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EXPONENTIALLY CONVERGENT METHOD FOR INTEGRAL NONLOCAL PROBLEM FOR THE ELLIPTIC EQUATION IN BANACH SPACE

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Резюме. Розглядається нелокальна задача з інтегральною умовою для еліптичного диференціального рівняння з необмеженим операторним коефіцієнтом в банаховому просторі. Побудовано та обгрунтовано експоненційно збіжний чисельний метод для наближеного розв'язку в припущенні, що операторний коефіцієнт A– секторіальний та виконані умови існування та єдиності розв'язку. Цей алгоритм базується на зображенні операторних функцій за допомогою інтеграла Данфорда-Коші вздовж гіперболи, що охоплює спектр A та використанні відповідної квадратурної формули, що містить невелику кількість резольвент. Ефективність запропонованого алгоритму демонструється на чисельному прикладі.

Abstract. Problem for the elliptic differential equation with an unbounded operator coefficient in Banach space and integral nonlocal condition is considered. An exponentially convergent algorithm is proposed and justified for the numerical solution of this problem under an assumption that operator coefficient A is strongly positive and some existence and uniqueness conditions are fulfilled. This algorithm is based on the representation of operator functions by a Dunford-Cauchy integral along a hyperbola, enveloping the spectrum of A, and on the proper quadratures involving small number of resolvents. The efficiency of the proposed algorithm is demonstrated on numerical example.

1. Introduction

Nonlocal boundary value problems naturally arise in mathematical modelling of many problems in engineering, physics, chemistry. These problems are interesting also from the point of view of mathematics as generalization of classical boundary value problems. Despite of a big amount of articles devoted to the nonlocal problems (see e.g. [1, 2, 7, 11]) and evidentially importance of such problems, the construction of highly precision and fast algorithms for their solution is still actual.

In this paper we consider the following nonlocal problem with integral condition:

$$\begin{aligned} &\frac{d^2u}{dx^2} - Au = 0, \quad x \in [0, X] \\ &u(0) = 0, \\ &\int_0^1 w(s)u(s)ds + u(1) = u_1, \end{aligned} \tag{1}$$

Key words. Nonlocal problem, differential equation with an operator coefficient in Banach space, exponentially convergent algorithms, nonlocal integral condition, elliptic equation.

where $w(s) \geq 0$ is a given function, $u_1 \in X$. The operator A with the domain $D(A)$ in a Banach space X is assumed to be densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin. The resolvent of A decays inversely proportional to $|z|$ at the infinity (see estimate (6) below).

Inhomogeneous problem related to (1) can be reduced to the homogeneous one by change of function in the following way. If we have

$$\begin{aligned} &\frac{d^2u}{dx^2} - Au = f(x), \quad x \in [0, X] \\ &u(0) = u_0, \\ &\int_0^1 w(s)u(s)ds + u(1) = u_1, \end{aligned} \tag{2}$$

with $f(x)$ being vector-valued function in the Banach space X then by putting $u(x) = v(x) + v_1(x)$, where

$$v_1(x) = \sinh(\sqrt{A}(1-x))\sinh^{-1}(\sqrt{A})u_0 + \int_0^1 G(x, s; A)f(s)ds,$$

$G(x, s; A)$ is a Green's function

$$G(x, s; A) = [\sqrt{A}\sinh\sqrt{A}]^{-1} \begin{cases} \sinh(x\sqrt{A})\sinh((1-s)\sqrt{A}) & x \leq s, \\ \sinh(s\sqrt{A})\sinh((1-x)\sqrt{A}) & x \geq s \end{cases}.$$

we obtain the following problem for $u(x)$

$$\begin{aligned} &\frac{d^2v}{dx^2} - Av = 0, \quad x \in (0, 1) \\ &v(0) = 0, \\ &\int_0^1 w(s)v(s)ds + v(1) = u_1 - \Phi, \end{aligned}$$

with

$$\Phi = \int_0^1 w(s)v_1(s)ds.$$

Note that an exponentially convergent numerical approximation for $v_1(x)$ was developed in [6], [5]. So, one can use this approximation to obtain $v_1(x)$ and then to find Φ.

It should be remark that various exponentially convergent methods were developed recently for problems with unbounded coefficients in Banach space [9], [6], [10], [12], [14], [17], [18]. These problems can be considered as metamodels of classical problems for partially differential equations such as parabolic elliptic and hyperbolic.

