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EXACT THREE-POINT DIFFERENCE SCHEME FOR SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS OF THE THIRD KIND

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Резюме. Для нелінійних звичайних диференціальних рівнянь другого порядку з похідною в правій частині та крайовими умовами третього роду побудовано та обґрунтовано точну триточкову різницеву схему на нерівномірній сітці. Доведено існування та єдиність розв'язку цієї схеми, збіжність методу простої ітерації для її розв'язування.

Abstract. Exact three-point difference scheme on a nonuniform grid for the second-order nonlinear ordinary differential equations with derivative in the right-hand side and boundary conditions of the third kind is constructed and justified. The existence and uniqueness of solution of this scheme, the convergence of the method of simple iteration for its solution are proved.

1. Introduction

The exact three-point difference scheme (ETDS) and three-point difference schemes (TDS) of high order accuracy on a uniform grid for the second-order nonlinear ordinary differential equations with no derivative in the right-hand side and Dirichlet boundary conditions is constructed and justified in [10, 11]. These results on a nonuniform grid were generalized and developed in [9] and for monotone boundary value problems in [1, 7]. Difference boundary conditions of the third kind is constructed in [6, 8].

In this chapter for the nonlinear boundary value problem (BVP)

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = -f\left(x, u, \frac{du}{dx}\right), \quad x \in (0,1), \tag{1}$$

$$k(0)\frac{du(0)}{dx} - \beta_1 u(0) = -\mu_1, \quad -k(1)\frac{du(1)}{dx} - \beta_2 u(1) = -\mu_2, \tag{2}$$

where $k(x), f(x, u, \xi)$ are given functions and $\beta_1, \beta_2, \mu_1, \mu_2$ are given numbers, exact three-point difference scheme is constructed. We prove the existence and the uniqueness of the solution of the ETDS and convergence of the method of simple iteration its solution for the operator of BVP (1), (2) with monotone conditions.

[†]*Key words.* Nonlinear boundary value problem, exact three-point difference schemes, method of simple iteration.

2. Existence and uniqueness of a solution

The function $u(x) \in W_2^1(0,1)$ is a week solution of problem (1), (2), if $\forall u(x), v(x) \in W_2^1(0,1)$ satisfies the relation

$$\int_0^1 k(x)\frac{du}{dx}\frac{dv}{dx}dx + (\beta_1 u(0) - \mu_1)v(0) + (\beta_2 u(1) - \mu_2)v(1) =$$

$$= \int_0^1 f\left(x, u, \frac{du}{dx}\right) v(x)dx.$$

Sufficient conditions for the existence and uniqueness of a weak solution of problem (1), (2) are given in the next theorem.

Theorem 1. *Let the following assumptions be satisfied*

$$0 < c_1 \le k(x) \le c_2 \quad \forall x \in [0,1]\ , \quad k(x) \in Q^1[0,1], \tag{3}$$

$$\begin{array}{l} f_{u\xi}(x) \equiv f\,(x,u,\xi) \in Q^0[0,1] \quad \forall u, \xi \in \mathbb{R}^1, \\ f_x\,(u,\xi) \equiv f\,(x,u,\xi) \in C(\mathbb{R}^2) \quad \forall x \in [0,1], \end{array} \tag{4}$$

$$|f\,(x,u,\xi) - f_0(x)| \le c(|u|)[g(x) + |\xi|] \quad \forall x \in [0,1],\ u, \xi \in \mathbb{R}^1, \tag{5}$$

$$[f\,(x,u,\xi) - f\,(x,v,\eta)]\,(u - v) \le 0 \quad \forall x \in [0,1],\ u, v, \xi, \eta \in \mathbb{R}^1, \tag{6}$$

$$\beta_1 > 0, \quad \beta_2 > 0, \tag{7}$$

then, the BVP (1), (2) has a unique solution $u(x) \in W_2^1(0,1)$*, with* $u(x), k(x)\dfrac{du}{dx} \in C[0,1]$.

Here $c(t)$ *is a continuous function,* $f_0(x) \in L_2(0,1)$, $g(x) \in L_1(0,1)$, c_1, c_2, c_3 *are constants,* $Q^p[0,1]$ *is the class of functions having p piece-wise continuous derivatives and a finite number of discontinuity points of first kind.*

Proof. Due to (4) and (5) the function $f\,(x,u,\xi)$ satisfies the Caratheodory conditions [3, p.63] and belongs to the class $L_1(0,1)$ (see e.g.[3, c.113]), we can define the operator $A\,(x,u)$ the identity

$$(A\,(x,u)\,, v) = \int_0^1 k(x)\frac{du}{dx}\frac{dv}{dx}dx - \int_0^1 \tilde{f}\left(x, u, \frac{du}{dx}\right) v(x)dx +$$

$$+ (\beta_1 u(0) - \mu_1)v(0) + (\beta_2 u(1) - \mu_2)v(1) \quad \forall u(x), v(x) \in W_2^1(0,1),$$

where

$$\tilde{f}(x,u,\xi) = f(x,u,\xi) - f_0(x).$$

Note that the function $u(x) \in W_2^1(0,1)$ is absolutely continuous on $[0,1]$, and its generalized derivative $\dfrac{du}{dx}$ is equal to the classical derivative almost everywhere on $[0,1]$ (see e.g. [3, c.74]). Thus, $u(x) \in C[0,1]$, $\dfrac{du}{dx} \in L_2(0,1)$.

Let us show that the operator $A\,(x,u)$ is bounded. Actually, taking into account the Cauchy-Bunyakovsky-Schwarz inequality, the conditions (3) and (5), the inequality $c(|u|) \le C_2$ for all $x \in [0,1]$ as well as $\|v\|_{C[0,1]} \le C_1\|v\|_{1,2,(0,1)}$ for all $v(x) \in W_2^1(0,1)$ (see e.g.[3, c.112]) we obtain

$$|(A(x,u),v)| \le \left\{\int_0^1 \left[k(x)\frac{du}{dx}\right]^2 dx\right\}^{1/2} \left\{\int_0^1 \left[\frac{dv}{dx}\right]^2 dx\right\}^{1/2} +$$
$$+\|v\|_{C[0,1]}\left[\int_0^1 \left|\tilde{f}\left(x,u,\frac{du}{dx}\right)\right| dx + (\beta_1+\beta_2)\|u\|_{C[0,1]} + |\mu_1+\mu_2|\right] \le$$
$$\le \Big[(c_2 + C_1^2(\beta_1+\beta_2))\,\|u\|_{1,2,(0,1)} +$$
$$+C_1\left\|\tilde{f}\right\|_{0,1,(0,1)} + C_1|\mu_1+\mu_2|\Big]\,\|v\|_{1,2,(0,1)} \le$$
$$\le \Big[(c_2 + C_1(C_2 + C_1(\beta_1+\beta_2)))\,\|u\|_{1,2,(0,1)} + C_1C_2\,\|g\|_{0,1,(0,1)} +$$
$$+C_1|\mu_1+\mu_2|]\,\|v\|_{1,2,(0,1)},$$

where

$$\|u\|_{C[0,1]} = \max_{x\in[0,1]}|u(x)|, \quad \|u\|_{0,1,(0,1)} = \int_0^1 |u(x)|dx,$$
$$\|u\|_{0,2,(0,1)} = \left[\int_0^1 (u(x))^2\,dx\right]^{1/2},$$
$$\|u\|_{1,2,(0,1)} = \left[\int_0^1 (u(x))^2\,dx + \int_0^1 \left(\frac{du}{dx}\right)^2 dx\right]^{1/2}.$$

If $u_n \to u_0$ in $W_2^1(0,1)$, then $\tilde{f}\left(x,u_n,\frac{du_n}{dx}\right) \to \tilde{f}\left(x,u_0,\frac{du_0}{dx}\right)$, $k(x)\frac{du_n}{dx} \to k(x)\frac{du_0}{dx}$ in $L_1(0,1)$ (see e.g.[3, c.113]). Thus, for $\forall v(x) \in W_2^1(0,1)$ we have

$$\lim_{n\to\infty}(A(x,u_n),v) = \lim_{n\to\infty}\left[\int_0^1 k(x)\frac{du_n}{dx}\frac{dv}{dx}dx - \int_0^1 \tilde{f}\left(x,u_n,\frac{du_n}{dx}\right)v(x)dx +\right.$$
$$+(\beta_1u_n(0)-\mu_1)v(0) + (\beta_2u_n(1)-\mu_2)v(1)] =$$
$$= \int_0^1 k(x)\frac{du_0}{dx}\frac{dv}{dx}dx - \int_0^1 \tilde{f}\left(x,u_0,\frac{du_0}{dx}\right)v(x)dx +$$
$$+(\beta_1u(0)-\mu_1)v(0) + (\beta_2u(1)-\mu_2)v(1) = (A(x,u_0),v),$$

i.e. the operator $A(x,u)$ is demicontinuous.

