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ON ACCURACY OF SOLVING SEMIDISCRETE ILL-POSED PROBLEMS IN SOBOLEV SPACES WITH ν-METHODS

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Резюме. Для розв'язування некоректної задачі у соболівських шкалах, отриманої в результаті застосування методу колокації до інтегрального рівняння Фредгольма першого роду, використано поєднання ν-методів з принципом балансу як апостеріорним правилом вибору параметра регуляризації.

Abstract. To solve ill-posed problem in Sobolev scales appearing as a result of application by a collocation method to Fredholm integral equation of the first kind a combination of ν-methods with balancing principle as an a-posteriori regularization parameter choice rule is used.

1. Introduction

Let us consider an equation

$$Af = g \tag{1}$$

with integral operator A defined as

$$Af(x) := \int_\Omega k(x,t) f(t) dt, \qquad x \in \Omega.$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary and kernel $k(x,t) : \Omega \times \Omega \to \mathbb{R}$ is such that A is compact operator with infinite dimensional range acting from $L_2 = L_2(\Omega)$ into L_2. Without loss of generality we may assume that $\|A\| \le 1$.

To guarantee a stable solution some regularization method should be used. In the paper we use ν-methods, but regularization process will be done not for original problem (1) but for semi-discrete equation obtained from it by collocation scheme. Let $X = \{x_1, \dots, x_n\} \subset \Omega$ be some set of pairwise distinct points. Consider an equation

$$A_X f = \bar{g}, \tag{2}$$

where $\bar{g} = \{g_1, \dots, g_n\}^T$, $g_j = g(x_j)$, and A_X is defined as

$$(A_X f)_j = Af(x_j), \quad 1 \le j \le n,$$

i.e. A_X is the restriction of A to set X ($A_X f = Af|_X$). To obtain good approximation to exact solution in the framework of ν-methods it is important to choose regularization parameter in properly way. In this case regularization parameter is the number of iteration step. As a rule we use balancing principle (see [5], [7]).

[†]*Key words.* Inverse problems, ν-methods, Sobolev scales, collocation method, a-posteriori parameter choice, error bound.

In practice exact right-hand side of (1) is usually unavailable and only noisy data vector $\bar{g}^\delta = \{g_1^\delta, \dots, g_n^\delta\}^T$ such that

$$|g_j - g_j^\delta| \leq \delta, \;\; j = \overline{1, n}$$

is known. Let n-dimensional Euclidean space $\mathbb{R}^n$ provided with standard norm $\|\cdot\|_{\mathbb{R}^n}$ and corresponding inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. Then the whole data error can be estimated as

$$\|\bar{g} - \bar{g}^\delta\|_{\mathbb{R}^n} \leq \delta\sqrt{n}.$$

Our aim is stable recovery of unknown solution of (2) from noisy values $\bar{g}^\delta$.

2. ν-methods in Sobolev scales

Following [2] we assume that A acts along scale of Sobolev spaces $\mathcal{H}^\tau$, $\tau \geq d/2$, with step $\alpha > 0$ i.e. there are constants $c^{''} \geq c^{'} \geq 0$ such that for fixed $\alpha \in \mathbb{R}$

$$c^{'}\|f\|_\tau \leq \|Af\|_{\tau+\alpha} \leq c^{''}\|f\|_\tau. \tag{3}$$

Recall that Sobolev space $H^\tau = H^\tau(\Omega)$ is completion in norm of space of square-summable function in Ω together with derivatives of order τ, and $\mathcal{H}^0 = L_2(\Omega)$.

For the first time ill-posed problems in Hilbert scales was considered in [6]. But in the paper we consider the case of discretization by projection methods. The first result in Hilbert scales for the case of discretization by collocation method was obtained recently in [2] where a-priori rule is used for regularization parameter choice. We consider aposteriory rule for choosing the parameter, i.e. without information about smoothing of exact solution.

