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ON THE COMBINATION OF SINGULAR AND HYPERSINGULAR BOUNDARY INTEGRAL EQUATIONS FOR THE NEUMANN BOUNDARY VALUE PROBLEM FOR AN ELLIPTIC EQUATION WITH VARIABLE COEFFICIENTS

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Резюме. Для чисельного розв'язування внутрішньої задачі Неймана для еліптичного рівняння зі змінними коефіцієнтами запропоновано підхід, який приводить до системи граничних інтегральних рівнянь з сингулярними і гіперсингулярними ядрами. Дискретизацію інтегральних рівнянь здійснено методом квадратур із використанням тригонометричних квадратурних формул інтерполяційного типу. Приведено приклади чисельних експериментів.

Abstract. We consider the interior Neumann boundary value problem for an elliptic equation with variable coefficients. For the numerical solution of this problem we develop an approach, which leads to a system of boundary integral equations with strong- and hypersingular kernels. The full discretization is realized by the quadrature method with use of quadrature rules based on trigonometrical interpolation. The results of numerical experiments are presented.

1. Introduction

The boundary integral equation method is an effective tool for theoretical investigations and numerical solution of various boundary value problems. For the use of direct or indirect integral equation approach it is extremely important to know the fundamental solution for the considered differential equation. This is not a big problem for the large number of equations with constant coefficients. But in the case of variable coefficients the fundamental solution is very difficult to find and therefore the integral equation method is not used very often for such kind of problems. However, it is possible to involve the parametrix which describes the main part of the fundamental solution and doesn't satisfy the equation. Note that in the case of elliptic equation of the second order the parametrix is also known as Levi's function [7, 8]. As a result, a given boundary value problem can be reduced to a boundary-domain integral equation. This approach doesn't contain the main advantage of integral equation method related to the decrease of the dimension of the differential problem. Therefore we investigate another approach which does not have this disadvantage. This approach has been applied in [2] for the case of the Dirichlet boundary value condition. Its idea consists in the following: we introduce a set of closed nonintersecting curves in the solution domain and consider the differential equation

†*Key words.* Elliptic equation with variable coefficients; Levi's functions; System of boundary integral equations; Strong and hyper-singularities; Quadrature method.

on these curves. Next we construct potentials with the Levi's function and reduce the given boundary value problem to boundary integral equations with various singularities in the kernels.

In this paper we extend the described approach to the case of the Neumann boundary value condition with the use of strong and hypersingular integral equations.

Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain with the boundary $\Gamma_0 \in C^3$. We search for the function $u : D \to \mathbb{R}$ which satisfies the elliptic equation

$$Lu(x) = \operatorname{div}(\sigma(x) \operatorname{grad} u(x)) = 0, \quad x \in D \tag{1}$$

and the Neumann boundary value condition

$$\sigma(x)\frac{\partial u}{\partial \nu}(x) = f(x), \quad x \in \Gamma_0. \tag{2}$$

Here ν is the outward unit normal on Γ_0, $\sigma \in L^\infty(\bar{D})$, $\sigma > 0$ and $f \in H^{-1/2}(\Gamma_0)$ are given functions and

$$\int_{\Gamma_0} f(y)ds(y) = 0.$$

It is known [9] that the solution $u \in H^1(D)$ of the problem (1), (2) can be determined uniquely up to an additive constant. Therefore we assume that the coordinate origin belongs to the domain D and add the condition $u(0) = 0$.

2. Modified problem and boundary integral equations

Definition 1. The function $P(x, y)$, $x, y \in D$ is called the parametrix (or Levi's function) of a differential operator L if

$$L_x P(x, y) = \delta(x - y) + R(x, y),$$

where δ is the Dirac function and the function R has weak singularity for $x = y$.