Let us show that the operator $A\left(x,u\right)$ is strongly monotone. Due to the conditions (3), (6) and (7), taking into account the Friedrichs inequality (see e.g. [2, c.187])

$$\int_0^1 u^2(x)dx \le \frac{16}{\pi^2}\int_0^1 \left(\frac{du}{dx}\right)^2 dx + \frac{4}{\pi}\left[u^2(0)+u^2(1)\right],$$

we obtain

$$\left(A\left(x,u\right)-A\left(x,v\right),u-v\right)=\int_0^1 k(x)\left[\frac{du}{dx}-\frac{dv}{dx}\right]^2 dx-$$
$$-\int_0^1 \left[f\left(x,u(x),\frac{du}{dx}\right)-f\left(x,v(x),\frac{dv}{dx}\right)\right]\left[u(x)-v(x)\right]dx+$$
$$+\beta_1(u(0)-v(0))^2+\beta_2(u(1)-v(1))^2 \ge c_1\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)}+$$
$$+\beta_1(u(0)-v(0))^2+\beta_2(u(1)-v(1))^2 \ge$$
$$\ge \min\left\{\frac{\pi^2 c_1}{16},\frac{\pi\beta_1}{4},\frac{\pi\beta_2}{4}\right\}\left\|u-v\right\|^2_{0,2,(0,1)}.$$

From the strong monotonicity follows the coerciveness of $A\left(x,u\right)$.

Thus, the Browder-Minty theorem (see [3, c.204]) guaranties the existence of a unique solution $u\in W_2^1(0,1)$ of problem (1), (2). $\square$

Since

$$k(x)\frac{du}{dx}=\int_0^x f\left(t,u,\frac{du}{dt}\right)dt+C$$

almost everywhere on [0,1] (see e.g. [3, c.134]), i.e. the flux $k(x)\dfrac{du}{dx}$ is the undefined Lebesgue integral, this function is absolutely continuous on [0,1], and the claim $k(x)\dfrac{du}{dx}\in C[0,1]$ is shown.

3. Existence of an exact three-point difference scheme

On the closed $(0,1)$ we introduce an nonuniform grid

$$\hat{\omega}_h=\left\{x_j\in(0,1),\quad j=1,2,...,N-1,\quad h_j=x_j-x_{j-1}>0,\quad \sum_{j=1}^N h_j=1\right\}$$

such that the discontinuity points of functions $k(x), f\left(x,u,\xi\right)$ coincide with the nodes of the grid $\hat{\omega}_h$. Denote by ρ the set of all discontinuity points and assume that N is such that $\rho\subseteq\hat{\omega}_h$. At points of discontinuity we use the continuity conditions for BVP(1), (2)

$$u(x_i-0)=u(x_i+0),\quad k(x)\frac{du}{dx}\Big|_{x=x_i-0} = k(x)\frac{du}{dx}\Big|_{x=x_i+0} \quad \forall x_i\in\rho.$$

We will use the following notation

$$e_\alpha^j=(x_{j-2+\alpha},x_{j-1+\alpha}),\quad \bar e_\alpha^j=[x_{j-2+\alpha},x_{j-1+\alpha}].$$

Consider the boundary value problems

$$\begin{aligned}&\frac{d}{dx}\left(k(x)\frac{dY_2^0(x,u)}{dx}\right)=-f\left(x,Y_2^0(x,u),\frac{dY_2^0(x,u)}{dx}\right),\quad x\in e_2^0,\\ &k(x_0)\frac{dY_2^0(x_0,u)}{dx}-\beta_1Y_2^0(x_0,u)=-\mu_1,\quad Y_2^0(x_1,u)=u(x_1),\end{aligned}\tag{8}$$

$$\begin{aligned}&\frac{d}{dx}\left(k(x)\frac{dY_\alpha^j(x,u)}{dx}\right)=-f\left(x,Y_\alpha^j(x,u),\frac{dY_\alpha^j(x,u)}{dx}\right),\quad x\in e_\alpha^j,\\ &Y_\alpha^j(x_{j-2+\alpha},u)=u(x_{j-2+\alpha}),\quad Y_\alpha^j(x_{j-1+\alpha},u)=u(x_{j-1+\alpha}),\\ &j=3-\alpha,4-\alpha,...,N-\alpha,\quad \alpha=1,2,\end{aligned}\tag{9}$$

$$\begin{aligned}&\frac{d}{dx}\left(k(x)\frac{dY_1^N(x,u)}{dx}\right)=-f\left(x,Y_1^N(x,u),\frac{dY_1^N(x,u)}{dx}\right),\quad x\in e_1^N,\\ &Y_1^N(x_{N-1},u)=u(x_{N-1}),\\ &-k(x_N)\frac{dY_1^N(x_N,u)}{dx}-\beta_2Y_1^N(x_N,u)=-\mu_2.\end{aligned}\tag{10}$$

Lemma 1. *Let the assumptions of Theorem* 1 *be satisfied. Then each of the problems (8)-(10) has a unique solution* $Y_\alpha^j(x,u)\in W_2^1(e_\alpha^j)$, $j=2-\alpha,3-\alpha,...,N+1-\alpha$, $\alpha=1,2$, *with*

$$Y_\alpha^j(x,u),k(x)\frac{dY_\alpha^j(x,u)}{dx}\in C(\bar e_\alpha^j)$$

and for the solution BVP (1), (2) it holds

$$u(x)=Y_\alpha^j(x,u),\quad x\in\bar e_\alpha^j.\tag{11}$$

Proof. We introduce the nonlinear operators for problems (8)-(10) by the equations

$$(A_2^0(x,Y_2^0),v)=\int_{x_0}^{x_1}k(x)\frac{dY_2^0(x,u)}{dx}\frac{dv(x)}{dx}dx-$$
$$-\int_{x_0}^{x_1}\tilde f\left(x,Y_2^0(x,u),\frac{dY_2^0(x,u)}{dx}\right)v(x)dx+(\beta_1Y_2^0(0,u)-\mu_1)v(0),$$

$$\left(A_\alpha^j\left(x, Y_\alpha^j\right), v\right)=$$
$$=\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x) \frac{dY_\alpha^j(x,u)}{dx} \frac{dv(x)}{dx} dx-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \tilde{f}\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right) v(x) dx,$$
$$(A_1^N(x, Y_1^N), v)=\int\limits_{x_{N-1}}^{x_N} k(x) \frac{dY_1^N(x,u)}{dx} \frac{dv(x)}{dx} dx-$$
$$-\int\limits_{x_{N-1}}^{x_N} \tilde{f}\left(x, Y_1^N(x,u), \frac{dY_1^N(x,u)}{dx}\right) v(x) dx+(\beta_2 Y_1^N(1,u)-\mu_2) v(1),$$
$$\tilde{f}(x,u,\xi)=f(x,u,\xi)-f_0(x),$$

that true for $\forall Y_\alpha^j(x,u), v(x) \in W_2^1(e_\alpha^j)$.