The aim of this paper is to construct an exponentially convergent approximation of a solution to problem (1). The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical algorithm for problem (1) is proposed and justified in section 3. The main result of this section is theorem 1 about the exponential convergence rate of the proposed discretization. The next section 4 is devoted to numerical examples which confirms theoretical results from the previous section.

2. Existence and representation of the solution

The solution of (1) can be formally represented as follows (see [5], [6]):

$$u(x) = E(x, \sqrt{A})u(1) = E(x, \sqrt{A}) \left[u_1 - \int_0^1 w(s)u(s)ds \right]. \tag{3}$$

where

$$E(x, \sqrt{A}) = \sinh(\sqrt{A}x) \sinh^{-1}(\sqrt{A}).$$

From the integral condition in (1) and formula (3) we obtain

$$\int_0^1 w(s)u(s)ds = \int_0^1 w(s)E(s, \sqrt{A})ds \left[u_1 - \int_0^1 w(s)u(s)ds \right],$$

or

$$\int_0^1 w(s)u(s)ds = \left[I + \int_0^1 w(s)E(s, \sqrt{A})ds \right]^{-1} \int_0^1 w(s)E(s, \sqrt{A})dsu_1,$$

in the case when $\left[I + \int_0^1 w(s)E(s, \sqrt{A})ds \right]^{-1}$ exists (sufficient conditions for the existence of this operator will be discussed later). Here I is the identity operator. So, we have

$$u(x) = E(x, \sqrt{A}) \left[I + \int_0^1 w(s)E(s, \sqrt{A})ds \right]^{-1} u_1 \tag{4}$$

Let the operator A from (1) be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ is situated in a sector Σ

$$\Sigma = \left\{ z = \rho_0 + r\mathrm{e}^{i\theta} : \quad r \in [0, \infty),\ \rho_0 > 0\ |\theta| < \varphi < \frac{\pi}{2} \right\}. \tag{5}$$

Additionally, the following estimate for the resolvent of A is valid

$$\|R_A(z)\| = \left\| (zI - A)^{-1} \right\| \leq \frac{M}{1 + |z|} \tag{6}$$

outside the sector and on its boundary Γ_Σ. The numbers ρ_0, φ are called the spectral characteristics of A.

We call the curve Γ_0 a spectral hyperbola:

$$\Gamma_0 = \{ z(\zeta) = \rho_0 \cosh\zeta - ib_0 \sinh\zeta : \ \zeta \in (-\infty, \infty),\ b_0 = \rho_0 \tan\varphi \}. \tag{7}$$

It has a vertex at $(\rho_0, 0)$ and asymptotes that are parallel to the rays of the spectral angle Σ.

A convenient representation of operator functions is the one through the Dunford-Cauchy integral (see e.g. [3, 8]) where the integration path plays an important role. Using the Dunford-Cauchy integral representation and (4) the solution to problem (1) can be written down as

$$u(x) = \frac{1}{2\pi i}\int_{\Gamma_I} \frac{E(x,\sqrt{z})}{1+\int_0^1 w(s)E(s,\sqrt{z})ds} R_A(z)u_1 dz = \qquad (8)$$
$$= \frac{1}{2\pi i}\int_{\Gamma_I} F(x,z)R_A(z)u_1 dz,$$

if $F(x,z)$ is analytic function inside the integration hyperbola Γ_I that envelopes Γ_0. To obtain uniformly convergent and numerically stable algorithm we shall modify this integral by changing the resolvent $R_A(z)$ to $R_A^1(z)$ that doesn't change the value of integral when $u_0 \in D(A^\alpha)$, $\alpha > 0$ (for the details see [4],[6]).