It is easy to make sure that for the operator in (1) the Levi's function has the form

$$P(x, y) = \frac{\ln|x - y|}{2\pi\sigma(y)}, \quad x, y \in \mathbb{R}^2, \ \ x \neq y$$

and the remainder function is

$$R(x, y) = \frac{(x - y)\cdot \operatorname{grad}\sigma(x)}{2\pi\sigma(y)|x - y|^2}, \quad x, y \in \mathbb{R}^2 \ \ x \neq y.$$

Now we introduce the set of smooth closed disjoint curves $\Gamma = \bigcup_{k=1}^N \Gamma_k$ in the domain D. Assume that all curves have following parametric representations

$$\Gamma_k = \{x_k(t) = (x_{1,k}(t), x_{2,k}(t)), t \in [0, 2\pi]\}, \quad k = 0, \dots, N,$$

where $x_k : \mathbb{R} \to \mathbb{R}^2$ are C^3 and 2π–periodic with $|x_k'(t)| > 0$ for all t.

We modify the problem (1), (2) as follows: find the function $\tilde{u} : \Gamma \to \mathbb{R}$, which satisfies the differential equation (1) on Γ and the boundary value condition (2).

Lets introduce the single layer potential

$$w(x) = \sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) P(x, y) ds(y), \quad x \in D \tag{3}$$

with unknown densities $\varphi_i \in L^2(\Gamma_i)$. Then from the equation (1) considered on Γ and the definition of the Levi's function we receive the system of integral equation

$$\varphi_k(x) + \sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) R(x, y) ds(y) = 0, x \in \Gamma_k,\ k = 1, \ldots, N.$$

The boundary value condition (2) also needs to be satisfied. In order to achieve this we will combine the representation (3) with a potential over the boundary Γ_0.

We can present the solution of the modified problem in the form

$$\tilde{u}(x) = \int_{\Gamma_0} \varphi_0(y) \sigma(y) \frac{\partial P(x, y)}{\partial \nu(y)} ds(y) + w(x), \quad x \in \Gamma. \tag{4}$$

Then from the definition of the Levi's function and properties of a logarithmic double layer potential the modified problem can be reduced to the system of boundary integral equations

$$\begin{cases} \varphi_k(x) + \displaystyle\sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) R(x, y) ds(y) + \\ \qquad + \displaystyle\int_{\Gamma_0} \varphi_0(y) \sigma(y) \frac{\partial R(x, y)}{\partial \nu(y)} ds(y) = 0, \quad x \in \Gamma_k, \\ \displaystyle\sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) \frac{\partial P(x, y)}{\partial \nu(x)} ds(y) + \\ \qquad + \displaystyle\frac{\partial}{\partial \nu(x)} \int_{\Gamma_0} \varphi_0(y) \sigma(y) \frac{\partial P(x, y)}{\partial \nu(y)} ds(y) = \frac{f(x)}{\sigma(x)}, \quad x \in \Gamma_0 \end{cases} \tag{5}$$

for $k = 1, \ldots, N$. Note here that in the case of $\sigma = 1$ the system (5) will be simplified to the integral equation

$$\frac{1}{2\pi} \frac{\partial}{\partial \nu(x)} \int_{\Gamma_0} \varphi(y) \frac{\partial \ln|x - y|}{\partial \nu(y)} ds(y) = f(x), x \in \Gamma_0. \tag{6}$$

It is known [1, 5], that the integral operator in this equation is not invertible. Therefore we replace the equation (6) by the following modification

$$\frac{1}{2\pi} \frac{\partial}{\partial \nu(x)} \int_{\Gamma_0} \varphi(y) \frac{\partial \ln|x - y|}{\partial \nu(y)} ds(y) + \alpha = f(x), x \in \Gamma_0, \int_{\Gamma_0} \varphi(y) ds(y) = 0. \tag{7}$$

Here $\varphi \in H^{1/2}(\Gamma_0)$ and $\alpha \in \mathbb{R}$ are unknown. Now the integral operator in (7) is invertible in corresponding Sobolev spaces [1, 5].