Let us show that the operators $A_2^0\left(x, Y_2^0\right)$, $A_\alpha^j\left(x, Y_\alpha^j\right)$, $A_1^N\left(x, Y_1^N\right)$ are bounded. Taking into account the Cauchy-Bunyakovsky-Schwarz inequality, the conditions (3), (5) with $c(|Y_\alpha^j(x,u)|) \leq C_2$, $\forall x \in \bar{e}_\alpha^j$ and inequality $\|v\|_{C(\bar{e}_\alpha^j)} \leq C_1\|v\|_{1,2,e_\alpha^j}$, $\forall v(x) \in W_2^1(e_\alpha^j)$, we obtain

$$\left|\left(A_2^0\left(x, Y_2^0\right), v\right)\right| \leq\left\{\int\limits_{x_0}^{x_1}\left[k(x) \frac{dY_2^0}{dx}\right]^2 dx\right\}^{1/2}\left\{\int\limits_{x_0}^{x_1}\left[\frac{dv}{dx}\right]^2 dx\right\}^{1/2}+$$
$$+\|v\|_{C(\bar{e}_2^0)}\left[\int\limits_{x_0}^{x_1}\left|\tilde{f}\left(x, Y_2^0, \frac{dY_2^0}{dx}\right)\right| dx+\beta_1\left\|Y_2^0\right\|_{C(\bar{e}_2^0)}+|\mu_1|\right] \leq$$
$$\leq\left[\left(c_2+C_1^2 \beta_1\right)\left\|Y_2^0\right\|_{1,2,e_2^0}+C_1\left\|\tilde{f}\right\|_{0,1,e_2^0}+C_1\left|\mu_1\right|\right]\|v\|_{1,2,e_2^0} \leq$$
$$\leq\left[\left(c_2+C_1\left(C_2+C_1 \beta_1\right)\right)\left\|Y_2^0\right\|_{1,2,e_2^0}+C_1 C_2\|g\|_{0,1,e_2^0}+C_1\left|\mu_1\right|\right]\|v\|_{1,2,e_2^0},$$
$$\left|\left(A_\alpha^j\left(x, Y_\alpha^j\right), v\right)\right| \leq\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[k(x) \frac{dY_\alpha^j}{dx}\right]^2 dx\right\}^{1/2}\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[\frac{dv}{dx}\right]^2 dx\right\}^{1/2}+$$
$$+\|v\|_{C(\bar{e}_\alpha^j)} \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left|\tilde{f}\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j}{dx}\right)\right| dx \leq$$
$$\leq\left[c_2\left\|Y_\alpha^j\right\|_{1,2,e_\alpha^j}+C_1\left\|\tilde{f}\right\|_{0,1,e_\alpha^j}\right]\|v\|_{1,2,e_\alpha^j} \leq$$
$$\leq\left[\left(c_2+C_1 C_2\right)\left\|Y_\alpha^j\right\|_{1,2,e_\alpha^j}+C_1 C_2\|g\|_{0,1,e_\alpha^j}\right]\|v\|_{1,2,e_\alpha^j},$$
$$\left|\left(A_1^N\left(x, Y_1^N\right), v\right)\right| \leq\left\{\int\limits_{x_{N-1}}^{x_N}\left[k(x) \frac{dY_1^N}{dx}\right]^2 dx\right\}^{1/2}\left\{\int\limits_{x_{N-1}}^{x_N}\left[\frac{dv}{dx}\right]^2 dx\right\}^{1/2}+$$

$$+\|v\|_{C(\bar{e}_1^N)}\left[\int\limits_{x_{N-1}}^{x_N}\left|\tilde{f}\left(x,Y_1^N,\frac{dY_1^N}{dx}\right)\right|dx+\beta_2\left\|Y_1^N\right\|_{C(\bar{e}_1^N)}+|\mu_2|\right]\leq$$

$$\leq\left[\left(c_2+C_1^2\beta_2\right)\left\|Y_1^N\right\|_{1,2,e_1^N}+C_1\left\|\tilde{f}\right\|_{0,1,e_1^N}+C_1\left|\mu_2\right|\right]\|v\|_{1,2,e_1^N}\leq$$

$$\leq\left[\left(c_2+C_1\left(C_2+C_1\beta_2\right)\right)\left\|Y_1^N\right\|_{1,2,e_1^N}+C_1C_2\|g\|_{0,1,e_1^N}+C_1\left|\mu_2\right|\right]\|v\|_{1,2,e_1^N}.$$

The demicontinuity of operators $A_2^0\left(x,Y_2^0\right), A_\alpha^j\left(x,Y_\alpha^j\right), A_1^N\left(x,Y_1^N\right)$ follows from the condition (5). Really (see[3, p.113]), if $Y_{\alpha n}^j(x,u)\to Y_{\alpha 0}^j(x,u)$ in $W_2^1(e_\alpha^j)$, then

$$\tilde{f}\left(x,Y_{\alpha n}^j(x,u),\frac{dY_{\alpha n}^j(x,u)}{dx}\right)\to\tilde{f}\left(x,Y_{\alpha 0}^j(x,u),\frac{dY_{\alpha 0}^j(x,u)}{dx}\right),$$
$$k(x)\frac{dY_{\alpha n}^j(x,u)}{dx}\to k(x)\frac{dY_{\alpha 0}^j(x,u)}{dx},$$
$$j=2-\alpha,3-\alpha,...,N+1-\alpha,\quad \alpha=1,2,$$

in space $L_1(e_\alpha^j)$. Thus, for $\forall v(x)\in W_2^1(e_\alpha^j)$

$$\lim_{n\to\infty}\left(A_2^0\left(x,Y_{2,n}^0\right),v\right)=\lim_{n\to\infty}\left[\int\limits_{x_0}^{x_1}k(x)\frac{dY_{2,n}^0}{dx}\frac{dv}{dx}dx-\right.$$
$$\left.-\int\limits_{x_0}^{x_1}\tilde{f}\left(x,Y_{2,n}^0,\frac{dY_{2,n}^0}{dx}\right)v(x)dx+\left(\beta_1Y_{2,n}^0(0,u)-\mu_1\right)v(0)\right]=$$
$$=\int\limits_{x_0}^{x_1}k(x)\frac{dY_{2,0}^0}{dx}\frac{dv}{dx}dx-\int\limits_{x_0}^{x_1}\tilde{f}\left(x,Y_{2,0}^0,\frac{dY_{2,0}^0}{dx}\right)v(x)dx+$$
$$+\left(\beta_1Y_{2,0}^0(0,u)-\mu_1\right)v(0)=\left(A_2^0\left(x,Y_{2,0}^0\right),v\right),$$

$$\lim_{n\to\infty}\left(A_\alpha^j\left(x,Y_{\alpha n}^j\right),v\right)$$
$$=\lim_{n\to\infty}\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}k(x)\frac{dY_{\alpha n}^j}{dx}\frac{dv(x)}{dx}dx-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\tilde{f}\left(x,Y_{\alpha n}^j,\frac{dY_{\alpha n}^j}{dx}\right)v(x)dx\right\}=$$
$$=\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}k(x)\frac{dY_{\alpha 0}^j}{dx}\frac{dv}{dx}dx-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\tilde{f}\left(x,Y_{\alpha 0}^j,\frac{dY_{\alpha 0}^j}{dx}\right)v(x)dx=$$
$$=\left(A_\alpha^j\left(x,Y_{\alpha 0}^j\right),v\right),$$

$$\lim_{n\to\infty}\left(A_1^N\left(x,Y_{1,n}^N\right),v\right)=$$
$$=\lim_{n\to\infty}\left[\int_{x_{N-1}}^{x_N} k(x)\frac{dY_{1,n}^N}{dx}\frac{dv}{dx}dx-\right.$$
$$\left.-\int_{x_{N-1}}^{x_N}\tilde{f}\left(x,Y_{1,n}^N,\frac{dY_{1,n}^N}{dx}\right)v(x)dx+\left(\beta_2 Y_{1,n}^N(1,u)-\mu_2\right)v(1)\right]=$$
$$=\int_{x_{N-1}}^{x_N} k(x)\frac{dY_{1,0}^N}{dx}\frac{dv}{dx}dx-\int_{x_{N-1}}^{x_N}\tilde{f}\left(x,Y_{1,0}^N,\frac{dY_{1,0}^N}{dx}\right)v(x)dx+$$
$$+\left(\beta_2 Y_{1,0}^N(1,u)-\mu_2\right)v(1)=\left(A_1^N\left(x,Y_{1,0}^N\right),v\right),$$

that operators $A_2^0\left(x,Y_2^0\right),A_\alpha^j\left(x,Y_\alpha^j\right),A_1^N\left(x,Y_1^N\right)$ are demicontinuous.

