{2}\right) + G(x_n; y_n)$ . If  $x_n, y_n \in U$ , then from the definition of the first order divided difference and (4), (6), (11), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| & = \|x_n - x^* - A_n^{-1}(F(x_n) + G(x_n) - F(x^*) - G(x^*))\| \leq \\ & \leq \|A_n^{-1}\| \left\| F(x_n) - F(x^*) - F'\left(\frac{x_n + x^*}{2}\right)(x_n - x^*) \right\| + \\ & + \|A_n^{-1}\| \left\| F'\left(\frac{x_n + x^*}{2}\right) - F'\left(\frac{x_n + y_n}{2}\right) \right\| \|x_n - x^*\| + \\ & + \|A_n^{-1}\| \|G(x_n; x^*) - G(x_n; y_n)\| \|x_n - x^*\| \leq \\ & \leq B \left[ (p_1 + q_1) \|y_n - x^*\| + \frac{p_2}{4(\alpha+1)(\alpha+2)} \|x_n - x^*\|^{1+\alpha} \right] \|x_n - x^*\| \end{aligned}$$

and

$$\begin{aligned} \|y_{n+1} - x^*\| & = \|x_{n+1} - x^* - A_n^{-1}(F(x_{n+1}) + G(x_{n+1}) - F(x^*) - G(x^*))\| \leq \\ & \leq \|A_n^{-1}\| \left\| \int_0^1 \left\{ F'(x^* + t(x_{n+1} - x^*)) - F'\left(\frac{x_n + y_n}{2}\right) \right\} dt \right\| \|x_{n+1} - x^*\| + \end{aligned}$$

$$\begin{aligned}
& + \|A_n^{-1}\| \|G(x_{n+1}; x^*) - G(x_n; y_n)\| \|x_{n+1} - x^*\| \leq \\
& \leq B(p_1 + q_1) (\|y_n - x^*\| + \|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x^*\|.
\end{aligned}$$

We prove that inequalities (8) and (9) are fulfilled. Taking  $n = 0$  above, we obtain

$$\|x_1 - x^*\| < B \left[ (p_1 + q_1)r_* + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r_*^{1+\alpha} \right] \|x_0 - x^*\| \leq \|x_0 - x^*\| < r_*$$

and

$$\begin{aligned}
\|y_1 - x^*\| & < 3B^2(p_1 + q_1) \left[ (p_1 + q_1)r_* + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r_*^{1+\alpha} \right] r_* \|x_0 - x^*\| \leq \\
& \leq \|x_0 - x^*\| < r_*.
\end{aligned}$$

Therefore,  $x_1, y_1 \in U$ . If  $\|x_n - x^*\| < r_*$  and  $\|y_n - x^*\| < r_*$  then from (7) – (9), it follows

$$\begin{aligned}
\|x_{n+1} - x^*\| & < B \left[ (p_1 + q_1)r_* + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r_*^{1+\alpha} \right] \|x_n - x^*\| \leq \\
& \leq \|x_n - x^*\| < \dots < r_*, \\
\|y_{n+1} - x^*\| & < 3B^2(p_1 + q_1) \left[ (p_1 + q_1)r_* + \right. \\
& \quad \left. + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r_*^{1+\alpha} \right] r_* \|x_n - x^*\| \leq \\
& \leq \|x_n - x^*\| < \dots < r_*.
\end{aligned}$$

So, iterative process (3) is well defined, the sequences  $\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0}$  belong to  $U$ . From the last inequalities and estimates (8) and (9) we can see that  $\{x_n\}_{n \geq 0}$  and  $\{y_n\}_{n \geq 0}$  converge to  $x^*$ .  $\square$

**Corollary 2.** *Let us suppose that the hypotheses of Theorem 1 hold. Then the iterative process (3) converges to a solution  $x^*$  of the equation (1) with the order of convergence  $1 + \sqrt{1 + \alpha}$ .*

*Proof.* We denote

$$a_n = \|x_n - x^*\|, \quad b_n = \|y_n - x^*\|, \quad C_1 = B(p_1 + q_1), \quad C_2 = \frac{Bp_2}{4(\alpha + 1)(\alpha + 2)}.$$