$$R_A^1(z) = (zI - A)^{-1} - \frac{I}{z}.$$

Therefore, one can obtain the following representation for the solution to problem (1):

$$u(x) = \frac{1}{2\pi i}\int_{\Gamma_I} F(x,z)R_A^1(z)u_1 dz. \qquad (9)$$

We choose the following hyperbola

$$\Gamma_I = \{z(\zeta) = a_I \cosh\zeta - ib_I \sinh\zeta : \ \zeta \in (-\infty,\infty)\}, \qquad (10)$$

for an integration contour that envelopes the spectrum of A, where the values of a_I, b_I are to be defined later. Using this hyperbola, we obtain from (9)

$$u(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} F(x,z(\zeta))R_A^1(\zeta)z'(\zeta)u_1 d\zeta = \int_{-\infty}^{\infty} \mathcal{F}(x,\zeta)d\zeta, \qquad (11)$$

with

$$z'(\zeta) = a_I \sinh\zeta - ib_I \cosh\zeta.$$

The next step toward a numerical algorithm is an approximation of (11) by an efficient quadrature formula. For this purpose we need to estimate the width of a strip around the real axis where the integrand in (11) admits analytical extension (with respect to ζ). The integration hyperbola Γ_I will be translated into the parametric set of hyperbolas with respect to ν after changing ζ to $\zeta + i\nu$

$$\Gamma(\nu) = \{z(\zeta,\nu) = a_I \cosh(\zeta + i\nu) - ib_I \sinh(\zeta + i\nu) : \ \zeta \in (-\infty,\infty)\}$$
$$= \{z(\zeta,\nu) = a(\nu)\cosh\zeta - ib(\nu)\sinh\zeta : \ \zeta \in (-\infty,\infty)\},$$

with

$$a(\nu) = a_I \cos\nu + b_I \sin\nu = \sqrt{a_I^2 + b_I^2}\sin(\nu + \phi/2),$$
$$b(\nu) = b_I \cos\nu - a_I \sin\nu = \sqrt{a_I^2 + b_I^2}\cos(\nu + \phi/2),$$
$$\cos\frac{\phi}{2} = \frac{b_I}{\sqrt{a_I^2 + b_I^2}}, \ \sin\frac{\phi}{2} = \frac{a_I}{\sqrt{a_I^2 + b_I^2}}.$$

The analyticity of the integrand in the strip

$$D_{d_1} = \{(\zeta,\nu) : \zeta \in (-\infty,\infty), |\nu| < d_1/2\},$$

with some d_1 could be violated if the resolvent or the part related to the nonlocal condition become unbounded. To avoid this we have to choose d_1 in a way such that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola is going through the point $(\rho_1, 0)$, for some $0 \le \rho_1 < \rho_0$. For $\nu = d_1/2$ it coincides with the spectral hyperbola and therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector. For $\nu = 0$ we have $\Gamma(0) = \Gamma_I$.

Such requirements for $\Gamma(\nu)$ imply the following system of equations

$$\begin{cases} a_I \cos(d_1/2) + b_I \sin(d_1/2) = \rho_0, \\ b_I \cos(d_1/2) - a_I \sin(d_1/2) = b_0 = \rho_0 \tan\varphi, \\ a_I \cos(-d_1/2) + b_I \sin(-d_1/2) = \rho_1, \end{cases}$$

from where we obtain

$$d_1 = \arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}}\right) - \varphi, \tag{12}$$

with $\cos\varphi = \frac{\rho_0}{\sqrt{\rho_0^2+b_0^2}}$, $\sin\varphi = \frac{b_0}{\sqrt{\rho_0^2+b_0^2}}$,

$$\begin{aligned} a_I &= \sqrt{\rho_0^2 + b_0^2}\cos\left(\frac{d_1}{2} + \varphi\right) = \rho_0 \frac{\cos\left(\frac{d_1}{2} + \varphi\right)}{\cos\varphi}, \\ b_I &= \sqrt{\rho_0^2 + b_0^2}\sin\left(\frac{d_1}{2} + \varphi\right) = \rho_0 \frac{\cos\left(\frac{d_1}{2} + \varphi\right)}{\cos\varphi}. \end{aligned} \tag{13}$$

For a_I and b_I defined as above the resolvent of the operator A is analytic in the strip D_{d_1} with respect to $w = \zeta + i\nu$ for any $t \ge 0$. Note, that for $\rho_1 = 0$ we have $d_1 = \pi/2 - \varphi$ as in [4].