Let us show that the operators $A_2^0\left(x,Y_2^0\right)$, $A_\alpha^j\left(x,Y_\alpha^j\right)$, $A_1^N\left(x,Y_1^N\right)$ are strongly monotone. Due to the conditions (4), (7), taking into account the Friedrichs inequalities (see e.g. [2, c.187])

$$\int_a^b u^2(x)dx\le\frac{16(b-a)^2}{\pi^2}\int_a^b\left(\frac{du}{dx}\right)^2dx+\frac{\pi(b-a)}{4}u^2(a),$$

$$\int_a^b u^2(x)dx\le\frac{16(b-a)^2}{\pi^2}\int_a^b\left(\frac{du}{dx}\right)^2dx+\frac{\pi(b-a)}{4}u^2(b)$$

we obtain

$$\left(A_2^0(x,Y_2^0)-A_2^0(x,\tilde{Y}_2^0),Y_2^0-\tilde{Y}_2^0\right)=\int_{x_0}^{x_1}k(x)\left(\frac{dY_2^0}{dx}-\frac{d\tilde{Y}_2^0}{dx}\right)^2dx-$$

$$-\int_{x_0}^{x_1}\left[f\left(x,Y_2^0,\frac{dY_2^0}{dx}\right)-f\left(x,\tilde{Y}_2^0,\frac{d\tilde{Y}_2^0}{dx}\right)\right]\left[Y_2^0(x,u)-\tilde{Y}_2^0(x,u)\right]dx+$$

$$+\beta_1\left(Y_2^0(0,u)-\tilde{Y}_2^0(0,u)\right)^2\ge c_1\left\|\frac{dY_2^0}{dx}-\frac{d\tilde{Y}_2^0}{dx}\right\|_{0,2,e_2^0}^2+$$

$$+\beta_1\left(Y_2^0(0,u)-\tilde{Y}_2^0(0,u)\right)^2\ge\min\left\{\frac{\pi^2c_1}{16},\frac{\pi\beta_1}{4}\right\}\left\|Y_2^0-\tilde{Y}_2^0\right\|_{0,2,e_2^0}^2$$

$$\left(A_\alpha^j(x,Y_\alpha^j)-A_\alpha^j(x,\tilde{Y}_\alpha^j),Y_\alpha^j-\tilde{Y}_\alpha^j\right)=\int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}k(x)\left(\frac{dY_\alpha^j}{dx}-\frac{d\tilde{Y}_\alpha^j}{dx}\right)^2dx-$$

$$-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[f\left(x,Y_\alpha^j,\frac{dY_\alpha^j}{dx}\right)-f\left(x,\tilde{Y}_\alpha^j,\frac{d\tilde{Y}_\alpha^j}{dx}\right)\right]\left[Y_\alpha^j(x,u)-\tilde{Y}_\alpha^j(x,u)\right]dx\geq$$

$$\geq c_1\left\|\frac{dY_\alpha^j}{dx}-\frac{d\tilde{Y}_\alpha^j}{dx}\right\|^2_{0,2,e_\alpha^j},$$

$$\left(A_1^N(x,Y_1^N)-A_1^N(x,\tilde{Y}_1^N),Y_1^N-\tilde{Y}_1^N\right)=\int\limits_{x_{N-1}}^{x_N}k(x)\left(\frac{dY_1^N}{dx}-\frac{d\tilde{Y}_1^N}{dx}\right)^2dx-$$

$$-\int\limits_{x_{N-1}}^{x_N}\left[f\left(x,Y_1^N,\frac{dY_1^N}{dx}\right)-f\left(x,\tilde{Y}_1^N,\frac{d\tilde{Y}_1^N}{dx}\right)\right]\left[Y_1^N(x,u)-\tilde{Y}_1^N(x,u)\right]dx+$$

$$+\beta_2\left(Y_1^N(1,u)-\tilde{Y}_1^N(1,u)\right)^2\geq c_1\left\|\frac{dY_1^N}{dx}-\frac{d\tilde{Y}_1^N}{dx}\right\|^2_{0,2,e_1^N}+$$

$$+\beta_2\left(Y_1^N(1,u)-\tilde{Y}_1^N(1,u)\right)^2\geq\min\left\{\frac{\pi^2c_1}{16},\frac{\pi\beta_2}{4}\right\}\left\|Y_1^N-\tilde{Y}_1^N\right\|^2_{0,2,e_1^N}$$

From the strong monotonicity follows the coerciveness of operators $A_2^0\left(x,Y_2^0\right)$, $A_\alpha^j\left(x,Y_\alpha^j\right)$, $A_1^N\left(x,Y_1^N\right)$.

Thus, the Browder-Minty theorem (see e.g.[3, p.204]) guaranties the existence of a unique solutions of problems (8)-(10).

Since

$$k(x)\frac{dY_\alpha^j(x,u)}{dx}=\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}f\left(t,Y_\alpha^j(t,u),\frac{dY_\alpha^j(t,u)}{dt}\right)dt+C,$$

then the function is absolutely continuous on $\bar{e}_\alpha^j$, that $k(x)\dfrac{dY_\alpha^j(x,u)}{dx}\in C(\bar{e}_\alpha^j)$, $j=2-\alpha,3-\alpha,...,N+1-\alpha,\ \alpha=1,2$.

Since $Y_\alpha^j(x,u)$ is the solution of (8)-(10), this function is also the solution of BVP (1), (2) which is unique due to the assumptions of our lemma. □

Now we are at the position to prove the next statement

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Then there exists the following ETDS for problem (1), (2)*

$$(au_{\bar{x}})_{\hat{x}}=-\hat{T}^x\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right),\quad x\in\hat{\omega}_h,\tag{12}$$

$$a_1u_{x,0}-\beta_1u_0=-\mu_1-h_1\hat{T}^{x_0}\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right),$$

$$-a_Nu_{\bar{x},N}-\beta_2u_N=-\mu_2-h_N\hat{T}^{x_N}\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right).$$

This ETDS has a unique solution $u(x)\ \forall x \in \hat{\omega}_h$ *which coincides with the solution (1), (2) at the points of the grid* $\hat{\omega}_h$*, where*

$$u_{\bar{x},j} = \frac{u_j - u_{j-1}}{h_j}, \quad u_{\hat{x},j} = \frac{u_{j+1} - u_j}{\hbar_j}, \quad \hbar_j = \frac{h_j + h_{j+1}}{2},$$

$$a(x_j) = \left[\frac{1}{h_j} V_1^j(x_j)\right]^{-1},$$

$$\hat{T}^{x_j}(w(\xi)) = [\hbar_j V_1^j(x_j)]^{-1} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) w(\xi) d\xi + [\hbar_j V_2^j(x_j)]^{-1} \int\limits_{x_j}^{x_{j+1}} V_2^j(\xi) w(\xi) d\xi,$$

$$\hat{T}^{x_0}(w(\xi)) = [h_1 V_1^1(x_1)]^{-1} \int\limits_{x_0}^{x_1} V_2^0(\xi) w(\xi) d\xi, \tag{13}$$

$$\hat{T}^{x_N}(w(\xi)) = [h_N V_1^N(x_N)]^{-1} \int\limits_{x_{N-1}}^{x_N} V_1^N(\xi) w(\xi) d\xi,$$

$$V_1^j(x) = \int\limits_{x_{j-1}}^{x} \frac{dx}{k(x)}, \quad V_2^j(x) = \int\limits_{x}^{x_{j+1}} \frac{dx}{k(x)}.$$

The function $u(x)$ *on the right-hand side of (12) is defined by (11) and depends only on* $u(x_j),\ j = 0, 1, ..., N$.