Let f^* be an exact solution of original problem (1). Then f^* also solves semi-discrete problem (2) and can be represented in the form

$$f^* = f_\delta + v_0,$$

where $f_\delta = A_X^\dagger \bar{g}$, $A_X^\dagger$ is the Moore-Penrose generalized inverse of A_X, and v_0 belongs to the null space of A_X.

We will obtain approximation to solution f_δ. Since A_X acts from $\mathcal{H}^\tau$ into $\mathbb{R}^n$ than $f_\delta \in \mathcal{H}^\tau$ for some $\tau > 0$.

In [3] was shown that always exists some continuous increasing index function $\phi(\lambda)$, $\lambda \in [0, 1]$, such that $\phi(0) = 0$ and

$$f_\delta = \phi(A_X^* A_X)v, \tag{4}$$

where $v \in \mathcal{H}^\tau$, $\|v\|_\tau \leq \rho$, $\rho > 0$, and $A_X^* : \mathbb{R}^n \to \mathcal{H}^\tau$ is the adjoint of A_X. Later we assume that (4) is fulfills.

Recall that ν-methods is the process of successive computation of elements f_k^δ, $k = 1, 2, \dots$ by the rule

$$f_k^\delta = p_k(A_X^* A_X)A_X^* \bar{g}^\delta,$$

where $\{p_k\}$ is some series of the polynomials of order $k-1$. Consider one more polynomial:

$$r_k(\lambda) := 1 - \lambda p_k(\lambda).$$

It is easy to obtain that for $f_k = p_k(A_X^* A_X)A_X^* \bar{g}$ we have

$$f_k - f_k^\delta = p_k(A_X^* A_X)A_X^*(\bar{g} - \bar{g}^\delta),$$

$$f_\delta - f_k = r_k(A_X^* A_X) f^\dagger,$$

$$\sup_{0\le\lambda\le1} \sqrt{\lambda} p_k(\lambda) \le 2k, \tag{5}$$

$$\sup_{0\le\lambda\le1} \lambda p_k(\lambda) \le 2, \tag{6}$$

$$|\lambda^\mu r_k(\lambda)| \le c_\mu k^{-2\mu}, \tag{7}$$

$$|r_k(\lambda)| \le 1, \tag{8}$$

where $\lambda \in [0,1]$, $c_\mu > 0$ is some constant, $0 < \mu \le \nu$.

3. Auxiliary assertions

Lemma 1. *If $\frac{\phi(t)}{t^{\nu-1/2}}$ is the decreasing function then estimations*

$$\|f_\delta - f_k\|_\tau \quad \le \quad \varkappa\rho\phi(k^{-2}), \tag{9}$$

$$\|A_X f_\delta - A_X f_k\|_{\mathbb{R}^n} \quad \le \quad c_\nu \rho k^{-1}\phi(k^{-2}), \tag{10}$$

are hold, where c and c_ν are some constants.

Proof. In [1, Theorem 6.15] the estimate

$$\|f_\delta - f_k\| \le \varkappa \|f_\delta - f_{\gamma_k,\nu}\|,$$

is obtained, where $f_{\gamma_k,\nu} = \sum_{i=1}^{\nu} \gamma_k^{i-1}(A_X^* A_X + \gamma_k I)^{-i} A_X^* \bar{g}$ is the approximate solution obtained by iterated Tikhonov method of order ν (ν is integer), $\gamma_k \in [(k+1)^{-2}, k^{-2}]$, and $\varkappa$ is a constant. On the other hand, it is easy to show that

$$\|f_\delta - f_{\gamma_k,\nu}\|_\tau \le \rho\phi(\gamma_k).$$

So, first statement of the Lemma is proved.

Further

$$\|A_X f_\delta - A_X f_k\|_{\mathbb{R}^n} = \|A_X(f_\delta - f_k)\|_{\mathbb{R}^n} = \|A_X r_k(A_X^* A_X) f_\delta\|_{\mathbb{R}^n} =$$
$$= \|A_X r_k(A_X^* A_X)\phi(A_X^* A_X) v\|_{\mathbb{R}^n} \le \|v\|_\tau \sup_{0\le\lambda\le1} \sqrt{\lambda} r_k(\lambda)\phi(\lambda).$$

To estimate expression in the right-hand side we consider two cases.

