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NONLOCAL PROBLEM FOR AN EVOLUTION FIRST ORDER EQUATION IN BANACH SPACE

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Резюме. Розглянуто двоточкову нелокальну задачу для диференціального еволюційного рівняння першого порядку з операторним коефіцієнтом у банаховому просторі. Запропоновано і обґрунтовано експонентціально збіжний алгоритм у припущенні, що операторний коефіцієнт є строго позитивний і виконуються деякі умови існування і єдиності. Алгоритм приводить до системи лінійних рівнянь, які можна розв'язати методом простої ітерації. Алгоритм забезпечує експонетціальну збіжність за часом, що в поєднанні з швидкими алгоритмами за просторовими змінними може бути ефективним для розв'язування таких задач. Ефективність пропонованих алгоритмів продемонстрована на чисельних експериментах.

Abstract. Two-points nonlocal problem for the first order differential evolution equation with an operator coefficient in a Banach space X is considered. An exponentially convergent algorithm is proposed and justified under the assumption that the operator coefficient is strongly positive and some existence and uniqueness conditions hold. This algorithm leads to a system of linear equations that can be solved by fixed-point iteration. The algorithm provides exponentially convergence in time that in combination with fast algorithms on spatial variables can be efficient for solving such problems. The efficiency of the proposed algorithms is demonstrated through numerical examples.

AMS Subject Classification: 65J10, 65M12, 65M15, 46N20, 46N40, 47N20, 47N40

1. Introduction

The m-point initial (nonlocal) problem for a differential equation with the nonlocal condition

$$u(t_0) + g(t_1; \dots; t_p; u) = u_0$$

and a given function g on a given point set $P = \{0 = t_0 < t_1 < \cdots < t_p\}$ is one of the important topics in the study of differential equations. Interest in such problems originates mainly from some physical problems with a control of the solution at P. For example, when the function $g(t_1; \dots; t_p; u)$ is linear we will have a periodic problem $u(t_0) = u(t_1)$. Problems with nonlocal conditions arise in the theory of physics of plasma [15], nuclear physics [10], mathematical chemistry [11], waveguides [8] etc. Two-point problem is also useful for considering the finale value problem [18].

Differential equations with operator coefficients in a Hilbert or Banach space can be considered as meta-models for systems of partial or ordinary differential equations and are suitable for investigating using the tools of the functional

[†]*Key words.* First order differential evolution equations in Banach space, nonlocal problem, unbounded operator coefficient, operator exponential, exponentially convergent algorithms.

analysis (see e.g. [4, 9]). Nonlocal problems can also be considered within this framework [2, 3].

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [5, 7, 12, 13, 16, 17, 22, 23] and the references therein). Methods from [7, 12, 13, 17, 22, 23] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type $O(\mathrm{e}^{-N^\alpha})$, $\alpha > 0$ with respect to a discretization parameter $N \to \infty$. For a given tolerance ε such discretization provides optimal or nearly optimal computational complexity [7].

In the present paper we consider the problem

$$\begin{aligned} &\frac{du(t)}{dt} + A_1(t)u(t) = f_1(t), \\ &u(0) + \alpha u(1) = \varphi, \end{aligned} \tag{1}$$

where $A_1(t)$ is a densely defined closed (unbounded) operator with the domain $D(A_1)$ independent of t in a Banach space X, φ is a given vector and $f_1(t)$ is a given vector-valued function, $\alpha \in \mathbb{R}$. We suppose that the operator $A_1(t)$ is strongly positive; i.e. there exists a positive constant M_R independent of t such that on the rays and outside a sector $\Sigma_\theta = \{z \in \mathbb{C} : 0 \le arg(z) \le \theta, \theta \in (0, \pi/2)\}$ the following estimate for a resolvent holds:

$$\|(zI - A_1(t))^{-1}\| \le \frac{M_R}{1 + |z|}. \tag{2}$$

This assumption implies that there exists a positive constant c_κ such that (see [6], p.103)

$$\|A_1^\kappa(t) e^{-sA_1(t)}\| \le c_\kappa s^{-\kappa}, \quad s > 0, \quad \kappa \ge 0. \tag{3}$$