Proof. Applying the operator $\hat{T}^{x_j}$ to both sides of equation (1) we obtain

$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = -\hat{T}^{x_j}\left(f\left(\xi, u(\xi), \frac{du(\xi)}{d\xi}\right)\right), \quad j = 0, 1, 2, ..., N,$$

where

$$\hat{T}^{x_0}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = \left[h_1 V_1^1(x_1)\right]^{-1} \int\limits_{x_0}^{x_1} V_2^0(\xi) \frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right] d\xi,$$

$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = \left[\hbar_j V_1^j(x_j)\right]^{-1} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) \frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right] d\xi +$$

$$+ \left[\hbar_j V_2^j(x_j)\right]^{-1} \int\limits_{x_j}^{x_{j+1}} V_2^j(\xi) \frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right] d\xi, \quad j = 1, 2, ..., N-1,$$

$$\hat{T}^{x_N}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = \left[h_N V_1^N(x_N)\right]^{-1} \int\limits_{x_{N-1}}^{x_N} V_1^N(\xi) \frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right] d\xi.$$

The integration by parts implies

$$\hat{T}^{x_0}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = \frac{1}{h_1}\left(a_1 u_{x,0} - \beta_1 u_0 + \mu_1\right),$$
$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = (a u_{\bar{x}})_{\hat{x},j}, \quad j = 1, 2, ..., N-1,$$
$$\hat{T}^{x_N}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right) = \frac{1}{h_N}\left(-a_N u_{\bar{x},N} - \beta_2 u_N + \mu_2\right),$$

which together with (8)-(10) proves the existence of the ETDS (12), (13).

To prove the uniqueness of the ETDS (12), (13) we consider the operator

$$A_h\left(x_j, u\right) = \\ = \begin{cases} -\dfrac{2}{h_1}\left(a_1 u_{x,0} - \beta_1 u_0 + \mu_1 - h_1 \hat{T}^{x_0}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right)\right), & j = 0, \\ -(a u_{\bar{x}})_{\hat{x},j} - \hat{T}^{x_j}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right), & j = 1, 2, ..., N-1, \\ \dfrac{2}{h_N}\left(a_N u_{\bar{x},N} + \beta_2 u_N - \mu_2 - h_N \hat{T}^{x_N}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right)\right), & j = N \end{cases}$$

which is defined in the finite-dimensional Hilbert space of grid functions $H(\hat{\bar{\omega}}_h)$, with the scalar products

$$(u, v)_{\hat{\bar{\omega}}_h} = \sum_{\xi \in \hat{\omega}_h} \hbar(\xi) u(\xi) v(\xi) + 0,5 h_1 u_0 v_0 + 0,5 h_N u_N v_N$$
$$(u, v)_{\hat{\omega}_h^+} = \sum_{\xi \in \hat{\omega}_h^+} h(\xi) u(\xi) v(\xi), \quad \hat{\omega}_h^+ = \hat{\omega}_h \cup x_N,$$

and the norms

$$\|u\|_{0,2,\hat{\bar{\omega}}_h} = (u, u)_{\hat{\bar{\omega}}_h}^{1/2}, \quad \|u\|_{0,2,\hat{\omega}_h^+} = (u, u)_{\hat{\omega}_h^+}^{1/2},$$
$$\|u\|_{1,2,\hat{\bar{\omega}}_h} = \left(\|u\|_{0,2,\hat{\bar{\omega}}_h}^2 + \|u_{\bar{x}}\|_{0,2,\hat{\omega}_h^+}^2\right)^{1/2}.$$

Due to condition (5) the operator $A_h\left(x, u\right)$ is continuous. Let as show that the operator $A_h\left(x, u\right)$ is strongly monotone. Actually, taking into account the equality

$$\sum_{\xi \in \hat{\bar{\omega}}_h} \hbar(\xi) \hat{T}^{\xi}(w(\eta)) g(\xi) = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \hat{g}(\eta) w(\eta) d\eta = \int_0^1 \hat{g}(\eta) w(\eta) d\eta,$$

$$\hat{g}(\eta) = g(x_j)\frac{V_1^j(\eta)}{V_1^j(x_j)} + g(x_{j-1})\frac{V_2^{j-1}(\eta)}{V_1^j(x_j)}, \quad x_{j-1} \le \eta \le x_j,$$

and the first difference Green's formula (see. [5, p.234]), we have

$$
\begin{aligned}
&(A_h(x,u)-A_h(x,v),u-v)_{\hat{\omega}_h}=\left(a(u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\\
&+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2-\\
&-\sum_{\xi\in\hat{\omega}_h}\hbar(\xi)\hat{T}^{\xi}\left(f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right)[u(\xi)-v(\xi)]=\\
&=\left(a(u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2-\\
&-\int_0^1[\hat{u}(\eta)-\hat{v}(\eta)]\left[f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right]d\eta
\end{aligned}
$$

where the functions $u(x)$and $v(x)$ are defined by (11). Then using (6), we have

$$
\begin{aligned}
&(A_h(x,u)-A_h(x,v),u-v)_{\hat{\omega}_h}=\left(a(u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\\
&+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2-\\
&-\int_0^1[u(\eta)-v(\eta)]\left[f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right]d\eta-\\
&-\int_0^1[\hat{u}(\eta)-\hat{v}(\eta)-u(\eta)+v(\eta)]\times\\
&\qquad\times\frac{d}{d\eta}\left\{k(\eta)\frac{d}{d\eta}[\hat{u}(\eta)-\hat{v}(\eta)-u(\eta)+v(\eta)]\right\}d\eta\geq\\
&\geq\left(a(u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2+\\
&+\int_0^1k(\eta)\left\{\frac{d}{d\eta}[\hat{u}(\eta)-\hat{v}(\eta)-u(\eta)+v(\eta)]\right\}^2d\eta.
\end{aligned}
$$

Since (see [5, p.244]) $\gamma_1\|u\|^2_{0,2,\hat{\omega}_h}\leq\left(u^2_{\bar{x}},1\right)_{\hat{\omega}_h^+}+\beta_1u_0^2+\beta_2u_N^2,\ \gamma_1>0$, then have

$$
\begin{aligned}
&(A_h(x,u)-A_h(x,v),u-v)_{\hat{\omega}_h}\geq\\
&\geq\left(a(u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2\geq\\
&\geq\max\{c_1,1\}\left[\left((u_{\bar{x}}-v_{\bar{x}})^2,1\right)_{\hat{\omega}_h^+}+\beta_1(u_0-v_0)^2+\beta_2(u_N-v_N)^2\right]\geq\\
&\geq\max\{c_1,1\}\gamma_1\|u-v\|^2_{0,2,\hat{\omega}_h},
\end{aligned}
\tag{14}
$$

i. e. the operator $A_h(x,u)$ is strongly monotone. This yields (see e.g.[4, p.461]) the uniqueness of the solution of the equation $A_h(x,u)=0$. □

Lemma 2. *Let the assumptions of Theorem* 1 *be satisfied and*

$$|f(x,u,\xi)-f(x,v,\eta)|\leq L\{|u-v|+|\xi-\eta|\}\quad\forall x\in(0,1),\ u,v,\xi,\eta\in\mathbb{R}^1.$$