1. $\lambda \le k^{-2}$. Due to (8) and increase of the function ϕ we have

$$\sqrt{\lambda} r_k(\lambda)\phi(\lambda) \le k^{-1}\phi(k^{-2}).$$

2. $k^{-2} \le \lambda$. Due to decrease of the function $\frac{\phi(t)}{t^{\nu-1/2}}$ and (7) we obtain

$$\sqrt{\lambda} r_k(\lambda)\phi(\lambda) = \sqrt{\lambda} r_k(\lambda)\lambda^{\nu-1/2}\frac{\phi(\lambda)}{\lambda^{\nu-1/2}} \le \sqrt{\lambda} r_k(\lambda)\lambda^{\nu-1/2}\frac{\phi(k^{-2})}{k^{-2(\nu-1/2)}} \le$$
$$\le \frac{\lambda^\nu r_k(\lambda)}{k^{-2\nu}} k^{-1}\phi(k^{-2}) \le c_\nu k^{-1}\phi(k^{-2}).$$

Hence,

$$\|A_X f_\delta - A_X f_k\|_{\mathbb{R}^n} \le c_\nu \rho k^{-1}\phi(k^{-2})$$

and Lemma is proved. □

Remark 7. *It is follows from decreasing of* $\frac{\phi(t)}{t^{\nu-1/2}}$ *that in the case of* $\phi(\gamma) = \gamma^{\beta}$ *the restriction is arisen* β*:* $0 \le \beta \le \nu - 1/2$*.*

Lemma 2. *Following estimations*

$$\|f_k - f_k^\delta\|_\tau \le 2k\delta\sqrt{n}, \tag{11}$$

$$\|A_X f_k - A_X f_k^\delta\|_{\mathbb{R}^n} \le 2\delta\sqrt{n}. \tag{12}$$

are hold.

Proof. Due to (5) we obtain the following estimation

$$\|f_k - f_k^\delta\|_\tau = \|p_k(A_X^* A_X)A_X^*(\bar{g} - \bar{g}^\delta)\|_\tau \le$$
$$\le \|\bar{g} - \bar{g}^\delta\|_{\mathbb{R}^n} \sup_{0\le\lambda\le1} \sqrt{\lambda} p_k(\lambda) \le 2k\delta\sqrt{n}$$

and the first statement of Lemma is proved.

Later due to (6) we have

$$\|A_X f_k - A_X f_k^\delta\|_{\mathbb{R}^n} = \|A_X p_k(A_X^* A_X)A_X^*(\bar{g} - \bar{g}^\delta)\|_{\mathbb{R}^n} \le$$
$$\le \|\bar{g} - \bar{g}^\delta\|_{\mathbb{R}^n} \sup_{0\le\lambda\le1} \lambda p_k(\lambda) \le 2\delta\sqrt{n}.$$

Thus, Lemma is proved. □

Define data density of the set X in domain Ω as

$$h := \sup_{x\in\Omega} \min_{x_i\in X} \|x - x_i\|_{\mathbb{R}^d}.$$

Below we need the sampling inequality obtained in [2, Theorem 4.8]. Namely, for arbitrary function $u \in \mathcal{H}^\theta = \mathcal{H}^\theta(\Omega)$, $\theta > d/2$ and sufficiently small h it is true

$$\|u\|_\sigma \le \kappa\left(h^{\theta-\sigma}\|u\|_\theta + h^{\frac{d}{2}-\sigma}\|u|_X\|_{\mathbb{R}^n}\right), \tag{13}$$

where $\sigma \in [0, \lfloor\theta\rfloor)$, and κ is some constant, doesn't depending on u and h.