Our further assumption is that there exists a real positive ω such that

$$\|e^{-sA_1(t)}\| \le e^{-\omega s} \quad \forall s,\ t \in [0, 1] \tag{4}$$

(see [14], Corollary 3.8, p.12, for corresponding assumptions on $A_1(t)$). Let us also assume that the following conditions are valid

$$\|[A_1(t) - A_1(s)]A_1^{-\gamma}(t)\| \le L_{1,\gamma}|t - s| \quad \forall t,\ s,\ 0 \le \gamma < 1, \tag{5}$$

$$\|A_1^\gamma(t) A_1^{-\gamma}(s) - I\| \le L_\gamma |t - s| \quad \forall t,\ s \in [0, 1]. \tag{6}$$

We suppose also that

$$f_1(t) \in C(0, 1; X). \tag{7}$$

The aim of this paper is to construct an exponentially convergent approximation for a solution to problem (1). The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical algorithm is presented in section 3. The main result of this section is theorem 1 about the convergence rate of the proposed discretization. In the next section 4 we present a numerical example which confirm theoretical results from the previous sections.

2. Existence and uniqueness of the solution

It is well known, that for $\alpha = 0$ the problem (1) has a unique solution under the assumptions (2)-(7) (se e.g. [14, 9]). This solution can be written down as follows:

$$u(t) = U(t,0)u(0) + \int_0^t U(t,s)f_1(s)ds = U(t,0)\varphi + \int_0^t U(t,s)f_1(s)ds, \quad (8)$$

where $U(t,s)$ is an evolution operator that corresponds to (1) for $\alpha = 0$.

Let us study conditions when there is a unique solution to the two-points problem (1). We have from (8)

$$u(1) = U(1,0)u(0) + \int_0^1 U(1,s)f_1(s)ds.$$

Substituting this expression into the nonlocal condition we obtain

$$u(0) = [I + \alpha U(1,0)]^{-1}\left[\varphi - \alpha \int_0^1 U(1,s)f_1(s)ds\right],$$

and for $u(t)$ we have

$$u(t) = U(t,0)\left[I + \alpha U(1,0)\right]^{-1}\left[\varphi - \alpha \int_0^1 U(1,s)f_1(s)ds\right] + \\ + \int_0^t U(t,s)f_1(s)ds.$$

It is necessary to establish conditions on α for the existence of $u(t)$. In fact, we have to explore when exists $[I + \alpha U(1,0)]^{-1}$. So, we obtain using estimate for $U(t,s)$ (see e.g. [14, 9]).

$$\left\|[I + \alpha U(1,0)]^{-1}\right\| \le [1 - |\alpha|\,\|U(1,0)\|]^{-1} \le [1 - |\alpha| M]^{-1} \le C,$$

for small enough α ($\alpha < M^{-1}$).

3. Numerical algorithm

We use the approach developed in [7] and [21] to construct numerical method for solving problem (1). First of all we change variable in (1) by $t \to \frac{1+t}{2}$ and for $v(t) = u\left(\frac{1+t}{2}\right)$ we have

$$\begin{aligned} &\frac{dv(t)}{dt} + A(t)v(t) = f(t), \\ &v(-1) + \alpha v(1) = \varphi, \end{aligned} \quad (9)$$

where $A(t) = \frac{1}{2}A_1\left(\frac{1+t}{2}\right)$, $f(t) = \frac{1}{2}f_1\left(\frac{1+t}{2}\right)$,

We choose a mesh $\omega_n = \{t_k,\ k = 0, ..., n\}$ of $n+1$ various points on $[-1,1]$ that are Chebyshev-Gauss-Lobatto nodes $t_k = \cos\left(\frac{n-k}{n}\pi\right)$ and set $\tau_k = t_k - t_{k-1}$. Let

$$\begin{aligned} \overline{A}(t) = &A_k = A(t_k), t \in (t_{k-1}, t_k], \quad k = \overline{1,n}, \\ &A_0 = A(-1). \end{aligned}$$

Let us rewrite the problem (9) in the equivalent form

$$\begin{aligned}&\frac{dv}{dt}+\overline{A}(t)v=[\overline{A}(t)-A(t)]v(t)+f(t), \quad t\in(-1,1)\\ &v(-1)=\varphi-\alpha v(1).\end{aligned} \tag{10}$$

Note that now all operators on the left hand side of these equations are constant on each subinterval and piece-wise constant on the whole interval $[-1,1]$.