Then the iteration method

$$B_h \frac{u^{(n)} - u^{(n-1)}}{\tau} + A_h\left(x, u^{(n-1)}\right) = 0, \quad x \in \hat{\bar{\omega}}_h,$$
$$u^{(0)}(x) = \frac{\mu_1 + \mu_2 + \mu_1\beta_2 V_1(1)}{\beta_1 + \beta_2 + \beta_1\beta_2 V_1(1)} \frac{V_2(x)}{V_1(1)} + \frac{\mu_1 + \mu_2 + \mu_2\beta_1 V_1(1)}{\beta_1 + \beta_2 + \beta_1\beta_2 V_1(1)} \frac{V_1(x)}{V_1(1)}, \tag{15}$$

$$B_h u = \begin{cases} -\dfrac{2}{h_1}\left(a_1 u_{x,0} - \beta_1 u_0\right), & j = 0, \\ -(a u_{\bar{x}})_{\hat{x},j}, & j = 1, 2, ..., N-1, \\ \dfrac{2}{h_N}\left(a_N u_{\bar{x},N} + \beta_2 u_N\right), & j = N, \end{cases}$$

$$A_h(x_j, u) =$$
$$= \begin{cases} -\dfrac{2}{h_1}\left(a_1 u_{x,0} - \beta_1 u_0 + \mu_1 - h_1 \hat{T}^{x_0}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right)\right), & j = 0, \\ -(a u_{\bar{x}})_{\hat{x},j} - \hat{T}^{x_j}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right), & j = 1, 2, ..., N-1, \\ \dfrac{2}{h_N}\left(a_N u_{\bar{x},N} + \beta_2 u_N - \mu_2 - h_N \hat{T}^{x_N}\left(f\left(\xi, u, \dfrac{du}{d\xi}\right)\right)\right), & j = N \end{cases}$$

with

$$\tau = \tau_0 = \left[\left(1 + 2L\left(\frac{2K_1K_2}{\gamma_1}\right)^{1/2}\right)\left(1 + \frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\right]^{-2},$$

$$K_1 = \max\left\{\frac{1}{c_1}\left(\frac{4}{\gamma_1} + \frac{c_2}{c_1}\right), \frac{4}{\gamma_1}\right\}, \quad K_2 = \max\left\{\frac{1}{c_1}, 1\right\},$$

$$\gamma_1 = \frac{8\left(\beta_1 + \beta_2 + \beta_1\beta_2\right)^2}{\left(2+\beta_1\right)\left(2+\beta_2\right)\left(2\beta_1 + 2\beta_2 + \beta_1\beta_2\right)}$$

converges in the space H_{B_h} *and the error estimate*

$$\left\|u^{(n)} - u\right\|_{B_h} \le q^n \left\|u^{(0)} - u\right\|_{B_h}, \tag{16}$$

where

$$q = \sqrt{1 - \tau_0}, \quad \|u\|_{B_h} = (B_h u, u)^{1/2}_{\hat{\bar{\omega}}_h}.$$

Proof. The operator B_h is selfadjoint and of positive definite $B_h = B_h^* > 0$. From the first difference Green's formula implies that

$$(B_h u, u)_{\hat{\bar{\omega}}_h} = \left(a u_{\bar{x}}^2, 1\right)_{\hat{\omega}_h^+} + \beta_1 u_0^2 + \beta_2 u_N^2,$$

and from (14) we obtain

$$\left(A_h(x, u) - A_h(x, v), u - v\right)_{\hat{\bar{\omega}}_h} \ge \|u - v\|^2_{B_h}. \tag{17}$$

Using the Cauchy-Bunyakovski-Schwarz inequality we can now deduce

$$
\begin{aligned}
&\left(A_h\left(x,u\right)-A_h\left(x,v\right),z\right)_{\hat{\hat{\omega}}_h}=(B_h(u-v),z)_{\hat{\hat{\omega}}_h}-\\
&-\sum_{\xi\in\hat{\hat{\omega}}_h}\hbar(\xi)T^{\xi}\left(f\left(\eta,u(\eta),\frac{du}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv}{d\eta}\right)\right)z(\xi)=\\
&=(B_h(u-v),z)_{\hat{\hat{\omega}}_h}-\int\limits_0^1\left[f\left(\eta,u(\eta),\frac{du}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv}{d\eta}\right)\right]\hat{z}(\eta)d\eta\le\\
&\le\|u-v\|_{B_h}\|z\|_{B_h}+\\
&+\left\{\int\limits_0^1\left[f\left(\eta,u(\eta),\frac{du}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv}{d\eta}\right)\right]^2d\eta\right\}^{1/2}\left\{\int\limits_0^1[\hat{z}(\eta)]^2d\eta\right\}^{1/2}\le\\
&\le\|u-v\|_{B_h}\|z\|_{B_h}+\sqrt{2}L\|u-v\|_{1,2,(0,1)}\|\hat{z}\|_{0,2,(0,1)}.
\end{aligned}
$$

Since $V_1^j(x)\le V_1^j(x_j),\ V_2^{j-1}(x)\le V_1^j(x_j)\ \forall x\in[x_{j-1},x_j]$, we have

$$
\begin{aligned}
\|\hat{z}\|^2_{0,2,(0,1)}&=\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left[z_j\frac{V_1^j(x)}{V_1^j(x_j)}+z_{j-1}\frac{V_2^{j-1}(x)}{V_1^j(x_j)}\right]^2dx\le\\
&\le 2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left\{z_j^2\left[\frac{V_1^j(x)}{V_1^j(x_j)}\right]^2+z_{j-1}^2\left[\frac{V_2^{j-1}(x)}{V_1^j(x_j)}\right]^2\right\}dx\le 4\|z\|^2_{0,2,\hat{\hat{\omega}}_h},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\left\|\frac{d\hat{z}}{dx}\right\|^2_{0,2,(0,1)}&\le\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left[\frac{1}{k(x)}\frac{1}{V_1^j(x_j)}\left(z_j-z_{j-1}\right)\right]^2dx\le\\
&\le\frac{c_2}{c_1^2}\sum_{j=1}^{N}h_ja_jz^2_{\bar{x},j}=\frac{c_2}{c_1^2}\left(az^2_{\bar{x}},1\right)_{\hat{\omega}_h^+}.
\end{aligned}
\tag{19}
$$

So,

$$
\begin{aligned}
\left(A_h\left(x,u\right)-A_h\left(x,v\right),z\right)_{\hat{\hat{\omega}}_h}&\le\|u-v\|_{B_h}\|z\|_{B_h}+\\
&+2\sqrt{2}L\|u-v\|_{1,2,(0,1)}\|z\|_{0,2,\hat{\hat{\omega}}_h}.
\end{aligned}
\tag{20}
$$

Let us now show that

$$
\|u-v\|_{1,2,(0,1)}\le\sqrt{K_1}\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}.
\tag{21}
$$

We write $u(x)=\tilde{u}(x)+\hat{u}(x)$, and reduce the problem

$$
\begin{aligned}
&\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]=-f\left(x,u,\frac{du}{dx}\right),\quad x\in e_1^j,\\
&u(x_{j-1})=u_{j-1},\quad u(x_j)=u_j,\quad j=1,2,...,N
\end{aligned}
$$

to

$$\frac{d}{dx}\left[k(x)\frac{d\tilde{u}}{dx}\right] = -f\left(x, \tilde{u}+\hat{u}, \frac{d\tilde{u}}{dx}+\frac{d\hat{u}}{dx}\right), \quad x \in e_1^j,$$
$$\tilde{u}(x_{j-1}) = 0, \quad \tilde{u}(x_j) = 0, \quad j = 1, 2, ..., N,$$

