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THE MIXED DIRICHLET-NEUMANN PROBLEM FOR THE ELLIPTIC EQUATION OF THE SECOND ORDER IN DOMAIN WITH THIN INCLUSION

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Резюме. Розглянуто змішану задачу Діріхле-Неймана для еліптичного рівняння другого порядку в обмеженій тривимірній Ліпшицевій області з тонким включенням, яке моделюється розімкнутою поверхнею. Гранична умова Діріхле задана на одній стороні цієї поверхні, а умова Неймана – на іншій. Введено функціональні простори в області із включенням та оператори сліду на розімкнутій Ліпшицевій поверхні. Доведено еквівалентність задачі у диференціальному формулюванні та відповідної варіаційної задачі. Досліджено питання існування та єдиності розв'язку поставленої задачі з неоднорідними граничними умовами у відповідних функціональних просторах.

Abstract. We consider Dirichlet-Neumann mixed boundary value problem for elliptic equation of the second order in three dimensional domain with thin inclusion which is presented by an open Lipschitz surface. The Dirichlet condition is posed on one side of the surface and the Neumann condition on the other side. Functional spaces in the domain with inclusion and corresponding trace operators on an open Lipschitz surface are introduced. We prove the equivalence of initial mixed boundary value problem and connected variational problem. As a result we obtain existence and uniqueness of solution of the posed problem with nonhomogeneous boundary conditions in appropriate functional spaces.

Introduction

Mixed boundary value problems for the second order elliptic equations in the case when on one part of closed boundary are given conditions of Dirichlet type and on another one conditions of Neumann type were considered in [2, 5, 9]. Boundary value problems in domains with thin inclusion as well as crack in solid bodies have a grate interest in applications. It's pretty convenient to present this thin object as an open double sided surface. Then for a mixed boundary value problem in unregular domain we have the Dirichlet conditions on one side of the open surface and the Neumann condition on the other one. Such kind of problems were considered in [3, 7] where the posed problems were reduced to systems of integral equations over the open boundary.

So far as domain with open surface is essentially unregular we have additional problems connected with definitions of corresponding trace maps and appropriate functional spaces [1, 2].

†*Key words.* Mixed boundary value problem; elliptic operator; variational problem; open surface.

In present paper we use a variational formulation of the posed mixed boundary value problem which gives us opportunity to obtain the existence and uniqueness of solution.

1. Functional spaces and trace operators

Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded connected Lipschitz domain. This means that its boundary Σ is locally the graph of a Lipschitz function [1, 2]. Let us note that Σ can be piecewise smooth and have edges and corners. $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$. We suppose that S is an open Lipschitz surface bounded by closed curve Γ, $\overline{S} = S \cup \Gamma$ and $\overline{S} \subset \Omega_+$. We denote $\Omega = \Omega_+ \setminus \overline{S}$ and consider S as a part of a some closed bounded Lipschitz surface $\Sigma_0 = \overline{S} \cup S_0$, $\Sigma_0 \subset \Omega_+$.

Since Σ and S are the Lipschitz surfaces almost everywhere we can define outward pointing vector of the normal $\vec{n}_x$, $x \in \Sigma$, and depend on the direction of $\vec{n}_x$, $x \in S$, we consider S as a double sided surface with sides S_+ and S_-.

In Ω_+ we consider the elliptic operator of the second order

$$Lu = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$

and connected bilinear form

$$a(u,v) = \int_\Omega \{\sum_{i,j=1}^{3} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j} + a_0 uv\}dx.$$

Here $a_{ij}, a_0 \in C^1(\overline{\Omega}_+)$ are real functions which satisfy the following conditions for $x \in \Omega_+$:

$$\sum_{i,j=1}^{3} a_{ij}t_it_j \geq c_1 \sum_{i=1}^{3} t_i^2, \quad t_i \in \mathbb{R},\ i = \overline{1,3}\,, c_1 > 0, \quad a_0(x) \geq c_2 > 0.$$