On each subinterval we can write down the equivalent to (10) integral equation

$$\begin{aligned}v(t)=&\mathrm{e}^{-A_k(t-t_{k-1})}v(t_{k-1})+\int_{t_{k-1}}^{t}\mathrm{e}^{-A_k(t-s)}\left[A_k-A(t)\right]v(s)ds+\\ &+\int_{t_{k-1}}^{t}\mathrm{e}^{-A_k(t-s)}f(s)ds, \quad t\in[t_{k-1},t_k], \quad k=\overline{2,n},\end{aligned} \tag{11}$$

$$\begin{aligned}v(t)=&\mathrm{e}^{-A_1(t+1)}\left[\varphi-\alpha v(1)\right]+\int_{-1}^{t}\mathrm{e}^{-A_1(t-s)}\left[A_1-A(t)\right]v(s)ds+\\ &+\int_{-1}^{t}\mathrm{e}^{-A_1(t-s)}f(s)ds, \quad t\in[-1,t_1].\end{aligned}$$

Let

$$P_n(t;v)=P_nv=\sum_{j=0}^{n}v(t_j)L_{j,n}(t),$$

be the interpolation polynomial for $v(t)$ on the mesh ω_n, $x=(x_0,...,x_n)$, $x_i\in X$ given vector and

$$P_n(t;y)=P_nx=\sum_{j=0}^{n}x_jL_{j,n}(t)$$

the polynomial that interpolates x where

$$L_{j,n}(s)=\frac{T_n'(s)(1-s^2)}{\frac{d}{ds}[(1-s^2)T_n'(s)]_{s=s_j}(s-s_j)}, \quad j=0,...,n$$

are the Lagrange fundamental polynomials. Substituting $P_n(s;x)$ for $v(s)$, x_k for $v(t_k)$ and then setting $t=t_k$ in (11) we obtain the following system of linear equations with respect to the unknown x_k :

$$\begin{aligned}&x_0+\alpha x_n=\varphi,\\ &x_k=\mathrm{e}^{-A_k\tau_k}x_{k-1}+\sum_{j=0}^{n}\alpha_{kj}x_j+\phi_k, \quad k=\overline{1,n},\end{aligned} \tag{12}$$

which represents our algorithm. Here we use the notations

$$\begin{aligned}\alpha_{kj}&=\int_{t_{k-1}}^{t_k}\mathrm{e}^{-A_k(t_k-s)}[A_k-A(s)]L_{j,n}(s)ds,\\ \phi_k&=\int_{t_{k-1}}^{t_k}\mathrm{e}^{-A_k(t_k-s)}f(s)ds, \quad k=\overline{1,n}, \quad j=\overline{0,n},\end{aligned}$$

and suppose that we have an algorithm to compute these coefficients.

For the error $z = (z_1, ..., z_n)$, with $z_k = v(t_k) - x_k$ we have the relations

$$z_0 + \alpha z_n = 0,$$
$$z_k = \mathrm{e}^{-A_k \tau_k} z_{k-1} + \sum_{j=0}^{n} \alpha_{kj} z_j + \psi_k, \quad k = \overline{1,n}, \tag{13}$$

where

$$\psi_k = \int_{t_{k-1}}^{t_k} \mathrm{e}^{-A_k(t_k - s)} [A_k - A(s)][v(s) - P_n(s;v)] ds, \quad k = \overline{1,n},$$

In order to represent algorithm (12) in a block-matrix form we introduce the matrix

$$S = \begin{pmatrix} I & 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha\sigma_0 \\ -\sigma_1 & I & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\sigma_2 & I & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\sigma_n & I \end{pmatrix}, \tag{14}$$

where $\sigma_0 = A_0^\gamma A_n^{-\gamma}$, $\sigma_k = \mathrm{e}^{-A_k \tau_k} A_k^\gamma A_{k-1}^{-\gamma}$, $k = \overline{1,n}$, the matrix $B = \{\tilde{\alpha}_{k,j}\}_{k,j=0}^n$ with $\tilde{\alpha}_{k,j} = A_k^\gamma \alpha_{k,j} A_j^{-\gamma}$, $k = \overline{1,n}$, $j = \overline{0,n}$, and $\tilde{\alpha}_{0,j} = 0$, $j = \overline{0,n}$, the vectors

$$\tilde{x} = \begin{pmatrix} A_0^\gamma x_0 \\ A_1^\gamma x_1 \\ \cdot \\ \cdot \\ A_n^\gamma x_n \end{pmatrix}, \ \phi = \begin{pmatrix} A_0^\gamma \varphi \\ A_1^\gamma \phi_1 \\ \cdot \\ \cdot \\ A_n^\gamma \phi_n \end{pmatrix}, \ \tilde{z} = \begin{pmatrix} A_0^\gamma z_0 \\ A_1^\gamma z_1 \\ \cdot \\ \cdot \\ A_n^\gamma z_n \end{pmatrix}, \ \psi = \begin{pmatrix} 0 \\ A_1^\gamma \psi_1 \\ \cdot \\ \cdot \\ A_n^\gamma \psi_n \end{pmatrix}. \tag{15}$$