By (8) and (9), we get

$$a_{n+1} \leq C_1 a_n b_n + C_2 a_n^{2+\alpha},$$

$$\begin{aligned}
b_{n+1} & \leq C_1(a_{n+1} + a_n + b_n)a_{n+1} \leq C_1(2a_n + b_n)a_{n+1} \leq \\
& \leq C_1(2a_n + C_1(2a_0 + b_0)a_n)a_{n+1} = C_1(2 + C_1(2a_0 + b_0))a_n a_{n+1},
\end{aligned}$$

Then for large  $n$  and  $a_{n-1} < 1$ , from previous inequalities, we obtain

$$\begin{aligned}
a_{n+1} & \leq C_1 a_n b_n + C_2 a_n^2 a_{n-1}^\alpha \leq \\
& \leq C_1^2 (2 + C_1(2a_0 + b_0)) a_n^2 a_{n-1} + C_2 a_n^2 a_{n-1}^\alpha \leq \\
& \leq [C_1^2 (2 + C_1(2a_0 + b_0)) + C_2] a_n^2 a_{n-1}^\alpha.
\end{aligned} \tag{12}$$

From (12) we can write down an equation of the convergence order of the iterative process (3):  $t^2 - 2t - \alpha = 0$ . The order of convergence is the unique positive solution  $t^* = 1 + \sqrt{1 + \alpha}$ . If  $\alpha = 1$ , we get that the iterative process (3) converges to the solution of the equation (1) with the order  $1 + \sqrt{2}$ .  $\square$

**Theorem 2.** *Let  $F$  and  $G$  be nonlinear operators, defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ .  $F$  is a Fréchet-differentiable operator,  $G$  is a continuous operator. We assume that  $U_0 = \{x : \|x - x_0\| \leq r_0\}$  is contained in  $D$ , the linear operator  $A_0 = F' \left( \frac{x_0 + y_0}{2} \right) + G(x_0; y_0)$ , where  $x_0, y_0 \in D$ , is invertible and the Lipschitz conditions are fulfilled*

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq 2p_0\|x - y\|, \quad (13)$$

$$\|A_0^{-1}(G(x; y) - G(u; v))\| \leq q_0(\|x - u\| + \|y - v\|). \quad (14)$$

Let's  $a, c$  ( $c > a$ ),  $r_0$  be non-negative numbers such that

$$\|x_0 - x_{-1}\| \leq a, \quad \|A_0^{-1}(F(x_0) + G(x_0))\| \leq c, \quad (15)$$

$$r_0 \geq c/(1 - \gamma), \quad (p_0 + q_0)(2r_0 - a) < 1,$$

$$\gamma = \frac{(p_0 + q_0)(r_0 - a) + 0.5p_0r_0}{1 - (p_0 + q_0)(2r_0 - a)}, \quad 0 \leq \gamma < 1.$$

Then the following inequalities hold for all  $n \geq 0$

$$\|x_n - x_{n+1}\| \leq t_n - t_{n+1}, \quad \|y_n - x_{n+1}\| \leq s_n - t_{n+1}, \quad (16)$$

$$\|x_n - x^*\| \leq t_n - t^*, \quad \|y_n - x^*\| \leq s_n - t^*, \quad (17)$$

where

$$t_0 = r_0, \quad s_0 = r_0 - a, \quad t_1 = r_0 - c, \\ t_{n+1} - t_{n+2} = \frac{(p_0 + q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1 - (p_0 + q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]}(t_n - t_{n+1}), \quad (18)$$

$$t_{n+1} - s_{n+1} = \frac{(p_0 + q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1 - (p_0 + q_0)[(t_0 - t_n) + (s_0 - s_n)]}(t_n - t_{n+1}), \quad (19)$$

$\{t_n\}_{n \geq 0}, \{s_n\}_{n \geq 0}$  are non-negative, decreasing sequences that converge to certain  $t^*$  such that  $r_0 - c/(1 - \gamma) \leq t^* < t_0$ ; sequences  $\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0}$  generated by the iterative process (3) are well defined, remain in  $U_0$  that converge to a solution  $x^*$  of equation (1).