We use the Hilbert spaces $H^1(\Omega_+)$ and $H^1(\Omega_+, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega_+)} = \int_{\Omega_+} \left\{|\nabla u|^2 + u^2\right\} dx, \quad (u,v)_{H^1(\Omega_+)} = \int_{\Omega_+} \left\{(\nabla u, \nabla v) + uv\right\} dx,$$

$$\|u\|^2_{H^1(\Omega_+,L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

$$(u,v)_{H^1(\Omega_+,L)} = (u,v)_{H^1(\Omega_+)} + (Lu,Lv)_{L_2(\Omega_+)}.$$

The following trace operators $\gamma^+_{0,\Sigma} : H^1(\Omega_+) \to H^{1/2}(\Sigma)$ and $\gamma^+_{1,\Sigma} : H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ are continuous and surjective [1, 4]. Here $\gamma^+_{1,\Sigma}u \in H^{-1/2}(\Sigma)$ and coincides with $\frac{\partial u}{\partial n_x}$ for $u \in C^1(\overline{\Omega}_+)$ where $\frac{\partial}{\partial n_x} = \sum_{i,j=1}^{3} \cos(\vec{n}_x, \vec{x}_i) a_{ij} \frac{\partial}{\partial x_j}$ is a conormal derivative, $\cos(\vec{n}_x, x_i)$ are the coordinates of the almost everywhere defined outward pointing vector of the normal $\vec{n}_x$ to Σ.

Let us denote by $C_0^\infty(\Omega)$ the class of infinitely differentiable functions with compact support in Ω. We introduce the Hilbert spaces $H^1(\Omega)$ and $H^1(\Omega, L)$

of real functions with norms

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \left\{|\nabla u|^2 + u^2\right\} dx, \tag{1}$$

$$\|u\|^2_{H^1(\Omega,L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)},$$

where derivatives $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$ are defined as

$$\left(\frac{\partial u}{\partial x_i}, \varphi\right)_{L_2(\Omega)} = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\left(u, \frac{\partial \varphi}{\partial x_i}\right)_{L_2(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$.

We consider some trace maps in Ω. We denote $\gamma^\pm_{0,S}$ and $\gamma^\pm_{1,S}$ the restrictions of trace maps $\gamma^\pm_{0,\Sigma_0}$ and $\gamma^\pm_{1,\Sigma_0}$ on S respectively [6]. Then we have $\gamma^\pm_{0,S} : H^1(\Omega) \to H^{1/2}(S)$ and $\gamma^\pm_{1,S} : H^1(\Omega, L) \to H^{-1/2}(S)$.

$$H^1_0(\Omega) = \left\{u \in H^1(\Omega) : \gamma^\pm_{0,S} u = 0, \;\; \gamma^+_{0,\Sigma} u = 0\right\}, \quad H^{-1}(\Omega) = (H^1_0(\Omega))'.$$

We also have that $H^1_0(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the norm (1).

In what follows we use the next trace maps: $[\gamma_{0,S}] = \gamma^+_{0,S} - \gamma^-_{0,S}$, $[\gamma_{1,S}] = \gamma^+_{1,S} - \gamma^-_{1,S}$. As it was shown in [6, 7] $[\gamma_{0,S}] : H^1(\Omega) \to H^{1/2}_{00}(S)$ and $[\gamma_{1,S}] : H^1(\Omega, L) \to H^{-1/2}_{00}(S)$, where $H^{1/2}_{00}(S) = \{g \in H^{1/2}(S) : p_0 g \in H^{1/2}(\Sigma_0)\}$. Here $p_0 g$ is extension by zero of the function g on S_0. The norm in $H^{1/2}_{00}(S)$ is given as $\|g\|_{H^{1/2}_{00}(S)} = \|p_0 g\|_{H^{1/2}(\Sigma_0)}$. $H^{-1/2}_{00}(S) = (H^{1/2}(S))'$, $H^{-1/2}(S) = (H^{1/2}_{00}(S))'$.