It is easy to check that for the (left) inverse

$$S^{-1} = \delta \left(R_1 - R_2 \right),$$

where

$$\delta = \left(I + \alpha\sigma_0\sigma_1 \ldots \sigma_n \right)^{-1},$$

$$R_1 = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ \sigma_1 & I & \cdots & 0 & 0 \\ \sigma_2\sigma_1 & \sigma_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \sigma_n \cdots \sigma_1 & \sigma_n \cdots \sigma_2 & \cdots & \sigma_n & I \end{pmatrix},$$

$$R_2 = \alpha s_0 \begin{pmatrix} 0 & \sigma_n \ldots \sigma_2 & \sigma_n \ldots \sigma_3 & \cdots & \sigma_n & I \\ 0 & 0 & \sigma_1\sigma_n \ldots \sigma_3 & \cdots & \sigma_1\sigma_n & \sigma_1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & \sigma_{n-1} \ldots \sigma_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Remark 10. *Using results of* [7] *one can get a parallel and sparse approximations with an exponential convergence rate of the operator exponentials contained in* S^{-1} *and as a consequence a parallel and sparse approximation of* S^{-1}.

We multiply the equations in (12) and the equation in (13) by A_k^γ, $k=\overline{0,n}$ and obtain

$$\begin{aligned}
&A_0^\gamma x_0+\alpha A_0^\gamma x_n=A_0^\gamma\varphi,\\
&A_k^\gamma x_k=\mathrm{e}^{-A_k\tau_k}A_k^\gamma x_{k-1}+\sum_{j=0}^{n}\tilde{\alpha}_{kj}A_j^\gamma x_j+A_k^\gamma\phi_k,\quad k=\overline{1,n},
\end{aligned}\tag{16}$$

$$\begin{aligned}
&A_0^\gamma z_0+\alpha A_0^\gamma z_n=0,\\
&A_k^\gamma z_k=\mathrm{e}^{-A_k\tau_k}A_k^\gamma z_{k-1}+\sum_{j=0}^{n}\tilde{\alpha}_{kj}A_j^\gamma z_j+A_k^\gamma\psi_k,\quad k=\overline{1,n},
\end{aligned}\tag{17}$$

Then systems (16), (17) can be written down in the matrix form using notations (14), (15) as

$$\begin{aligned}
S\tilde{x}&=B\tilde{x}+\phi,\\
S\tilde{z}&=B\tilde{z}+\psi.
\end{aligned}\tag{18}$$

Next, for a vector $v=(v_1,v_2,...,v_n)^T$ and a block operator matrix $A=\{a_{ij}\}_{i,j=1}^n$ we introduce a vector norm

$$|||v|||\equiv|||v|||_1=\max_{1\le k\le n}\|v_k\|,$$

and the consistent matrix norm

$$|||A|||\equiv|||A|||_1=\max_{1\le i\le n}\sum_{j=1}^{n}\|a_{i,j}\|.$$

Due to (6) we have

$$|||A_k^\gamma A_{k-1}^{-\gamma}|||=|||A_k^\gamma A_{k-1}^{-\gamma}-I+I|||\le 1+L_\gamma\tau_k,$$

$$\|\sigma_0\|=\|A_0^\gamma A_n^{-\gamma}\|\le 1+L_\gamma T.$$

In our case $T=2$. So, we have the following, using these estimates

$$\|\sigma_k\|=\|\mathrm{e}^{-A_k\tau_k}A_k^\gamma A_{k-1}^{-\gamma}\|\le\mathrm{e}^{-\omega\tau_k}\|A_k^\gamma A_{k-1}^{-\gamma}\|\le\mathrm{e}^{-\omega\tau_k}\left(1+L_\gamma\tau_k\right),$$

$$\|\delta\|=\left\|\Big(I+\alpha\sigma_0\sigma_1\ldots\sigma_n\Big)^{-1}\right\|\le\Big(1-|\alpha|\,\|\sigma_0\|\,\|\sigma_1\|\,\|\sigma_2\|\ldots\|\sigma_n\|\Big)^{-1}\le$$

$$\le\Big(1-|\alpha|\,\left(1+2L_\gamma\right)\mathrm{e}^{-\omega\tau_1}\left(1+L_\gamma\tau_1\right)\mathrm{e}^{-\omega\tau_2}\left(1+L_\gamma\tau_2\right)\ldots\mathrm{e}^{-\omega\tau_n}\left(1+L_\gamma\tau_n\right)\Big)^{-1}$$

$$\le\left(1-|\alpha|\,\left(1+2L_\gamma\right)\mathrm{e}^{-2\omega}\left(1+\frac{2L_\gamma}{n}\right)^n\right)^{-1}\le$$

$$\le\Big(1-|\alpha|\,\left(1+2L_\gamma\right)\mathrm{e}^{-2\omega}\mathrm{e}^{2L_\gamma}\Big)^{-1}\le c,$$

for α small enough.