*Proof.* Firstly, we prove, by mathematical induction, that the following inequalities hold for all  $k \geq 0$

$$t_{k+1} \geq s_{k+1} \geq t_{k+2} \geq r_0 - \frac{c}{1 - \gamma} \geq 0, \quad (20)$$

$$t_{k+1} - t_{k+2} \leq \gamma(t_k - t_{k+1}), \quad t_{k+1} - s_{k+1} \leq \gamma(t_k - t_{k+1}). \quad (21)$$

From (18), (19) for  $k = 0$  we obtain

$$t_1 - t_2 = \frac{(p_0 + q_0)(s_0 - t_1) + 0.5p_0(t_0 - t_1)}{1 - (p_0 + q_0)[(t_0 - t_1) + (s_0 - s_1)]}(t_0 - t_1) \leq \gamma(t_0 - t_1),$$

$$t_1 - s_1 = [(p_0 + q_0)(s_0 - t_1) + 0.5p_0(t_0 - t_1)](t_0 - t_1) \leq \gamma(t_0 - t_1),$$

$$\begin{aligned}
t_2 &\geq r_0 - c - \frac{(p_0 + q_0)s_0 + 0.5p_0t_0}{1 - (p_0 + q_0)[t_0 + s_0]}c = \\
&= r_0 - (1 + \gamma)c = r_0 - \frac{(1 - \gamma^2)c}{1 - \gamma} \geq r_0 - \frac{c}{1 - \gamma} \geq 0, \\
t_1 &\geq t_2, \quad s_1 \geq t_2, \quad t_1 \geq s_1 \geq t_2 \geq r_0 - \frac{c}{1 - \gamma} \geq 0.
\end{aligned}$$

Let us suppose that inequalities (20) and (21) hold for  $k = 0, 1, \dots, n - 1$ . Then for  $k = n$  we obtain

$$\begin{aligned}
t_{n+1} - t_{n+2} &= \frac{(p_0 + q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1 - (p_0 + q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]}(t_n - t_{n+1}) \leq \\
&\leq \frac{(p_0 + q_0)s_n + 0.5p_0t_n}{1 - (p_0 + q_0)[t_0 + s_0]}(t_n - t_{n+1}) \leq \gamma(t_n - t_{n+1}), \\
t_{n+1} - s_{n+1} &= \frac{(p_0 + q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1 - (p_0 + q_0)[(t_0 - t_n) + (s_0 - s_n)]}(t_n - t_{n+1}) \leq \\
&\leq \frac{(p_0 + q_0)s_n + 0.5p_0t_n}{1 - (p_0 + q_0)[t_0 + s_0]}(t_n - t_{n+1}) \leq \gamma(t_n - t_{n+1})
\end{aligned}$$

and

$$\begin{aligned}
t_{n+1} &\geq s_{n+1} \geq t_{n+2} \geq t_{n+1} - \gamma(t_n - t_{n+1}) \geq \\
&\geq r_0 - \frac{1 - \gamma^{n+2}}{1 - \gamma}c \geq r_0 - \frac{c}{1 - \gamma} \geq 0.
\end{aligned}$$

So, we prove, that sequences  $\{t_n\}_{n \geq 0}$  and  $\{s_n\}_{n \geq 0}$  are non-negative, decreasing sequences and converge to  $t^*$  such that  $t^* \geq 0$ .

Let us prove, by mathematical induction, that the iterative process (3) is well defined and inequalities (16) hold for all  $n \geq 0$ .

Using (15) and  $t_0 - t_1 = c$ , we prove that (16) hold for  $n = 0$ .

Let denote  $A_n = F'\left(\frac{x_n + y_n}{2}\right) + G(x_n; y_n)$ . Using Lipschitz conditions (13) and (14), we have

$$\begin{aligned}
&\|I - A_0^{-1}A_{n+1}\| = \|A_0^{-1}[A_0 - A_{n+1}]\| \leq \\
&\leq \|A_0^{-1}\left[F'\left(\frac{x_0 + y_0}{2}\right) - F'\left(\frac{x_{n+1} + y_{n+1}}{2}\right)\right]\| + \\
&\quad + \|A_0^{-1}[G(x_0; y_0) - G(x_{n+1}; y_{n+1})]\| \leq \\
&\leq 2p_0\left(\frac{\|x_0 - x_{n+1}\|}{2} + \frac{\|y_0 - y_{n+1}\|}{2}\right) + q_0(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \leq \\
&\leq (p_0 + q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \leq \\
&\leq (p_0 + q_0)(t_0 - t_{n+1} + s_0 - s_{n+1}) \leq \\
&\leq (p_0 + q_0)(t_0 + s_0) = (p_0 + q_0)(2r_0 - a) < 1.
\end{aligned}$$