Let us denote $H^1_S(\Omega) = \{u \in H^1(\Omega) : \gamma^-_{0,S} u = 0\}$. If $u \in H^1_S(\Omega)$ then $\gamma^+_{0,S} u \in H^{1/2}_{00}(S)$ [8].

In [6] we obtained the first Green's formula for domain with an open surface which in presented case for $u \in H^1(\Omega, L)$ and $v \in H^1(\Omega)$ has the following form:

$$a(u, v) = (Lu, v)_{L_2(\Omega)} + \langle \gamma^+_{1,S} u, [\gamma_{0,S}] v\rangle + \langle [\gamma_{1,S}] u, \gamma^-_{0,S}] v\rangle + \langle \gamma^+_{1,\Sigma} u, \gamma^+_{0,\Sigma} v\rangle. \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ are relations of duality between $H^{1/2}_{00}(S)$ and $H^{-1/2}(S)$, $H^{1/2}(S)$ and $H^{-1/2}_{00}(S)$, $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ respectively.

We assume that Ω_1 is a Lipschitz domain bounded by the closed surface Σ_0. $\overline{\Omega}_1 = \Omega_1 \cup \Sigma_0$, $\Omega_2 = \Omega_+ \setminus \overline{\Omega}_1$. We denote by u_i the restriction of $u \in H^1(\Omega)$ to Ω_i, $i = 1, 2$. It's obviously that $u_i \in H^1(\Omega_i)$, $i = 1, 2$.

In ([8], Lemma 5) we obtained the next proposition.

Lemma 1. *Let $u \in H^1(\Omega)$. Then the norm (1) can be presented in the following form:*

$$\|u\|^2_{H^1(\Omega)} = \|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)}$$

and this norm doesn't depend on the choice of S_0.

Lemma 2. *The operator $\gamma_{0,S} = (\gamma^+_{0,\Sigma}, \gamma^+_{0,S}) : H^1_S(\Omega) \to H^{1/2}(\Sigma) \times H^{1/2}_{00}(S)$ is continuous and surjective.*

Proof. Let $g \in H^{1/2}(\Sigma)$, $g_0 \in H^{1/2}_{00}(S)$ are arbitrary functions. We denote by $\tilde{g}_0 \in H^{1/2}(\Sigma_0)$ the extension g_0 by zero on S_0. Operator $\gamma^+_{0,\Sigma_0} : H^1(\Omega_1) \to H^{1/2}(\Sigma_0)$ is continuous and surjective [1]. Thus we have function $u_1 \in H^1(\Omega_1)$ with trace meaning $\gamma^+_{0,\Sigma_0} u_1 = \tilde{g}_0$ and

$$\|\tilde{g}_0\|_{H^{1/2}(\Sigma_0)} = \|g_0\|_{H^{1/2}_{00}(S)} \le c_1 \|u_1\|_{H^1(\Omega_1)}. \tag{3}$$

Analogously there exists $u_2 \in H^1(\Omega_2)$ that $\gamma^-_{0,\Sigma_0} u_2 = 0$, $\gamma^+_{0,\Sigma} u_2 = g$ and

$$\|g\|_{H^{1/2}(\Sigma)} \le c_2 \|u_2\|_{H^1(\Omega_2)}. \tag{4}$$

As a result we obtained the function $u \in L_2(\Omega)$ with restrictions $u_i \in H^1(\Omega_i)$ to Ω_i, $i = 1, 2$. Then $[\gamma_{0,S_0}]u = \gamma^+_{0,S_0} u_1 - \gamma^-_{0,S_0} u_2 = 0$ and by ([8], Lemma 4) we have $u \in H^1(\Omega)$. Since $\gamma^-_{0,S} u = 0$ it follows that $u \in H^1_S(\Omega)$.