Thus we consider the following final system of integral equations related to (5)

$$\begin{cases} \varphi_k(x) + \displaystyle\sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) R(x,y) ds(y) + \\ \qquad + \displaystyle\int_{\Gamma_0} \varphi_0(y)\sigma(y) \frac{\partial R(x,y)}{\partial \nu(y)} ds(y) = 0, \quad x \in \Gamma_k, \\ \\ \displaystyle\sum_{i=1}^{N} \int_{\Gamma_i} \varphi_i(y) \frac{\partial P(x,y)}{\partial \nu(x)} ds(y) + \\ \qquad + \displaystyle\frac{\partial}{\partial \nu(x)} \int_{\Gamma_0} \varphi_0(y)\sigma(y) \frac{\partial P(x,y)}{\partial \nu(y)} ds(y) + \alpha = \frac{f(x)}{\sigma(x)}, \quad x \in \Gamma_0, \\ \\ \displaystyle\int_{\Gamma_0} \varphi_0(y) ds(y) = 0. \end{cases}$$

Taking into account the form of Levi's and remainder functions we can rewrite this system in the following parametric form

$$\begin{cases} (1-\delta_{k0})\mu_k(t) + \displaystyle\frac{1}{2\pi} \sum_{i=0}^{N} \int_0^{2\pi} \mu_i(\tau) H_{k,i}(t,\tau) d\tau + \\ \qquad + \delta_{k0}\alpha = g_k(t), \ k = 0, \dots, N, \\ \\ \displaystyle\int_0^{2\pi} \mu_0(\tau) d\tau = 0 \end{cases} \tag{8}$$

with unknown densities $\mu_k(t) = \varphi_k(x_k(t))$, $k = 0, \dots, N$ and an unknown constant α and with right hand sides

$$g_k(t) = \begin{cases} 0 & k = 1, \dots, N, \\ \\ \dfrac{f(x_0(t))}{\sigma(x_0(t))} & k = 0, \end{cases}$$

and 2π-periodic kernels $H_{k,i}(t,\tau) = 2\pi R(x_k(t), x_i(\tau))|x_i'(\tau)|$, $k = 1, \dots, N$, $i = 1, \dots, N$,

$$H_{k,0}(t,\tau) = \frac{2(x_{k,1}(t) - x_{0,1}(\tau))(x_{k,2}(t) - x_{0,2}(\tau))}{|x_k(t) - x_0(\tau)|^4} \times$$

$$\times (x_2'(\tau)\sigma_{x_2}'(x_k(t)) - x_1'(\tau)\sigma_{x_1}'(x_k(t))) + \frac{(x_{k,1}(t) - x_{0,1}(\tau))^2}{|x_k(t) - x_0(\tau)|^4} -$$

$$- \frac{(x_{k,2}(t) - x_{0,2}(\tau))^2 (x_1'(\tau)\sigma_{x_2}'(x_k(t)) + x_2'(\tau)\sigma_{x_1}'(x_k(t)))}{|x_k(t) - x_0(\tau)|^4} -$$

$$- \frac{(x_k(t) - x_0(\tau)) \cdot \operatorname{grad} \sigma(x_k(t)) \nu(x_0(\tau)) \cdot \operatorname{grad} \sigma(x_0(\tau))}{|x_k(t) - x_0(\tau)|^2 \sigma(x_0(\tau))},$$

$$H_{0,k}(t,\tau) = |x_k'(\tau)| \frac{(x_0(t) - x_k(\tau)) \cdot \nu(x_0(t))}{|x_0(t) - x_k(\tau)|^2 \sigma(x_k(\tau))}$$

for $k = 1, \dots, N$ and

$$H_{0,0}(t,\tau) = \left\{ -\frac{\nu(x_0(t)) \cdot \nu(x_0(\tau))}{|x_0(t) - x_0(\tau)|^2} + \right.$$

$$+ \frac{2\nu(x_0(t)) \cdot (x_0(t) - x_0(\tau)) \nu(x_0(\tau)) \cdot (x_0(t) - x_0(\tau))}{|x_0(t) - x_0(\tau)|^4} -$$

$$\left. - \frac{\nu(x_0(t)) \cdot (x_0(t) - x_0(\tau)) \nu(x_0(\tau)) \cdot \operatorname{grad} \sigma(x_0(\tau))}{|x_0(t) - x_0(\tau)|^2 \sigma^2(x_0(\tau))} \right\} |x_0'(\tau)|.$$