Taking into account (13) we can similarly write the equations for $a(\nu)$, $b(\nu)$ on the whole interval $-\frac{d_1}{2} \le \nu \le \frac{d_1}{2}$

$$\begin{aligned} a(\nu) &= a_I \cos\nu + b_I \sin\nu = \sqrt{\rho_0^2 + b_0^2}\cos\left(\frac{d_1}{2} + \varphi\right)\cos(\nu) \\ &\quad + \sqrt{\rho_0^2 + b_0^2}\sin\left(\frac{d_1}{2} + \varphi\right)\sin(\nu) = \sqrt{\rho_0^2 + b_0^2}\cos\left(\frac{d_1}{2} + \varphi - \nu\right), \\ b(\nu) &= b_I \cos\nu - a_I \sin\nu = \sqrt{\rho_0^2 + b_0^2}\sin\left(\frac{d_1}{2} + \varphi\right)\cos(\nu) \\ &\quad - \sqrt{\rho_0^2 + b_0^2}\cos\left(\frac{d_1}{2} + \varphi\right)\sin(\nu) = \sqrt{\rho_0^2 + b_0^2}\sin\left(\frac{d_1}{2} + \varphi - \nu\right), \end{aligned}$$

$$\rho_1 \le a(\nu) \le \rho_0, \quad b_0 \le b(\nu) \le \sqrt{b_0^2 + \rho_0^2 - \rho_1^2},$$

with d_1, defined by (12).

Now, let us establish a condition on $w(s)$, that guaranties the existence of operator related to nonlocal condition from (4). For this to be true the

expression
$$\left[1+\int_0^1 w(s)E(s,\sqrt{z})ds\right]$$
related to nonlocal condition have to be bounded away from zero inside the integration hyperbola Γ_I.
$$\left|1+\int_0^1 w(s)E(s,\sqrt{z(\zeta)})ds\right| \geq 1-\left|\int_0^1 w(s)E(s,\sqrt{z(\zeta)})ds\right| \geq$$
$$\geq 1-\|w(s)\|_{C[0,1]}\int_0^1 |E(s,\sqrt{z(\zeta)})|ds \geq 1-\|w(s)\|_{C[0,1]}\int_0^1 \frac{\cosh(s\sqrt{a_I})}{\sinh(\sqrt{a_I})}ds =$$
$$= 1-\frac{\|w(s)\|_{C[0,1]}}{\sqrt{a_I}},$$
because (see [16])
$$\left|\frac{\sinh(\sqrt{z(\zeta)}x)}{\sinh(\sqrt{z(\zeta)})}\right| \leq \frac{\cosh(x\sqrt{a_I})}{\sinh\sqrt{a_I}}.$$
Similarly to above one can obtain more rough estimate
$$\left|1+\int_0^1 w(s)E(s,\sqrt{z(\zeta)})ds\right| \geq 1-\|w(s)\|_{C[0,1]}\int_0^1 \frac{\cosh(s\sqrt{a_I})}{\sinh(\sqrt{a_I})}ds \geq$$
$$\geq 1-\frac{\|w(s)\|_{C[0,1]}}{\sinh(\sqrt{a_I})}$$
Therefore, we have
$$\left|1+\int_0^1 w(s)E(s,\sqrt{z(\zeta)})ds\right|^{-1} \leq C_1,$$
in the case when
$$\|w(s)\|_{C[0,1]} < \sqrt{a_I}, \tag{14}$$
or, alternatively
$$\|w(s)\|_{C[0,1]} < \sqrt{a_I}, \tag{15}$$
where a_I is defined in (13).

So, we can summarize all of the above in the following lemma.

Lemma 1. *Let A be a densely defined strongly positive operator. If one of the conditions* (14) *or* (15) *is valid then there exists a unique solution to problem* (1) *that can be represented by* (9).

Further, let us establish conditions for the existence of the solution to (1) in the case when the operator A is self-adjoint positive definite. To achieve that we have to choose d_1 in a way that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola $\Gamma(\nu)$ remains in the right half-plane of complex plane. For $\nu = -d_1/2$ the corresponding hyperbola turns into the line parallel to the imaginary axis. For

$\nu = d_1/2$ it coincides with the ray that lies on the real axis having a vertex at ρ_0. These requirements imply the following system of equations

$$\begin{cases} a_I \cos(d_1/2) + b_I \sin(d_1/2) = \rho_0, \\ b_I \cos(d_1/2) - a_I \sin(d_1/2) = 0, \\ a_I \cos(-d_1/2) + b_I \sin(-d_1/2) = 0, \end{cases}$$

which has the solution

$$a_I = b_I = \frac{\rho_0}{\sqrt{2}},$$

$$d_1 = \frac{\pi}{2}$$

The condition (14) then becomes

$$\|w(s)\|_{C[0,1]} < \sqrt{\frac{\rho_0}{\sqrt{2}}}, \tag{16}$$

that is sufficient condition of existence solution to (1) in the case of self-adjoint positive operator A.