In order to prove continuity of the trace map $\gamma_{0,S}$ we consider function $u \in H^1_S(\Omega)$ with $\gamma^+_{0,\Sigma} u = g \in H^{1/2}(\Sigma)$ and $\gamma^+_{0,S} u = g_0 \in H^{1/2}_{00}(S)$.

Then from (3), (4) and Lemma 1 we obtain

$$\|g\|_{H^{1/2}(\Sigma)} + \|g_0\|_{H^{1/2}_{00}(S)} \le c_1 \|u_1\|_{H^1(\Omega_1)} + c_2 \|u_2\|_{H^1(\Omega_2)} \le c\|u\|_{H^1(\Omega)}.$$

Here c, c_1, c_2 are some positive constants. □

2. Mixed boundary value problem and it's variational formulation

Let us state a Dirichlet-Neumann mixed boundary value problem in domain Ω.

Problem M. Find a function $u \in H^1(\Omega, L)$ that satisfies

$$Lu = h, \quad \gamma^-_{0,S} u = g, \quad \gamma^+_{1,S} u = f, \quad \gamma^+_{1,\Sigma} u = z.$$

Here $h \in L_2(\Omega)$, $g \in H^{1/2}(S)$, $f \in H^{-1/2}(S)$, $z \in H^{-1/2}(\Sigma)$ are given functions.

A partial case of the problem M when $g = 0$ we denote as problem M_0.

With problem M_0 it's closely connected the following variational problem.

Problem VM_0. Find a function $u \in H^1_S(\Omega)$ that satisfies

$$a(u, v) = l(v) \tag{5}$$

for every $v \in H^1_S(\Omega)$.

Here

$$l(v) = (h, v)_{L_2(\Omega)} + \langle f, \gamma^+_{0,S} v\rangle + \langle z, \gamma^+_{0,\Sigma} v\rangle, \tag{6}$$

$h \in L_2(\Omega)$, $f \in H^{-1/2}(S)$, $z \in H^{-1/2}(\Sigma)$ are given functions.

Lemma 3. *Bilinear form $a(u, v) : H^1_S(\Omega) \times H^1_S(\Omega) \to \mathbb{R}$ is continuous and $H^1_S(\Omega)$-elliptic.*

Proof. Since $H^1_S(\Omega)$ is a subspace of $H^1(\Omega)$ this lemma is a corollary of ([8], Lemma 7). □

Theorem 1. *Problems M_0 and VM_0 are equivalent.*

Proof. Let $u \in H^1_S(\Omega)$ be a solution of problem M_0. Then from the first Green's formula (1) for any $v \in H^1_S(\Omega)$ we have

$$a(u,v) = (h,v)_{L_2(\Omega)} + \langle f, \gamma^+_{0,S}v\rangle + \langle z, \gamma^+_{0,\Sigma}\rangle.$$

Thus u is a solution of problem VM_0.

Let now $u \in H^1_S(\Omega)$ be a solution of problem VM_0. Since $H^1_0(\Omega)$ is a subspace of $H^1_S(\Omega)$ for any $v \in H^1_0(\Omega)$ from (5) we obtain

$$a(u,v) = (h,v)_{L_2(\Omega)}.$$

But as it was shown in [6, 8] for any $u \in H^1(\Omega)$ and $v \in H^1_0(\Omega)$ we have $a(u,v) = \langle Lu, v\rangle$, where $\langle\cdot,\cdot\rangle$ is relations of duality between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. Thus $\langle Lu, v\rangle = \langle h, v\rangle$ or $\langle Lu - h, v\rangle = 0$ for any $v \in H^1_0(\Omega)$. It means that $Lu = h$. So far as $f \in L_2(\Omega)$ we get $u \in H^1(\Omega, L)$.