4. Error estimate

Theorem 1. *Let (3) is true. Then for any discrete set* X *with sufficiently small data density* h *there is constant* $c_1 > 0$ *such that*

$$\|f_\delta - f_k^\delta\|_{L_2} \le c_1(h^\tau\left(\varkappa\rho\phi(k^{-2}) + 2\delta k\sqrt{n}\right) + \tag{14}$$
$$+h^{\frac{d}{2}-\alpha}\left(c_\nu\rho k^{-1}\phi(k^{-2}) + 2\delta\sqrt{n}\right)).$$

Proof. First of all we estimate $\|f_\delta - f_k^\delta\|_\tau$. Due to (9) and (11) we have

$$\|f_\delta - f_k^\delta\|_\tau \le \|f_\delta - f_k\|_\tau + \|f_k - f_k^\delta\|_\tau \le \varkappa\rho\phi(k^{-2}) + 2k\delta\sqrt{n}. \tag{15}$$

Using the sampling inequality (13) with $u = A(f_\delta - f_k^\delta)$, $\sigma = \alpha$ and $\theta = \tau + \alpha$ we obtain

$$\|A(f_\delta - f_k^\delta)\|_\alpha \le \kappa\left(h^\tau\|A(f_\delta - f_k^\delta)\|_{\tau+\alpha} + h^{\frac{d}{2}-\alpha}\|A(f_\delta - f_k^\delta)|_X\|_{\mathbb{R}^n}\right).$$

Now we apply condition (3) to last inequality

$$c'\|f_\delta - f_k^\delta\|_{L_2} \le \kappa\left(c''h^\tau\|f_\delta - f_k^\delta\|_\tau + h^{\frac{d}{2}-\alpha}\|A(f_\delta - f_k^\delta)|_X\|_{\mathbb{R}^n}\right).$$

Taking into account that $Af|_X = A_X f$, we obtain

$$\|f_\delta - f_k^\delta\|_{L_2} \le c_1 \left(h^\tau \|f_\delta - f_k^\delta\|_\tau + h^{\frac{d}{2}-\alpha} \|A_X(f_\delta - f_k^\delta)\|_{\mathbb{R}^n} \right),$$

where $c_1 = \frac{\kappa}{c'} \max\{1, c''\}$.

Considering estimates (10), (12), (15) we have Theorem's statement. □

Let partition of the set X is uniform, i.e. $h = \chi n^{-\frac{1}{d}}$ for some constant χ. Then inequality (14) can be rewritten as

$$\|f_\delta - f_k^\delta\|_{L_2} \le \Phi(k) + \Psi(k),$$

where

$$\Phi(k) := c_2 \rho \left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}-\frac{1}{2}} k^{-1} \phi(k^{-2}) + \chi^\tau n^{-\frac{\tau}{d}} \phi(k^{-2}) \right),$$

$$\Psi(k) := c_2 \left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}} \delta + \chi^\tau n^{-\frac{\tau}{d}} k \delta \sqrt{n} \right),$$

and $c_2 = c_1 \max\{2, \varkappa, c_\nu\}$.

It is obvious that due to the monotonicity of ϕ the function Φ is increasing and Ψ is decreasing. Herewith optimal value of the regularization parameter $\gamma = \gamma_{opt}$ balances functions Φ and Ψ, i.e. $\Phi(\gamma_{opt}) = \Psi(\gamma_{opt})$ and

$$\|f_\delta - f_{k_{opt}}^\delta\|_{L_2} \le 2\Phi(k_{opt}).$$

In the case of unknown function ϕ such apriory rule for choosing regularization parameter is inapplicable so it is necessary to use one of the aposteriory rules. As a rule we use balancing principle.

Take into consideration following sets

$$\Delta_N = \left\{1, \ldots, N, \qquad N \asymp (\delta\sqrt{n})^{-1}\right\}, \tag{16}$$

and

$$M^+(\Delta_N) = \left\{ k \in \Delta_N : \|f_k^\delta - f_l^\delta\|_{L_2} \le 4\Psi(l), \quad l = k, \ldots, N \right\}.$$

To obtain approximate solution we use as regularization parameter such element

$$k = k_+ := \min\left\{k \in M^+(\Delta_N)\right\}.$$

Let us consider one more set

$$M(\Delta_N) := \{k \in \Delta_N : \quad \Phi(k) \le \Psi(k)\}$$

and define

$$k_* := \min\{k \in M(\Delta_N)\}.$$

Without loss of generality we assume that $M(\Delta_N) \ne \emptyset$ and $\Delta_N \backslash M(\Delta_N) \ne \emptyset$.