3. Numerical algorithm

First of all we approximate integral $\int_0^1 w(s)E(s,\sqrt{z})ds$ in (8) using exponentially convergent quadrature. For such approximation one can uses Gauss, Clenshaw-Curtis or Sinc quadrature formulas for integrals over bounded intervals. For analytical integrands these quadratures provide exponential rate of convergence. The Gauss quadrature is of the order $O(\rho^{-2n})$ and the Clenshaw-Curtis quadrature is of the order $O(\rho^{-n})$ where ρ is the sum of the semiminor and semimajor axis lengths of Bernstein ellipse [15]. The Sinc quadrature has the rate of convergence of the order $O(\mathrm{e}^{-\sqrt{n}})$ [13] and are well suited for integrals over unbounded intervals. Its convergence order may be either $O(\mathrm{e}^{-\sqrt{n}})$ or $O(\mathrm{e}^{-n/\ln n})$ depending on the analytical properties of integrands. We use the Gauss quadrature for the integral

$$\mathcal{I} = \int_0^1 w(s)E(s,\sqrt{z(\zeta)})ds \approx \frac{1}{2}\sum_{j=0}^{n} \omega_j w(\xi_j)E(\xi_j,\sqrt{z(\zeta)}) = \mathcal{I}_n, \tag{17}$$

$$\xi_j = \frac{1}{2}(\theta_j + 1),$$

where $\{\theta_j\}$ is a set of $n+1$ roots of the Legendre polynomial $P_{n+1}(x)$ and $\{\omega_j\}$ is a set of weights related to the Gauss quadrature rule. Note that θ_j and ω_j can be precomputed using fast algorithms (see [15]).

Therefore we obtain from (11)

$$u(x) \approx u_n(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} F_n(x,z(\zeta))z'(\zeta)R_A^1(\zeta)u_1 d\zeta = \int_{-\infty}^{\infty} \mathcal{F}_n(x,\zeta)d\zeta, \tag{18}$$

where

$$F_n(z(\zeta),A) = \frac{E(x,\sqrt{z(\zeta)})}{1+\mathcal{I}_n}$$

For the error estimate we have

$$\left|\frac{1}{1+\mathcal{I}} - \frac{1}{1+\mathcal{I}_n}\right| = \left|\frac{\mathcal{I}-\mathcal{I}_n}{(1+\mathcal{I})(1+\mathcal{I}_n)}\right|.$$

Due to (14) we have

$$\frac{1}{|1+\mathcal{I}|} \le C.$$

$$\frac{1}{|1+\mathcal{I}_n|} \le \frac{1}{1-\left|\frac{1}{2}\sum\limits_{j=0}^{n}\omega_j w(\xi_j)E(\xi_j,\sqrt{z(\zeta)})\right|} \le$$

$$\le \frac{1}{1-\frac{\|w(s)\|_{C[0,1]}}{2}\sum\limits_{j=0}^{n}\omega_j E(\xi_j,\sqrt{z(\zeta)})} \le \frac{1}{1-\frac{\|w(s)\|_{C[0,1]}}{\sinh\sqrt{a_I}}} \le c = const, \tag{19}$$

in the case when (15) is valid. Consequently we arrive at the estimate

$$\left|\frac{1}{1+\mathcal{I}} - \frac{1}{1+\mathcal{I}_n}\right| \le c\,|\mathcal{I}-\mathcal{I}_n|\,.$$

Normalized hyperbolic sin-function $E(x,z)$ is analytical with respect to x in all complex plane. So, smoothness of the integrand in $\mathcal{I}$ is governed by $w(s)$. Using theorem 19.3 from [15] we can deduce that if $w(\frac{1}{2}(s+1))$ is analytic in $[-1,1]$ and analytically continuable to the open Bernstein ellipse where $\left|w(\frac{1}{2}(s+1))E(\frac{1}{2}(s+1),z)\right| \le M$ then

$$|\mathcal{I}-\mathcal{I}_n| \le \frac{144M\rho^{-2n}}{35(\rho^2-1)}, \qquad n \ge 2. \tag{20}$$