In Lemma 2 we showed that the trace operator $\gamma_{0,S} = (\gamma^+_{0,\Sigma}, \gamma^+_{0,S}) : H^1_S(\Omega) \to H^{1/2}(\Sigma) \times H^{1/2}_{00}(S)$ is surjective. Using (2), (5) and $Lu = h$ we have

$$\langle \gamma^+_{1,S}u - f, \gamma^+_{0,S}v\rangle + \langle \gamma^+_{1,\Sigma}u - z, \gamma^+_{0,\Sigma}v\rangle = 0$$

which is valid for an arbitrary $v \in H^1_S(\Omega)$. Thus $\gamma^+_{1,S}u = f$ and $\gamma^+_{1,\Sigma}u = z$. It gives us that u is a solution of problem M_0. □

Theorem 2. *Problem VM_0 has a unique solution for arbitrary $h \in L_2(\Omega)$, $f \in H^{-1/2}(S)$, $z \in H^{-1/2}(\Sigma)$.*

Proof. Lemma 3 gives us that the bilinear form $a(u,v) : H^1_S(\Omega) \times H^1_S(\Omega) \to \mathbb{R}$ is continuous and $H^1_S(\Omega)$-elliptic. Let's show that the functional $l : H^1_S(\Omega) \to \mathbb{R}$ given by (6) is continuous. If $v \in H^1_S(\Omega)$ then using Lemma 2 we have:

$$|l(v)| \le \|h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \|f\|_{H^{-1/2}(S)}\|\gamma^+_{0,S}v\|_{H^{1/2}_{00}(S)} +$$

$$\|z\|_{H^{-1/2}(\Sigma)}\|\gamma^+_{0,\Sigma}v\|_{H^{1/2}(\Sigma)} \le \|h\|_{L_2(\Omega)}\|v\|_{H^1(\Omega)} + c_1\|f\|_{H^{-1/2}(S)}\|v\|_{H^1(\Omega)} +$$

$$c_2\|z\|_{H^{-1/2}(\Sigma)}\|v\|_{H^1(\Omega)} \le c\|v\|_{H^1(\Omega)},$$

where c, c_1, c_2 some positive constants which do not depend on v. Then by the Lax-Milgram Lemma we obtain what was to be proved. □

Corollary 4. *Problem M_0 has a unique solution for arbitrary $h \in L_2(\Omega)$, $f \in H^{-1/2}(S)$, $z \in H^{-1/2}(\Sigma)$.*

Lemma 4. *For every $g \in H^{1/2}(S)$ there exists function $w \in H^1(\Omega, L)$ that $\gamma^-_{0,S}w = g$.*

Proof. From [8] it follows that for every $g \in H^{1/2}(S)$ there exists $u_1 \in H^1(\Omega)$ and $Lu_1 = 0$, $\gamma^-_{0,S}u_1 = g$. Analogously for any $h_2 \in L_2(\Omega)$ we have a function $u_2 \in H^1(\Omega)$ that $Lu_2 = h_2$ and $\gamma^-_{0,S}u_2 = 0$. Thus we obtain a class of function $w = u_1 + u_2$ that $w \in H^1(\Omega, L)$ and $\gamma^-_{0,S}w = g$. □

Now we consider problem M which differs from problem M_0 only by non-homogeneous boundary condition on S_-. Lemma 4 gives us the function $w \in H^1(\Omega, L)$ which satisfies boundary condition $\gamma^-_{0,S}w = g$. Let the function u_1 be a solution of problem M_0:

$$Lu_1 = h_1, \quad \gamma^-_{0,S}u_1 = 0, \quad \gamma^+_{1,S}u_1 = f_1, \quad \gamma^+_{1,\Sigma}u_1 = z_1,$$

where $h_1 = h - Lw$, $f_1 = f - \gamma^+_{1,S}w$, $z_1 = z - \gamma^+_{1,\Sigma}w$. Then the function $u = u_1 + w$ is a solution of problem M. The preceding considerations imply the following assertion.

Theorem 3. *Problem M has a unique solution for arbitrary $h \in L_2(\Omega)$, $g \in H^{1/2}(S)$, $f \in H^{-1/2}(S)$, $z \in H^{-1/2}(\Sigma)$.*

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