As we see some kernels in (8) have various singularities. We split the strong singularity in $H_{\ell,\ell}$, $\ell = 1, \dots, N$ in the following form

$$H_{\ell,\ell}(t,\tau) = H_{\ell,\ell}^{(1)}(t,\tau) \cot \frac{\tau - t}{2}$$

with smooth kernels

$$H_{\ell,\ell}^{(1)}(t,\tau) = \begin{cases} \tan \dfrac{\tau - t}{2} H_{\ell,\ell}(t,\tau) & \text{for} \quad t \neq \tau, \\[2ex] \dfrac{x_0'(t) \cdot \operatorname{grad} \sigma(x_0(t))}{2\sigma(x_0(t)) |x_0'(t)|} & \text{for} \quad t = \tau. \end{cases}$$

To handle the hypersingularity in the kernel $H_{0,0}$ we rewrite it as

$$H_{0,0}(t,\tau) = -\frac{1}{4|x_0'(t)| \sin^2 \dfrac{t-\tau}{2}} + \tilde{H}_{0,0}(t,\tau),$$

where

$$\tilde{H}_{0,0}(t,\tau) = H_{0,0}(t,\tau) + \frac{1}{4|x_0'(t)| \sin^2 \dfrac{t-\tau}{2}}$$

with the diagonal term

$$\tilde{H}_{0,0}(t,t) = -\frac{\nu(x_0(t)) \cdot x_0''(t)}{2|x_0'(t)|^4} + \frac{\nu(x_0(t)) \cdot x_0''(t) \; \nu(x_0(t)) \cdot \operatorname{grad} \sigma(x_0(t))}{\sigma(x_0(t)) |x_0'(t)|} +$$

$$+ \frac{|x_0'(t)|^4 - 2|x_0'(t)|^2 x_0'(t) \cdot x_0'''(t) + 3(x_0'(t) \cdot x_0''(t))^2}{12|x_0'(t)|^5} -$$

$$- \frac{3(x_{0,1}'(t) x_{0,2}''(t) - x_{0,2}'(t) x_{0,1}''(t))^2}{12|x_0'(t)|^5}.$$

Based on the uniqueness results of the boundary value problem (1)–(2) and the Riesz-Schauder theory for compact operators [6] we have the following result about well-posedness for the system of 2π-periodic integral equations (8).

Theorem 1. *Let $p > 1/2$. For every $f \in H^p[0, 2\pi]$ the system (8) posses an unique solution $\mu_0 \in H^{p+1}[0, 2\pi]$ and $\mu_k \in H^p[0, 2\pi]$, $k = 1, \dots, N$.*

3. Quadrature method

We begin by describing the appropriate quadrature rules. For this we consider trigonometric interpolation with $2n$ equidistant nodal points

$$t_j^{(n)} = \frac{j\pi}{n}, \quad j = 0, \dots, 2n-1$$

with respect to the $2n$-dimensional space of trigonometric polynomials, and use the following quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k^{(n)}), \tag{9}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{k=0}^{2n-1}\tilde{R}_k^{(n)}(t)\,f(t_k^{(n)}), \tag{10}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\cot\frac{\tau-t}{2}d\tau \approx \sum_{k=0}^{2n-1}\tilde{T}_k^{(n)}(t)\,f(t_k^{(n)}), \tag{11}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f'(\tau)\cot\frac{\tau-t}{2}d\tau \approx \sum_{k=0}^{2n-1}T_k^{(n)}(t)\,f(t_k^{(n)}). \tag{12}$$