If $w(s)$ and its derivatives up to $w^{(\nu-1)}$ are absolutely continuous and $w^{(\nu)}$ has a bounded variation V then

$$|\mathcal{I}-\mathcal{I}_n| \le \frac{32V}{15\pi\nu(n-2\nu-1)^{2\nu+1}}, \qquad n > 2\nu+1. \tag{21}$$

Supposing $u_1 \in D(A^\alpha)$, $0<\alpha<1$ it was shown in [6] that

$$\begin{aligned}\left\|E(x,\sqrt{z(\zeta)})z'(\zeta)R_A^1(\zeta)u_1\right\| \le (1+M)K\frac{b_I}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha} \times \\ \times\, \mathrm{e}^{(x-1)\sqrt{a_I\cosh\xi}-\alpha|\xi|}\|A^\alpha u_1\|, \\ \xi\in\mathbb{R}, \quad x\in(0,1],\end{aligned} \tag{22}$$

where K is a constant that depends on α, M is a constant from resolvent estimate (6).

The part responsible for the nonlocal condition in (18) is estimated by (19). Thus, we obtain the following estimate for $\mathcal{F}_n(x,\xi)$:

$$\begin{aligned}\|\mathcal{F}_n(x,\zeta)\| \le C(\varphi,\alpha)\mathrm{e}^{(x-1)\sqrt{a_I\cosh\xi}-\alpha|\xi|}\|A^\alpha u_1\|, \\ C(\varphi,\alpha) = (1+M)qK\frac{b_I}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha}, \quad \zeta\in\mathbb{R}, \quad x\in(0,1].\end{aligned} \tag{23}$$

Next we approximate integral (18) by the Sinc-quadrature formula [6, 13]:

$$u_{n,N}(x) = h \sum_{k=-N}^{N} \mathcal{F}_n(x, z(kh)), \tag{24}$$

with the error

$$\begin{aligned} \|\eta_N(\mathcal{F}_n, h)\| &= \|u_n(x) - u_{n,N}(x)\| \\ \leq \left\| u_n(x) - h \sum_{k=-\infty}^{\infty} \mathcal{F}_n(x, z(kh)) \right\| &+ \left\| h \sum_{|k|>N} \mathcal{F}_n(x, z(kh)) \right\| \\ &\leq \frac{1}{4\pi} \frac{\mathrm{e}^{-\pi d/h}}{\sinh(\pi d/h)} \|\mathcal{F}_n\|_{\mathbf{H}^1(D_d)} \\ + \frac{C(\varphi,\alpha) h \|A^\alpha u_1\|}{2\pi} &\sum_{k=N+1}^{\infty} \mathrm{e}^{(x-1)\sqrt{a_I \cosh(kh)} - \alpha k h}. \end{aligned}$$

Here $\mathbf{H}^1(D_d)$ is a space of all vector-valued functions $\mathcal{F}$ analytic in the strip D_d introduced similarly to [13] in [6]. Due to [6]

$$\begin{aligned} &\|E(x, \sqrt{z(\zeta)}) z'(\zeta) R_A^1(\zeta) u_1\|_{\mathbf{H}^1(D_{d_1})} \leq \|A^\alpha u_1\| [C_-(\varphi,\alpha) \\ &+ C_+(\varphi,\alpha)] \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|\xi|} d\xi = C(\varphi,\alpha) \|A^\alpha u_1\| \end{aligned} \tag{25}$$

with

$$C(\varphi,\alpha) = \frac{2}{\alpha} [C_+(\varphi,\alpha) + C_-(\varphi,\alpha)],$$

$$C_\pm(\varphi,\alpha) = c \tan\left(\frac{d_1}{2} + \varphi \pm \frac{d_1}{2}\right) \left(\frac{2\cos\varphi}{\rho_0 \cos\left(\frac{d_1}{2} + \varphi \pm \frac{d_1}{2}\right)} \right)^\alpha.$$

$$d = d_1 - \delta,$$

for an arbitrary small positive δ.