In order to estimate the norm of matrix S we must estimate the norms of matrices R_1, R_2. In [7] it was proved that for a matrix similar to R_1 the estimate $|||R_1|||\le cn$ holds true. Let us estimate the norm of matrix R_2.

$$|||R_2||| \le (1+2c)\left(1+e^{-\omega\tau}(1+c\tau)+\cdots+[e^{-\omega\tau}(1+c\tau)]^{n-1}\right) \le$$
$$\le (1+2c)\left(1+(1+c\tau)+\cdots+(1+c\tau)^{n-1} \le \frac{(1+c\tau)^n-1}{c\tau}\right) \le$$
$$\le (1+2c)\,\frac{e^{2c}}{c\tau} \le cn.$$

Using these estimates we obtain that

$$|||S^{-1}||| \le cn. \tag{19}$$

It was proved an estimate for the matrix B in [7]:

$$|||B||| \le cn^{\gamma-2}\ln(n). \tag{20}$$

So we can formulate the following assertion

Lemma 1. *Let assumptions* (2)-(6) *are fulfilled. Then estimates* (19), (20) *hold true.*

Using (18) we have

$$\begin{aligned} \tilde{x} &= \left[E - S^{-1}B\right]^{-1} S^{-1}\phi, \\ \tilde{z} &= \left[E - S^{-1}B\right]^{-1} S^{-1}\psi, \end{aligned} \tag{21}$$

where E is a diagonal matrix with unit operators I on diagonal. Using lemma 1 we obtain that

$$|||S^{-1}B||| \le cn^{\gamma-1}\ln(n) \to 0,\ n\to\infty. \tag{22}$$

It means that for n large enough there exists the matrix $\left[E - S^{-1}B\right]^{-1}$ and

$$\left|\left|\left|\left[E - S^{-1}B\right]^{-1}\right|\right|\right| \le c.$$

Consequently we obtain the following stability estimates from (21) using lemma 1:

$$\begin{aligned} |||\tilde{x}||| &\le cn|||\phi|||, \\ |||\tilde{z}||| &\le cn|||\psi|||. \end{aligned} \tag{23}$$

Let Π_n be the set of all polynomials in t with vector coefficients of degree less or equal than n. In complete analogy with [1, 19, 20] the following Lebesgue inequality for vector-valued functions can be proved

$$\|u(t) - P_n(t;u)\|_{C[-1,1]} \equiv \max_{t\in[-1,1]} \|u(t) - P_n(t;u)\| \le (1+\Lambda_n)E_n(u),$$

with the error of the best approximation of u by polynomials of degree not greater than n

$$E_n(u) = \inf_{p\in\Pi_n} \max_{t\in[-1,1]} \|u(t) - p(t)\|.$$

Now, we can go over to the main result of this section.

Theorem 1. *Let the assumptions of Lemma 1 with $\gamma < 1$ hold, then there exists a positive constant c such that*

1. *For n large enough it holds*
$$|||\tilde{z}||| \leq cn^{\gamma-1} \cdot \ln n \cdot E_n(A_0^\gamma v),$$
where v is the solution of (9)*;*
2. *The first equation in* (18) *can be written in the form*
$$\tilde{x} = S^{-1}B\tilde{x} + S^{-1}\phi,$$
and can be solved by the fixed point iteration
$$\tilde{x}^{(k+1)} = S^{-1}B\tilde{x}^{(k)} + S^{-1}\phi,\ k = 0, 1, ...;\ \tilde{x}^{(0)} - arbitrary,$$
with the convergence rate of an geometrical progression with the denominator $q \leq cn^{\gamma-1}\ln(n) < 1$ for n large enough.