Theorem 2. *Let the set Δ_N is defined as (16). Then for regularization parameter $k = k_+$ following estimate*

$$\|f_\delta - f_{k_+}^\delta\|_{L_2} \le 6q\Phi(k_{opt}), \tag{17}$$

holds, where $2 \ge q \ge \frac{k_+}{k_+ - 1}$.

Proof. From the beginning we show that $k_* \le k_+$. For any element $l > k_*$ we have

$$\begin{aligned} \|f^\delta_{k_*} - f^\delta_l\|_{L_2} &\le \|f_\delta - f^\delta_{k_*}\|_{L_2} + \|f_\delta - f^\delta_l\|_{L_2} \\ &\le \Phi(k_*) + \Psi(k_*) + \Phi(l) + \Psi(l) \\ &\le 2\Phi(k_*) + \Psi(k_*) + \Psi(l) \\ &\le 3\Psi(k_*) + \Psi(l) \le 4\Psi(l). \end{aligned}$$

So, $k_* \in M^+(\Delta_N)$ and by the definition $k_* \ge k_+$.

Define the unknown norm using $\Psi(k_*)$

$$\begin{aligned} \|f_\delta - f^\delta_{k_+}\|_{L_2} &\le \|f_\delta - f^\delta_{k_*}\|_{L_2} + \|f^\delta_{k_*} - f^\delta_{k_+}\|_{L_2} \\ &\le 6\Psi(k_*). \end{aligned}$$

Due to monotonicity of the function Ψ for $2 \ge q \ge \frac{k_+}{k_+-1} > 1$ we have

$$\begin{aligned} \Psi(k_*) &= c_2\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}}\delta + \chi^\tau n^{-\frac{\tau}{d}} k_* \delta\sqrt{n}\right) \\ &\le qc_2\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}}\delta + \chi^\tau n^{-\frac{\tau}{d}} \frac{k_*}{q} \delta\sqrt{n}\right) \\ &= q\Psi(\tfrac{k_*}{q}). \end{aligned}$$

It follows from the definitions of the elements k_*, k_{opt} that $k_* \ge k_{opt} \ge k_* - 1$. Then

$$\|f_\delta - f^\delta_{\gamma_+}\|_{L_2} \le 6\Psi(k_*) \le 6q\Psi(k_*/q) \le 6q\Psi(k_{opt}) = 6q\Phi(k_{opt})$$

and Theorem is proved. □

Corollary 3. *For $\theta(k) = \phi(k^{-2})k^{-1}$ the estimate*

$$\|f_\delta - f^\delta_{m,\gamma_+}\|_{L_2} \le 6q\Phi\left(\theta^{-1}\left(\frac{\delta\sqrt{n}}{\rho}\right)\right),$$

is true. In particular for $\phi(\gamma) = \gamma^\beta$ with $0 < \beta \le \nu - 1/2$

$$\|f_\delta - f^\delta_k\|_{L_2} \le 6qc_2\left(\chi^{\frac{d}{2}-\alpha}\delta n^{\frac{\alpha}{d}} + \chi^\tau \rho^{\frac{1}{2\beta+1}} n^{-\frac{\tau}{d}} (\delta\sqrt{n})^{\frac{2\beta}{2\beta+1}}\right). \tag{18}$$

Proof. By the definition of k_{opt} it holds that $\Phi(k_{opt}) = \Psi(k_{opt})$, i.e.

$$\rho k^{-1}\phi(k^{-2})\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}-\frac{1}{2}} + \chi^\tau n^{-\frac{\tau}{d}} k\right) = \delta\sqrt{n}\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}-1/2} + \chi^\tau n^{-\frac{\tau}{d}} k\right).$$

Then $k_{opt}^{-2} = \theta^{-1}\left(\frac{\delta\sqrt{n}}{\rho}\right)$.