By Banach lemma on invertible operator, it follows that  $A_{n+1}$  is invertible and

$$\|A_{n+1}^{-1}A_0\| \leq \left[1 - (p_0 + q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|)\right]^{-1}.$$

Let us prove that iterative process (3) is well defined for  $k = n + 1$ . From the definition of the first order divided difference and (13), (14), we obtain

$$\begin{aligned} & \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| = \\ & = \|A_0^{-1}[F(x_{n+1}) + G(x_{n+1}) - F(x_n) - G(x_n) - A_n(x_{n+1} - x_n)]\| \leq \\ & \leq \left\|A_0^{-1}\left[\int_0^1 \left\{F'(x_{n+1} + t(x_n - x_{n+1})) - F'\left(\frac{x_n + y_n}{2}\right)\right\} dt\right]\right\| \|x_n - x_{n+1}\| + \\ & \quad + \|A_0^{-1}[G(x_n; y_n) - G(x_n; x_{n+1})]\| \|x_n - x_{n+1}\| \leq \\ & \leq 2p_0 \left[ \|x_n - x_{n+1}\| \int_0^1 \left|t - \frac{1}{2}\right| dt + \frac{\|y_n - x_{n+1}\|}{2} \right] \|x_n - x_{n+1}\| + \\ & \quad + q_0 \|y_n - x_{n+1}\| \|x_n - x_{n+1}\| = \\ & = (p_0 + q_0) \|y_n - x_{n+1}\| \|x_n - x_{n+1}\| + 0.5p_0 \|x_n - x_{n+1}\|^2. \end{aligned}$$

Hence, using (16), we have

$$\begin{aligned} & \|x_{n+1} - x_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \leq \\ & \leq \|A_{n+1}^{-1}A_0\| \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \|x_n - x_{n+1}\| \leq \\ & \leq \frac{(p_0 + q_0) \|y_n - x_{n+1}\| + 0.5p_0 \|x_n - x_{n+1}\|}{1 - (p_0 + q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|)} \|x_n - x_{n+1}\| \leq \\ & \leq \frac{(p_0 + q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1 - (p_0 + q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]} (t_n - t_{n+1}) = t_{n+1} - t_{n+2}, \\ & \|x_{n+2} - y_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+2}) + G(x_{n+2}))\| \leq \\ & \leq \|A_{n+1}^{-1}A_0\| \|A_0^{-1}(F(x_{n+2}) + G(x_{n+2}))\| \|x_n - x_{n+1}\| \leq \\ & \leq \frac{(p_0 + q_0) \|y_{n+1} - x_{n+2}\| + 0.5p_0 \|x_{n+1} - x_{n+2}\|}{1 - (p_0 + q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|)} \|x_{n+1} - x_{n+2}\| \leq \\ & \leq \frac{(p_0 + q_0)(s_{n+1} - t_{n+2}) + 0.5p_0(t_{n+1} - t_{n+2})}{1 - (p_0 + q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]} (t_{n+1} - t_{n+2}) = s_{n+2} - t_{n+2}. \end{aligned}$$

So, iterative process (3) is well defined and (15) holds for all  $n \geq 0$ . From this it follows

$$\|x_n - x_k\| \leq t_n - t_k, \quad \|y_n - x_k\| \leq s_n - t_k, \quad \|y_n - y_k\| \leq s_n - s_k, \quad 0 \leq n \leq k, \quad (22)$$

i.e.,  $\{x_n\}_{n \geq 0}$  and  $\{y_n\}_{n \geq 0}$  are fundamental sequences in a Banach space  $X$ . From (22) for  $k \rightarrow \infty$  it follows inequalities (17). Let's show that  $x^*$  is solution of equation (1). Indeed,

$$\begin{aligned} & \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \leq \\ & \leq (p_0 + q_0) \|y_n - x_{n+1}\| \|x_n - x_{n+1}\| + 0.5p_0 \|x_n - x_{n+1}\|^2 \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