The weight functions are given by

$$\begin{aligned}
\tilde{R}_k^{(n)}(t) &= -\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_k^{(n)}) - \frac{1}{2n^2}\cos n(t-t_k^{(n)}),\\
\tilde{T}_k^{(n)}(t) &= -\frac{1}{n}\sum_{m=1}^{n-1}\sin m(t-t_k^{(n)}) - \frac{1}{2n}\sin n(t-t_k^{(n)}),\\
T_k^{(n)}(t) &= -\frac{1}{n}\sum_{m=1}^{n-1}m\cos m(t-t_k^{(n)}) - \frac{1}{2}\cos n(t-t_k^{(n)}).
\end{aligned}$$

These quadratures are obtained by replacing f with its trigonometric interpolation polynomial and then integrating exactly [3, 6]. Note that some of given quadratures coincide with quadrature formulas used in the method of discrete charges [4].

Thus we use quadrature rules (9),(11) and (12) to approximate three types of integrals in the system of integral equations (8) and collocate the approximate equations to obtain the linear system

$$\mathbf{A}\tilde{\mu} = \mathbf{b}$$

with matrix coefficients

$$A_{k,0}^{(ij)} = \frac{1}{2n}H_{k,0}(t_i^{(n)}, t_j^{(n)}), \quad k = 1, \dots, N,$$

$$A_{0,k}^{(ij)} = \begin{cases} \dfrac{1}{2n} H_{0,k}(t_i^{(n)}, t_j^{(n)}), & k = 1, \dots, N, \\ -\dfrac{1}{2|x_0(t_i^{(n)})|} T_j(t_i^{(n)}) + \dfrac{1}{2M} \tilde{H}_{0,0}(t_i^{(n)}, t_j^{(n)}), & k = 0, \end{cases}$$

$$A_{k,\ell}^{(ij)} = \begin{cases} \dfrac{1}{2n} H_{k,\ell}(t_i^{(n)}, t_j^{(n)}), & k \neq \ell, \\ \tilde{T}_j(t_i^{(n)}) H_{k,\ell}(t_i^{(n)}, t_j^{(n)}), & k = \ell \end{cases}$$

and $A_{0,0}^{(2n,j)} = 1$, $A_{0,0}^{(i,2n)} = 1$ and with the right hand side $b_k^{(i)} = g_k(t_i^{(n)})$, $k = 0, \dots, N$, $i = 0, \dots, 2n-1$ and $b_0^{(2n)} = 0$.

To find the numerical solution of the modified problem we parametrize the reprezentation (4)

$$\tilde{u}(x_k(t)) = \frac{1}{2\pi} \sum_{\ell=0}^{N} \int_0^{2\pi} \mu_\ell(\tau) L_{k,\ell}(t,\tau) d\tau, \tag{13}$$

where $L_{k,\ell}(t,\tau) = \frac{\pi}{n} |x'_\ell(\tau)| P(x_k(t), x_\ell(\tau))$ for $\ell, k = 1, \dots, N$ and

$$L_{k,0}(t,\tau) = -\frac{(x_k(t) - x_0(\tau)) \cdot \nu(x_0(\tau))}{\sigma(x_0(\tau)) |x_k(t) - x_0(\tau)|^2} -$$

$$-\frac{\operatorname{grad} \sigma(x_0(\tau)) \cdot \nu(x_0(\tau))}{\sigma^2(x_0(\tau))} \ln |x_k(t) - x_0(\tau)|.$$

As we see the kernels $L_{\ell,\ell}$ have logarithmic singularity and we split it in the following form

$$L_{\ell,\ell}(t,\tau) = L_{\ell,\ell}^1(t,\tau) \ln \left(4 \sin \frac{t-\tau}{2} \right) + L_{\ell,\ell}^2(t,\tau)$$

with

$$L_{\ell,\ell}^1(t,\tau) = \frac{|x'_\ell(\tau)|}{2\sigma(x_\ell(\tau))}$$

and

$$L_{\ell,\ell}^2(t,\tau) = \begin{cases} L_{\ell,\ell}(t,\tau) - L_{\ell,\ell}^1(t,\tau) \ln \left(4 \sin \dfrac{t-\tau}{2} \right) & \text{for} \quad t \neq \tau, \\ \dfrac{|x'_\ell(t)|}{\sigma(x_\ell(t))} \ln |x'_\ell(t)| & \text{for} \quad t = \tau. \end{cases}$$