Taking into account that for $\phi(\gamma) = \gamma^\beta$ we have $\theta^{-1}(\gamma) = \gamma^{\frac{2}{2\beta+1}}$, then from (17) we obtain (18). □

Remark 8. *In view of the data error estimation*

$$\|\bar{g} - \bar{g}^\delta\|_{\mathbb{R}^n} \le \delta\sqrt{n}$$

it is natural to assume that $\delta\sqrt{n} \ll 1$*, or, what is the same,* $n \ll \delta^{-2}$*. If* n *can be chosen at will, then, as it has been shown in* [2, Corollary 4.13]*, under the condition* $\alpha + \tau > d/2$*, an optimal choice is* $n \simeq \delta^{-\frac{d}{\alpha+\tau}}$*. However, it is very often, that the amount of available noisy data is limited such that one should deal with*

$$n \ll \delta^{-\frac{d}{\alpha+\tau}}.$$

For such n *using a-priori parameter choice* $\tilde{\gamma} = \delta n^{-\frac{\alpha+\tau-d}{d}}$ *suggested in* [2, Corollary 4.11] *one has the following error bound*

$$\begin{aligned} \|f_\delta - f_{\tilde{\gamma}}^\delta\|_{L_2} &\le \tilde{C}\left(n^{-\frac{\tau}{d}} + \delta n^{\frac{\alpha}{d}} + \sqrt{\delta} n^{\frac{\alpha-\tau}{2d}}\right) \\ &= O(n^{-\frac{\tau}{d}}). \end{aligned}$$

At the same time, from Corollary 1 it follows that a-posteriori parameter choice $k = k_+$ *allows a higher order error bound. Indeed, keeping in mind that*

$$n^{-\frac{\tau}{d}} \gg \delta n^{-\frac{\alpha}{d}}, \quad n^{-\frac{\tau}{d}} \gg \sqrt{\delta} n^{-\frac{\alpha-\tau}{2d}}$$

from (18) we have

$$\|f_\delta - f_{k_+}^\delta\|_{L_2} \ll n^{-\frac{\tau}{d}}.$$

Remark 9. *Recall that we are looking for the solution* f^+ *of a normally solvable problem (2). It is well known (see, for example,* [1, Section 3.3]*) that in such situation the error bound for direct reconstruction of* f^+ *from noisy data is determined by* $\frac{\varepsilon}{\lambda_n}$*, where* ε *is a given data error level of the right-hand side and* λ_n *is the smallest singular value of* A_X*. In view of the condition (3) it is natural to assume that in our case it holds* $\lambda_n \sim n^{-\frac{\alpha}{d}}$*. Then, keeping in mind* $\varepsilon = \delta\sqrt{n}$ *we obtain*

$$\frac{\varepsilon}{\lambda_n} \sim \delta n^{\frac{\alpha}{d}+\frac{1}{2}}. \tag{19}$$

At the same time, from (18) it follows that for $\delta^{-1} \le n^{\frac{1}{2}+\frac{(2\beta+1)(\alpha+\tau)}{d}}$

$$\|f_\delta - f_{k_+}^\delta\|_{L_2} \le O(\delta n^{\frac{\alpha}{d}+\frac{1}{2}}). \tag{20}$$

Comparing (19) and (20) one can conclude that, if the amount n *of available discrete data is sufficiently large such that* $n \ll \delta^{-2}$ *but*

$$\delta^{-\frac{2d}{2(2\beta+1)(\alpha+\tau)+d}} \ll n,$$

or (see Remark 2)

$$\delta^{-\frac{2d}{2(2\beta+1)(\alpha+\tau)+d}} \ll n \ll \delta^{-\frac{d}{\alpha+\tau}}$$

then the regularized solution $f_{k_+}^\delta$ *allows a better error bound (in the sense of order) than the direct reconstruction.*

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