So,  $H(x^*) = 0$ . □

**Remark 5.** *If we choose  $F(x) = 0$ ,  $p_1 = 0$ ,  $p_2 = 0$  then the estimates (8) and (9) reduce to similar ones in [8] for the case  $\alpha = 1$ .*

**Remark 6.** *If the divided difference of the operator  $G$  satisfies the condition (6), i.e. the operator  $G(x; y)$  is Lipschitz continuous, then  $G$  is Fréchet-differentiable.*

### 3. NUMERICAL EXPERIMENTS

For the numerical investigation we choose the equation and the systems of equations considered in [1, 4, 5, 6, 7].

**Example 1.**

$$e^{x-0.5} - 1.05 + 0.2x|x - 1| = 0,$$

$$x^* = 0.5.$$

**Example 2.**

$$3x^2y - y^2 - 1 + |x - 1| = 0,$$

$$x^4 + xy^3 - 1 + |y| = 0,$$

$$(x^*; y^*) \approx (0.894655; 0.327827).$$

**Example 3.**

$$x^2 - y + 1 + \frac{1}{9}|x - 1| = 0,$$

$$y^2 + x - 7 + \frac{1}{9}|y| = 0,$$

$$(x^*; y^*) \approx (1.15936; 2.36182).$$

**Example 4.**

$$z^2(1 - y) - xy + |y - z^2| = 0,$$

$$z^2(x^3 - x) - y^2 + |3y^2 - z^2 + 1| = 0,$$

$$6xy^3 + y^2z^2 - xy^2z + |x + z - y| = 0,$$

$$(x^*; y^*; z^*) = (-1; 2; 3).$$

Let  $X = Y = \mathbb{R}^m$ ,  $m = 1, 2, 3$ . In this case the first order divided difference  $G(x; y)$  is a matrix of dimension  $m \times m$ . Its elements are calculated as [8]

$$G(x; y)_{i,j} = \frac{G_i(x^1, \dots, x^j, y^{j+1}, \dots, y^m) - G_i(x^1, \dots, x^{j-1}, y^j, \dots, y^m)}{x^j - y^j},$$

$$i, j = \overline{1, m}.$$

In calculations we use the norm  $\|x\|_\infty = \max_{1 \leq i \leq m} |x^i|$ . In the following Tables there are results obtained by methods (3) and (2) in particular, for such cases

$$x_{n+1} = x_n - [F'(x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots, \quad (23)$$



$$x_{n+1} = x_n - [F'(x_n) + G(x_{n-1}; x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots, \quad (24)$$

$$x_{n+1} = x_n - [H(x_{n-1}; x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots \quad (25)$$

TABLE 1. Numbers of iterations for solving equations with initial points  $x_0 = 1 \cdot d$ ,  $x_{-1} = y_0 = 2 \cdot d$  – for Example 1,  $x_0 = (1, 0)d$ ,  $x_{-1} = y_0 = (5, 5)d$  – for Example 2

$d$	$\varepsilon$	Example 1				Example 2			
		(23)	(24)	(25)	(3)	(23)	(24)	(25)	(3)
1	$10^{-5}$	5	5	6	5	11	4	5	5
	$10^{-15}$	6	7	8	6	33	6	9	6
10	$10^{-5}$	14	15	20	13	19	13	18	12
	$10^{-15}$	15	17	22	14	41	15	21	13
100	$10^{-5}$	104	105	–	88	27	21	30	19
	$10^{-15}$	105	107	–	89	49	23	32	20

The calculations were conducted in MATLAB 7.1. Iterations were stopped after conditions  $\|x_{n+1} - x_n\|_\infty \leq \varepsilon$  and  $\|H(x_{n+1})\|_\infty \leq \varepsilon$  were satisfied. Sign "–" means, that in this case the solution was not possible to be found. We examined the convergence of the considered method for such variants of choice of the additional initial approximation  $y_0$ : for Example 1 –  $x_{-1} = y_0 = 2 \cdot d$ , for Examples 2, 3  $y_0$  was chosen as  $x_{-1}$  in the works [1, 5, 6, 7] and  $x_{-1}^i = y_0^i = x_0^i + 10^{-4}$ ,  $i = 1, 2, 3$  – for Example 4.