Considering (3), (6) and using a Lipschitz condition we get

$$\frac{\pi^2 c_1}{\pi^2+1}\left\|\tilde{u}-\tilde{v}\right\|^2_{0,2,e_1^j} \le c_1 \left\|\frac{d\tilde{u}}{dx}-\frac{d\tilde{v}}{dx}\right\|^2_{1,2,e_1^j} \le \int\limits_{x_{j-1}}^{x_j} k(x)\left[\frac{d\tilde{u}}{dx}-\frac{d\tilde{v}}{dx}\right]^2 dx =$$

$$= \int\limits_{x_{j-1}}^{x_j}\left[f\left(x, \tilde{u}(x)+\hat{u}(x), \frac{d\tilde{u}}{dx}+\frac{d\hat{u}}{dx}\right) - \right.$$
$$\left. -f\left(x, \tilde{v}(x)+\hat{v}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{v}}{dx}\right)\right]\left[\tilde{u}(x)-\tilde{v}(x)\right] dx =$$

$$= \int\limits_{x_{j-1}}^{x_j}\left[f\left(x, \tilde{u}(x)+\hat{u}(x), \frac{d\tilde{u}}{dx}+\frac{d\hat{u}}{dx}\right) - \right.$$
$$\left. -f\left(x, \tilde{v}(x)+\hat{u}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{u}}{dx}\right)\right]\left[\tilde{u}(x)-\tilde{v}(x)\right] dx +$$
$$+ \int\limits_{x_{j-1}}^{x_j}\left[f\left(x, \tilde{v}(x)+\hat{u}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{u}}{dx}\right) - \right.$$
$$\left. -f\left(x, \tilde{v}(x)+\hat{v}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{v}}{dx}\right)\right]\left[\tilde{u}(x)-\tilde{v}(x)\right] dx$$

$$\le \left\{\int\limits_{x_{j-1}}^{x_j}\left[f\left(x, \tilde{v}(x)+\hat{u}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{u}}{dx}\right) - \right.\right.$$
$$\left.\left. -f\left(x, \tilde{v}(x)+\hat{v}(x), \frac{d\tilde{v}}{dx}+\frac{d\hat{v}}{dx}\right)\right]^2 dx\right\}^{1/2} \times$$
$$\times \left\{\int\limits_{x_{j-1}}^{x_j}\left[\tilde{u}(x)-\tilde{v}(x)\right]^2 dx\right\}^{1/2} \le$$
$$\le \sqrt{2}L\left\|\hat{u}-\hat{v}\right\|_{1,2,e_1^j}\left\|\tilde{u}-\tilde{v}\right\|_{1,2,e_1^j},$$

Hence we get

$$\left\|\tilde{u}-\tilde{v}\right\|_{1,2,e_1^j} \le \frac{\sqrt{2}(1+\pi^2)L}{\pi^2 c_1}\left\|\hat{u}-\hat{v}\right\|_{1,2,e_1^j}.$$

So taking into account inequalities (18), (19) and inequality (see [5, p.244])

$$\gamma_1\left\|u\right\|^2_{0,2,\hat{\bar{\omega}}_h} \le \left(u^2_{\bar{x}}, 1\right)_{\hat{\omega}_h^+} + \beta_1 u_0^2 + \beta_2 u_N^2,$$

we have

$$\|u-v\|_{1,2,(0,1)} \le \|\tilde{u}-\tilde{v}\|_{1,2,(0,1)} + \|\hat{u}-\hat{v}\|_{1,2,(0,1)} \le$$
$$\le \left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|\hat{u}-\hat{v}\|_{1,2,(0,1)} \le$$
$$\le \left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\left(4\|u-v\|^2_{0,2,\hat{\omega}_h}+\frac{c_2}{c_1^2}\left(a\left(u_{\bar{x}}-v_{\bar{x}}\right)^2,1\right)_{\hat{\omega}_h^+}\right)^{1/2} \le$$
$$\le \left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\left\{\frac{1}{c_1}\left(\frac{4}{\gamma_1}+\frac{c_2}{c_1}\right)\left(a\left(u_{\bar{x}}-v_{\bar{x}}\right)^2,1\right)_{\hat{\omega}_h^+}+\right.$$
$$\left.+\frac{4}{\gamma_1}\left[\beta_1\left(u_0-v_0\right)^2+\beta_2\left(u_N-v_N\right)^2\right]\right\}^{1/2} \le$$
$$\le \sqrt{K_1}\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}.$$

Based on the (18), (21) from inequality (20) we obtain

$$\left(A_h(x,u)-A_h(x,v),z\right)_{\hat{\omega}_h} \le \|u-v\|_{B_h}\|z\|_{B_h}+$$
$$+2L\sqrt{2K_1}\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}\|z\|_{0,2,\hat{\omega}_h} \le$$
$$\le \|u-v\|_{B_h}\|z\|_{B_h}+2L\left(\frac{2K_1}{\gamma_1}\right)^{1/2}\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}\times$$
$$\times\left(\frac{1}{c_1}\left(az_{\bar{x}}^2,1\right)_{\hat{\omega}_h^+}+\beta_1 z_0^2+\beta_2 z_N^2\right)^{1/2} \le$$
$$\le \left(1+2L\left(\frac{2K_1K_2}{\gamma_1}\right)^{1/2}\right)\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}\|z\|_{B_h}.$$

Setting $z=B_h^{-1}\left(A_h(x,u)-A_h(x,v)\right)$, we obtain

$$\left\|B_h^{-1}\left(A_h(x,u)-A_h(x,v)\right)\right\|_{B_h} \le$$
$$\le \left(1+2L\left(\frac{2K_1K_2}{\gamma_1}\right)^{1/2}\right)\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\|u-v\|_{B_h}. \tag{22}$$

Implies from (22), (17)

$$\left(A_h(x,u)-A_h(x,v),B_h^{-1}\left(A_h(x,u)-A_h(x,v)\right)\right)_{\hat{\omega}_h} \le$$
$$\le \left[\left(1+2L\left(\frac{2K_1K_2}{\gamma_1}\right)^{1/2}\right)\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\right]^2\|u-v\|^2_{B_h} \le$$
$$\le \left[\left(1+2L\left(\frac{2K_1K_2}{\gamma_1}\right)^{1/2}\right)\left(1+\frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\right]^2\times$$
$$\left(A_h(x,u)-A_h(x,v),u-v\right)_{\hat{\omega}_h}.$$

Then (see [5, p.502]) the iteration method (15) converges in the space H_{B_h} as well as the estimate (16). □

Note that the space H_{B_h} coincides with the space $L_2(\hat{\bar{\omega}}_h)$ and the conditions of equivalence of norms are executed.

$$\gamma_1 \|u\|_{0,2,\hat{\bar{\omega}}_h} \leq \|u\|_{B_h} \leq \gamma_2 \|u\|_{0,2,\hat{\bar{\omega}}_h} .$$

Lemma 3. *Let the assumptions of Lemma 2 be satisfied. Then the method of simple iteration (15) and in addition to (16) the following estimate holds*

$$\left\| k\frac{du^{(n)}}{dx} - k\frac{du}{dx} \right\|_{0,2,\hat{\bar{\omega}}_h} \leq M \left\| u^{(n)} - u \right\|_{B_h} \leq Mq^n,$$

where

$$\|u\|_{0,2,\hat{\bar{\omega}}_h} = \left\{ \sum_{j=1}^{N-1} \hbar_j u_j^2 + \frac{1}{2} h_1 u_0^2 + \frac{1}{2} h_N u_N^2 \right\}^{1/2} = \left\{ \frac{1}{2} \sum_{j=1}^{N} h_j (u_j^2 + u_{j-1}^2) \right\}^{1/2} .$$