Proof. For $\tilde{z}$ we have the second estimate in (23). The norm of the first summand on the right hand side of this inequality can be estimated in the following way

$$|||\psi||| = \max_{1\leq k\leq n}\left\| \int_{t_{k-1}}^{t_k} \left\{ A_k^\gamma \mathrm{e}^{-A_k(t_k-s)}[A_k - A(s)]\times \right.\right.$$
$$\left.\left. \times A_k^{-\gamma}(A_k^\gamma A_0^{-\gamma})(A_0^\gamma v(s) - P_n(s; A_0^\gamma v)) \right\} ds \right\| \leq$$
$$\leq c \max_{1\leq k\leq n} \int_{t_{k-1}}^{t_k} |t_k - s|^{-\gamma}|t_k - s|\, \|A_0^\gamma v(s) - P_n(s; A_0^\gamma v)\| ds \leq$$
$$\leq c\tau_{max}^{2-\gamma}\, \|A_0^\gamma u(s) - P_n(\cdot; A_0^\gamma v)\|_{C[-1,1]} \leq c\tau_{max}^{2-\gamma}(1 + \Lambda_n)E_n(A_0^\gamma v).$$

So, we obtain
$$|||\psi||| \leq cn^{\gamma-2} \cdot \ln n \cdot E_n(A_0^\gamma u), \tag{24}$$
Now, the first assertion of the theorem follows from (23), (24). The second one follows from (18) and (22). ☐

TABL. 1. The error in the case $n = 4$, $x = 0.5$

Point t	ε
-1	0.00005276
-0.70710678	0.00097645
0	0.00063440
0.70710678	0.00029592
1	0.00010552

4. EXAMPLES

Let us consider the following problem
$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t)u(x,t) = f(x,t), \\ &u(0,t) = u(1,t) = 0, \\ &u(x,-1) + \alpha u(x,1) = \varphi(x), \end{aligned} \tag{25}$$

TABL. 2. The error in the case $n = 6$, $x = 0.5$

Point t	ε
-1	8.12568908E-7
-0.86602540	0.00010146
-0.5	0.00030932
0	0.00022136
0.5	0.00013419
0.86602540	0.00007182
1	0.00000162

TABL. 3. The error in the case $n = 8$, $x = 0.5$

Point t	ε
-1	0.00000117
-0.92387953	0.00000613
-0.70710678	0.00004544
-0.38268343	0.00005753
0	0.00004745
0.38268343	0.00003362
0.70710678	0.00002096
0.92387953	0.00000846
1	0.00000235

TABL. 4. The error in the case $n = 12$, $x = 0.5$

Point t	ε
-1	0.49451310e-8
-0.96592582	0.14687232e-7
-0.86602540	0.23393074e-6
-0.70710678	0.54494052e-6
-0.5	0.76722515e-6
-0.25881904	0.82803283e-6
0	0.76362937e-6
0.25881904	0.63174173e-6
0.5	0.47173110e-6
0.70710678	0.30381367e-6
0.86602540	0.14341583e-6
0.96592582	0.21271757e-7
1	0.98902621e-8

with $f(x,t) = \mathrm{e}^{-\pi^2(1+t)}\sin(\pi x)(1+t)$, $\alpha = 0.5$, $\varphi(x) = \left(1 + 0.5\mathrm{e}^{-2\pi^2}\right)\sin(\pi x)$, $q(x,t) = 1+t$. Then, the solution of this problem is $u(x,t) = \mathrm{e}^{-\pi^2(1+t)}\sin(\pi x)$.

TABL. 5. The error in the case $n = 16$, $X = 0.5$

Point t	ε
-1	0.20628738e-11
-0.98078528	0.28602854e-10
-0.92387953	0.48425552e-9
-0.83146961	0.14258845e-8
-0.70710678	0.25968220e-8
-0.55557023	0.36339719e-8
-0.38268343	0.42916820e-8
-0.19509032	0.44975339e-8
0	0.43045006e-8
0.19509032	0.38169887e-8
0.38268343	0.31414290e-8
0.55557023	0.23686579e-8
0.70710678	0.15787207e-8
0.83146961	0.85640040e-9
0.92387953	0.30309439e-9
0.98078528	0.16809109e-10
1	0.41257476e-11

The problem (25) can be written down in the form (9) where the operator $A(t)$ is defined by

$$D(A(t)) = D(A) = \{v \in H^2(0,1) : v(0) = 0,\ v(1) = 0\},$$
$$A(t)v = -\frac{\partial^2 v}{\partial x^2} + (1+t)v.$$

Coefficients of the system (16) were calculated by using the Fourier series expansion. The results of calculation presented in tables confirm our theory above.

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