The obtained results show that the methods (24) and (3) differ a little for the initial points that are close to the solution. But the method (3) converge faster than (2) for the initial points with  $d = 100$ . In this case  $\|x_0 - x^*\|$  takes the largest value. The method (23) has the lowest speed of convergence.

TABLE 2. Numbers of iterations for solving equations with initial points  $x_0 = (1, 1)d$ ,  $x_{-1} = y_0 = (0.9, 1.1)d$  – for Example 3,  $x_0 = (-2, 3, 5)d$ ,  $x_{-1}^i = y_0^i = x_0^i + 10^{-4}$  – for Example 4

$d$	$\varepsilon$	Example 3				Example 4			
		(23)	(24)	(25)	(3)	(23)	(24)	(25)	(3)
1	$10^{-5}$	6	5	6	5	85	7	10	7
	$10^{-15}$	13	7	9	6	266	10	12	8
10	$10^{-5}$	8	7	9	6	102	10	25	14
	$10^{-15}$	15	9	11	7	284	20	27	16
100	$10^{-5}$	11	11	14	9	110	28	39	23
	$10^{-15}$	18	12	16	10	292	30	41	24

In Table 3 the numerical results are presented for the example 1 with  $\varepsilon = 10^{-10}$ , where  $n$  is the iteration number,  $x_n$  is the approximate value for  $x^*$ ,

TABL. 3. Numerical results for the Example 1:  $x_0 = 1, y_0 = 2$ 

$n$	$x_n$	$ x_n - x_{n-1} $	$ H(x_n) $
0	1		0.5987212707001
1	0.8079964212227	0.1920035787772	0.3417237602029
2	0.5200746907444	0.28792173047835	0.02019694382837
3	0.5000182789519	0.02005641179247	$1.827905217970 \cdot 10^{-5}$
4	0.5000000000006	$1.827895124595 \cdot 10^{-5}$	$6.967343368913 \cdot 10^{-13}$
5	0.5	$6.967759702547 \cdot 10^{-13}$	$4.163336342344 \cdot 10^{-17}$

$|x_n - x_{n-1}|$  is the norm of correction and  $|H(x_n)|$  is the norm of deviation on every step of the iterative process (3).

Now we verify whether the hypothesis of Theorem 2 are satisfied. The research are carried out for the example 1. Since  $m = 1$  than  $\|\cdot\|_\infty = |\cdot|$ . In [9] we showed that the following estimates hold for all  $x, y \in [0; 1]$

$$\begin{aligned} |A_0^{-1}(F'(x) - F'(y))| &\leq |A_0^{-1}| |F'(x) - F'(y)| \leq \frac{e^{0.5}}{|A_0|} |x - y|, \\ |A_0^{-1}(G(x, y) - G(u, v))| &\leq |A_0^{-1}| |(G(x, y) - G(u, v))| \leq \\ &\leq \frac{1}{5|A_0|} (|x - u| + |y - v|). \end{aligned}$$

Hence  $p_0 = \frac{e^{0.5}}{2|A_0|}$  and  $q_0 = \frac{1}{5|A_0|}$ . Let us choose  $x_0 = 0.43, y_0 = 0.47$ . Then we get

$$\begin{aligned} \frac{1}{|A_0|} &= 1.049985813745361, \quad p_0 = 0.8655669725276801, \\ q_0 &= 0.2099971627490723, \quad c = 0.07201451611773883, \quad a = 0.04. \end{aligned}$$

Let us choose  $r_0 = 0.1$ . Then, according to formulas (18) and (19), we get

$$\begin{aligned} t_0 &= 0.1000000000000000, \quad s_0 = 0.0600000000000000, \\ t_1 &= 0.0798548388226117, \quad s_1 = 0.02326130579394141, \\ t_2 &= 0.0226355142098747, \quad s_2 = 0.02261740817032270, \\ t_3 &= 0.02261727501017343, \dots, \quad t^* \approx 0.02261727484294557, \\ &0.01720355125317807 < t^* < 0.1, \quad \gamma = 0.1302221628134378 < 1. \end{aligned}$$

The solution  $x^*$  is obtained in 3 iterations with  $\varepsilon = 10^{-5}$ .