Proof. Taking into account equality

$$k\frac{du}{dx}\bigg|_{x=x_j} = a_j u_{\bar{x},j} + \frac{1}{V_1^j(x_j)} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) \frac{d}{d\xi} \left[k(\xi) \frac{du}{d\xi} \right] d\xi =$$

$$= a_j u_{\bar{x},j} - \frac{1}{V_1^j(x_j)} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) f\left(\xi, u(\xi), \frac{du}{d\xi}\right) d\xi,$$

$$k\frac{du}{dx}\bigg|_{x=x_{j-1}} = a_j u_{\bar{x},j} - \frac{1}{V_2^{j-1}(x_{j-1})} \int\limits_{x_{j-1}}^{x_j} V_2^{j-1}(\xi) \frac{d}{d\xi} \left[k(\xi) \frac{du}{d\xi} \right] d\xi =$$

$$= a_j u_{\bar{x},j} + \frac{1}{V_1^j(x_j)} \int\limits_{x_{j-1}}^{x_j} V_2^{j-1}(\xi) f\left(\xi, u(\xi), \frac{du}{d\xi}\right) d\xi,$$

the inequality $(a+b)^2 \leq 2(a^2+b^2)$ as well as the Cauchy-Bunyakovsky-Schwarz inequality and a Lipschitz condition we obtain

$$\left\| k\frac{du^{(n)}}{dx} - k\frac{du}{dx} \right\|_{0,2,\hat{\bar{\omega}}_h} = \left\{ \frac{1}{2} \sum_{j=1}^{N} h_j \left[a_j u_{\bar{x},j}^{(n)} - a_j u_{\bar{x},j} - \right.\right.$$

$$\left.- \frac{1}{V_1^j(x_j)} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) \left(f\left(\xi, u^{(n)}(\xi), \frac{du^{(n)}}{d\xi}\right) - f\left(\xi, u(\xi), \frac{du}{d\xi}\right) \right) d\xi \right]^2 +$$

$$+\frac{1}{2}\sum_{j=1}^{N} h_j \left[a_j u_{\bar{x},j}^{(n)} - a_j u_{\bar{x},j} + \frac{1}{V_1^j(x_j)}\times\right.$$

$$\left.\left.\times\int\limits_{x_{j-1}}^{x_j} V_2^{j-1}(\xi)\left(f\left(\xi, u^{(n)}(\xi), \frac{du^{(n)}}{d\xi}\right) - f\left(\xi, u(\xi), \frac{du}{d\xi}\right)\right)d\xi\right]^2\right\}^{1/2} \leq$$

$$\leq \left\{2\sum_{j=1}^{N} h_j \left[a_j u_{\bar{x},j}^{(n)} - a_j u_{\bar{x},j}\right]^2 + \sum_{j=1}^{N} h_j [V_1^j(x_j)]^{-2}\times\right.$$

$$\times\left[\int\limits_{x_{j-1}}^{x_j} V_1^j(\xi)\left(f\left(\xi, u^{(n)}(\xi), \frac{du^{(n)}}{d\xi}\right) - f\left(\xi, u(\xi), \frac{du}{d\xi}\right)\right)d\xi\right]^2 +$$

$$+\sum_{j=1}^{N} h_j [V_1^j(x_j)]^{-2}\times$$

$$\left.\times\left[\int\limits_{x_{j-1}}^{x_j} V_2^{j-1}(\xi)\left(f\left(\xi, u^{(n)}(\xi), \frac{du^{(n)}}{d\xi}\right) - f\left(\xi, u(\xi), \frac{du}{d\xi}\right)\right)d\xi\right]^2\right\}^{1/2} \leq$$

$$\leq \left\{2\sum_{j=1}^{N} h_j \left[a_j u_{\bar{x},j}^{(n)} - a_j u_{\bar{x},j}\right]^2 + \sum_{j=1}^{N} h_j \int\limits_{x_{j-1}}^{x_j} \frac{\left[V_1^j(\xi)\right]^2 + \left[V_2^{j-1}(\xi)\right]^2}{\left[V_1^j(\xi) + V_2^{j-1}(\xi)\right]^2} d\xi\times\right.$$

$$\left.\times\int\limits_{x_{j-1}}^{x_j}\left[f\left(\xi, u^{(n)}(\xi), \frac{du^{(n)}}{d\xi}\right) - f\left(\xi, u(\xi), \frac{du}{d\xi}\right)\right]^2 d\xi\right\}^{1/2} \leq$$

$$\leq \left\{2\sum_{j=1}^{N} h_j a_j^2 \left[u_{\bar{x},j}^{(n)} - u_{\bar{x},j}\right]^2 +\right.$$

$$\left.+L^2\sum_{j=1}^{N} h_j \int\limits_{x_{j-1}}^{x_j}\left[\left|u^{(n)}(\xi) - u(\xi)\right| + \left|\frac{du^{(n)}}{d\xi} - \frac{du}{d\xi}\right|\right]^2 d\xi\right\}^{1/2} \leq$$

$$\leq \sqrt{2c_2}\left(a\left(u_{\bar{x}}^{(n)} - u_{\bar{x}}\right)^2, 1\right)_{\hat{\omega}_h^+} + \sqrt{2}L\left\|u^{(n)} - u\right\|_{1,2,(0,1)}.$$

Then based on the inequality (21) and Lemma 2 we have

$$\left\|k\frac{du^{(n)}}{dx} - k\frac{du}{dx}\right\|_{0,2,\hat{\bar{\omega}}_h} \leq \sqrt{2}\left[\sqrt{c_2} + \sqrt{K}_1\left(1 + \frac{\sqrt{2}(1+\pi^2)L}{c_1\pi^2}\right)\right]\times$$

$$\times \left\| u^{(n)} - u \right\|_{B_h} = M_1 \left\| u^{(n)} - u \right\|_{B_h} \le M q^n.$$

So, in this page ETDS is constructed and justified, which you can develop (see [6]) a three-point difference schemes of high order accuracy for the numerical solution of the BVP (1), (2). ☐

Bibliography

1. Gnativ L. B. Generalized three-point difference schemes of high order of accuracy for nonlinear ordinary differential equations of second order / L. B. Gnativ, M. V. Kutniv, A. I. Chukhrai // Journal of Mathematical Sciences.– 2010.– Vol. 167, № 1.– P. 62-75.
2. Rektorys K. Variational Methods in Mathematical Physics and Engineering / K. Rektorys.– Moscow: Mir, 1985 (in Russian).
3. Kufner A. Nonlinear Differential Equations / A. Kufner, S. Fucik.– Moscow: Nauka, 1988 (in Russian).
4. Trenogin V. A. Functional Analysis / V. A. Trenogin.– Moscow: Nauka, 1980 (in Russian).
5. Samarskii A. A. Numerical Methods for Grid Equations / A. A. Samarskii, E. S. Nikolaev.– Moscow: Nauka, 1978 (in Russian).
6. Gavrilyuk I. P. Exact and Truncated Difference Schemes for Boundary Value ODEs. (International Seris of Numerical Mathematics Vol. 159) / I. P. Gavrilyuk, M. Hermann, V. L. Makarov., M. V. Kutniv.– Basel: Springer AG, 2011.
7. Kutniv M. V. Accurate three-point difference schemes for second-order monotone ordinary differential equations and their implementation / M. V. Kutniv // Computational Mathematics and Mathematical Physics.– 2000.– Vol. 40, № 3.– P. 368-382.
8. Kutniv M. V. Tree-point difference schemes of high accuracy order for second order nonlinear ordinary differential equations with the boundary conditions of the third kind / M. V. Kutniv // Visnyk of Lviv University. Series Applied Mathematics and Computer Science.– 2002.– № 4.– P. 61-66 (in Ukrainian).
9. Kutniv M. V. Modified three-point difference schemes of high-accuracy order for second order nonlinear ordinary differential equations / M. V. Kutniv // Computational Methods in Applied Mathematics.– 2003.– Vol. 3, № 2.– P. 287-312.
10. Kutniv M. V. Accurate three-point difference schemes for second-order nonlinear ordinary differential equations and their implementation / M. V. Kutniv, V. L. Makarov, A. A. Samarskii // Computational Mathematics and Mathematical Physics.– 1999.– Vol. 39, № 1.– P. 45-60.
11. Makarov V. L. Exact three-point difference schemes for second-order nonlinear ordinary differential equations and their implementation / V. L. Makarov, A. A. Samarskii // Soviet Math. Dokl.– 1991.– Vol. 41, № 4.– P. 495-500.

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