Now according to (13) and using quadratures (9) and (10) we have the following formula for the numerical solution of the modified problem

$$\tilde{u}_n(x_k(t)) = \sum_{\ell=1}^{N} \sum_{i=0}^{2n-1} \tilde{\mu}_\ell^{(i)} \tilde{L}_{k,\ell}(t, t_i^{(n)}),$$

where

$$\tilde{L}^2_{k,\ell}(t,t_i^{(n)}) = \begin{cases} \dfrac{1}{2n} L_{k,\ell}(t,t_i^{(n)}) & \text{for} \quad \ell \neq k, \\ \\ L^1_{\ell,\ell}(t,t_i^{(n)})\tilde{R}_i^{(n)}(t) + \dfrac{1}{2n} L^2_{\ell,\ell}(t,t_i^{(n)}) & \text{for} \quad \ell = k. \end{cases}$$

4. Numerical examples

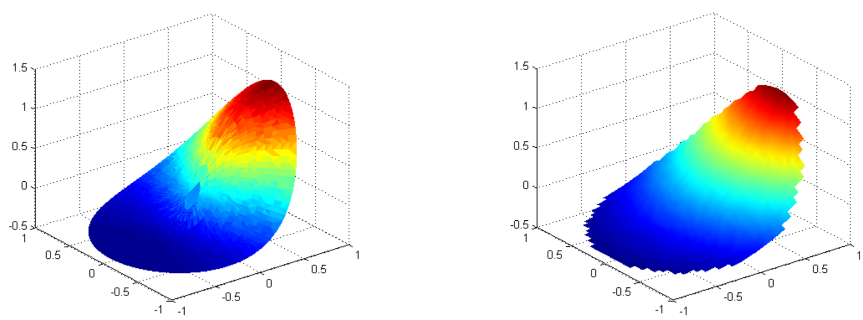
Example 1. We consider the domain D bounded by a circle Γ_0 with the radius $R = 1$. The given function σ and f are given as

$$\sigma(x) = 1 + x_1^2 + x_2^2,\ x \in D \quad \text{and} \quad f(x) = x_1 e^{x_1}\cos x_2 - x_2 e^{x_1}\sin x_2,\ x \in \Gamma_0.$$

The numerical solution of the boundary value problem (1),(2) received by proposed method is presented in the Fig. 1a. Here we used the following discretization parameters $n = 64$ and $N = 13$ and the set of curves

$$\Gamma_k = \left\{ x_k(t) = (1 - \frac{k}{N+1})(\cos t, \sin t), 0 \leq t \leq 2\pi \right\},\ k = 0, \dots, N.$$

The numerical result obtained by FEM method by PDE Toolbox in Matlab is illustrated in Fig. 1b. As we see both results are sufficiently close.



a) Numerical solution by BIEM b) Numerical solution by FEM

FIG. 1. Results of numerical experiments for the example 1

Example 2. Assume that the boundary curve Γ_0 and the set of curves Γ (see Fig. 2a) have the parametric representation

$$\Gamma_k = \left\{ x_k(t) = r(t)(1 - \frac{k}{N+1})(\cos t, \sin t), 0 \leq t \leq 2\pi \right\},\ k = 0, \dots, N$$

with the radial function

$$r(t) = \left(\left(\frac{1}{2}\cos t\right)^{10} + \left(\frac{2}{3}\sin t\right)^{10} \right)^{-0.1}.$$