TABL. 4. Numerical results for the Example 1

$n$	$ x_{n-1} - x_n $	$t_{n-1} - t_n$	$ y_{n-1} - x_n $	$s_{n-1} - t_n$
1	$7.0617898 \cdot 10^{-2}$	$7.2014516 \cdot 10^{-2}$	$3.0617898 \cdot 10^{-2}$	$3.2014516 \cdot 10^{-2}$
2	$6.1790108 \cdot 10^{-4}$	$5.3499697 \cdot 10^{-3}$	$1.8418431 \cdot 10^{-5}$	$6.2579158 \cdot 10^{-4}$
3	$3.4257955 \cdot 10^{-9}$	$1.8239200 \cdot 10^{-5}$	$6.1617378 \cdot 10^{-13}$	$1.3316015 \cdot 10^{-7}$

Thus for the given values hypothesis of the Theorem 2 are satisfied (See Tabl.4). According to this theorem, the iterative process (2) is well-defined, remains in  $U_0$  and converges to the solution  $x^* \in U_0$ .

## BIBLIOGRAPHY

1. Argyros I. K. A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space / I. K. Argyros // *J. Math. Anal. Appl.*– 2004.– Vol. 298.– P. 374-397.
2. Argyros I. K. Improving the rate of convergence of Newton methods on Banach spaces with a convergence structure and applications / I. K. Argyros // *Appl. Math. Lett.*– 1997.– Vol. 6.– P. 21-28.
3. Argyros I. K. On the convergence of modified Newton methods for solving equations containing a non-differentiable term / I. K. Argyros, H. Ren // *J. Comp. App. Math.*– 2009.– Vol. 231.– P. 897-906.
4. Chen X. On the convergence of Broyden-like methods for nonlinear equations with non-differentiable terms / X. Chen // *Ann. Inst. Statist. Math.*– 1990.– Vol. 42, № 2.– P. 387-401.
5. Hernandez M. A. A uniparametric family of iterative processes for solving nondifferentiable operators / M. A. Hernandez, M. J. Rubio // *J. Math. Anal. Appl.*– 2002.– Vol. 275.– P. 821-834.
6. Hernandez M. A. The Secant method for nondifferentiable operators / M. A. Hernandez, M. J. Rubio // *Appl. Math. Lett.*– 2002.– Vol. 15.– P. 395-399.
7. Ren H. A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators / H. Ren, I. K. Argyros // *J. Appl. Math. Comp.*– 2010.– Vol. 34, № 1-2.– P. 39-46.
8. Shakhno S. M. On an iterative algorithm with superquadratic convergence for solving nonlinear operator equations / S. M. Shakhno // *J. Comp. App. Math.*– 2009.– Vol. 231.– P. 222-235.
9. Shakhno S. M. Twopoint method for solving nonlinear equation with nondifferentiable operator / S. M. Shakhno, H. P. Yarmola // *Matematychni Studii.*– 2010.– Vol. 36, № 2.– P. 213-220 (in Ukrainian).
10. Werner W. Über ein Verfahren der Ordnung  $1 + \sqrt{2}$  zur Nullstellenbestimmung / W. Werner // *Numer. Math.*– 1979.– Vol. 32.– P. 333-342.
11. Zabrejko P. P. The majorant method in the theory of Newton-Kantorovich approximations and the Pta'k error estimates / P. P. Zabrejko, D. F. Nguen // *Numerical Functional Analysis and Optimization.*– 1987.– Vol. 9, № 5-6.– P. 671-684.

STEPAN SHAKHNO, HALINA YARMOLA,  
IVAN FRANKO NATIONAL UNIVERSITY OF LVIV,  
1, UNIVERSYTETS'KA STR., LVIV, 79000, UKRAINE

Received 12.04.2012