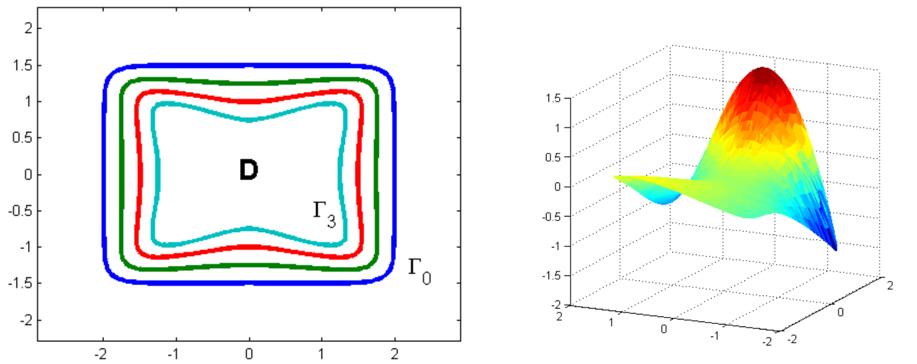
Let

$$\sigma(x) = 1 + e^{0.3(x_1^2 + x2)},\ x \in D$$

and

$$f(x) = e^{x_1}\left(\cos x_2 \nu_1(x) - \sin x_2 \nu_2(x)\right), \ x \in \Gamma_0.$$

The numerical solution obtained via proposed method is given in Fig. 2b.



a) Set of curves Γ b) Numerical solution by BIEM

FIG. 2. Results of numerical experiments for the example 2

We considered the numerical solution of the interior planar Neumann boundary value problem for an elliptic differential equation of second order with variable coefficients. The proposed method is based on boundary integral equations. First we approximated the given problem by a modified problem on the introduced set of closed curves in the solution domain. Then the potentials with Levi's function are used for the modified problem. As result the system of boundary integral equations with singular and hypersingular kernels is received. The full discretization is realized by trigonometric quadrature method. The presented numerical examples confirmed the applicability of the proposed method.

BIBLIOGRAPHY

1. Atkinson K. The Numerical Solution of Integral Equations of the Second Kind / K. Atkinson.– Cambridge: Cambridge University Press, 1997.
2. Babenko Ch. On the numerical solution of a direct EIT problem by boundary integral equation method / Ch. Babenko, R. Chapko // Visnyk Lviv. Universytetu. Seriya prykladna matematyka ta informatyka.– 2012.– Vol. 18.– P. 22-30 (in Ukrainian).
3. Chapko R. On a quadrature method for a logarithmic integral equation of the first kind / R. Chapko, R. Kress // In Agarwal, ed.: World Scientific Series in Applicable Analysis.– Vol. 2. Contributions in Numerical Mathematics.– 1993.– P. 127-140.
4. Gandel Yu. V. Mathematical Questions of Discrete Charges Method / Yu. V. Gandel, T. S. Polanskaya.– Kharkiv: Kharkiv. University, 1991 (in Russian).
5. Kieser R. On a full discretization scheme for a hypersingular boundary integral equation over smooth curves / R. Kieser, B. Kleemann, R. Ratsfeld // Zeitschrieft für Analysis und Ihre Anwendungen.– 1992.– Vol. 11.– P. 385-396.
6. Kress R. Linear Integral Equations / R. Kress. 2nd. ed. Berlin: Springer-Verlag, 1999.
7. Miranda C. Partial Differential Equations of Elliptic Type / C. Miranda.– New York: Springer-Verlag, 1970.

8. Pomp A. The boundary-domain integral method for elliptic systems. With applications in shells / A. Pomp // Lecture Notes in Mathematics.– Vol. 1683, Berlin: Springer-Verlag, 1998.
9. McLean W. Strongly Elliptic Systems and Boundary Integral Operators / W. McLean.– Cambridge: Cambridge University Press, 2000.
10. Mikhailov S. E. Analysis of extended boundary-domain integral and integro-differential equations of some variable-coefficient BVP / S. E. Mikhailov // Advances in Boundary Integral Methods. Proceedings of the 5th UK Conference on Boundary Integral Methods (Edited by Ke Chen), University of Liverpool Publ., UK.– 2005.– P. 106-125.

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