It is obvious that in the case of (15) the part responsible for the nonlocal condition is bounded in D_d. It allows us to obtain

$$\|\mathcal{F}_n(x, \cdot)\|_{\mathbf{H}^1(D_d)} \leq C(\varphi, \alpha, \delta) \|A^\alpha u_1\|.$$

So, we end up with the error for $\eta_N(\mathcal{F}_n, h)$

$$\|\eta_N(\mathcal{F}_n, h)\| \leq \frac{c\|A^\alpha u_1\|}{\alpha} \left\{ \frac{\mathrm{e}^{-\frac{\pi d_1}{h}}}{\sinh(\frac{\pi d_1}{h})} + \mathrm{e}^{(x-1)\sqrt{a_I} \cosh(\frac{(N+1)h}{2}) - \alpha(N+1)h} \right\} \tag{26}$$

where the constant c does not depend on h, N, x.

Equalizing both exponentials gives us

$$\frac{\pi d_1}{h} = \alpha(N+1)h,$$

$$h = \sqrt{\frac{\pi d_1}{\alpha(N+1)}}, \tag{27}$$

this leads to the following error estimate

$$\|\eta_N(\mathcal{F}_n,h)\| \le \frac{c}{\alpha}\mathrm{e}^{\left(-\sqrt{\pi d_1 \alpha(N+1)}\right)}\|A^\alpha u_1\| \tag{28}$$

The first summand in the argument of $\mathrm{e}^{(x-1)\sqrt{a_I}\cosh\left(\frac{(N+1)h}{2}\right)-\alpha(N+1)h}$ from (26) contributes mainly to the error in the case $x<1$. Setting for such case $h = c_1 \ln N/N$ with some positive constant c_1 we obtain for a fixed x the following estimate:

$$\|\eta_N(\mathcal{F}_n,h)\| \le c\left[\mathrm{e}^{-\pi d_1 N/(c_1 \ln N)} + \mathrm{e}^{-c_1(x-1)\sqrt{a_I}N/2-c_1\alpha\ln N}\right]\|A^\alpha u_1\|. \tag{29}$$

Thus, we have proven the following theorem.

Theorem 1. *Let A be a densely defined strongly positive operator, $u_1 \in D(A^\alpha)$, $\alpha \in (0,1)$ and condition (15) is valid. Then Sinc-quadrature (24) represents an approximation to $u_n(x)$. It provides the convergence of exponential order uniformly with respect to x presented by estimate (28) for the step size h defined in (27). The approximation has the convergence rate (29) for the case $x<1$ and $h = c_1 \ln N/N$.*

Remark 5. *The integration curve Γ_I is symmetric with respect to the real axis. Therefore $z(-kh) = \overline{z(kh)}$ and $z'(-kh) = -\overline{z'(kh)}$. Approximation (24) can be rewritten in the form*

$$u_{n,N}(x) = \frac{h}{2\pi i}\mathcal{F}_n(x,z(0)) + Re\left[\sum_{k=1}^{N} h\frac{\mathcal{F}_n(x,z(kh))}{\pi i}\right],$$

which reduce the number of resolvent calculations by the factor of two.

Now we can turn our attention to the full error estimate.

$$\varepsilon_1 = \|u(x)-u_n(x)\| = \left\|\int_{-\infty}^{\infty}[\mathcal{F}(x,\zeta)-\mathcal{F}_n(x,\zeta)]\,d\zeta\right\| \le$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|E(x,\sqrt{z(\zeta)})z'(\zeta)\right|\left|\frac{\mathcal{I}}{1+\mathcal{I}}-\frac{\mathcal{I}_n}{1+\mathcal{I}_n}\right|\left\|R_A^1(\zeta)u_1\right\|d\zeta,$$

By virtue of (22) it can be transformed to

$$\varepsilon_1 = \frac{(1+M)Kb_Ic}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha}\|A^\alpha u_1\|\,|\mathcal{I}-\mathcal{I}_n|\int_{-\infty}^{\infty}\mathrm{e}^{(x-1)\sqrt{a_I\cosh\zeta}-\alpha|\zeta|}d\zeta \le$$

$$\le \frac{2(1+M)Kb_Ic}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha}\|A^\alpha u_1\|\,|\mathcal{I}-\mathcal{I}_n|\int_{0}^{\infty}\mathrm{e}^{-\alpha|\zeta|}d\zeta =$$

$$= C\,\|A^\alpha u_1\|\,|\mathcal{I}-\mathcal{I}_n|\,.$$

Then for the full error estimate we have

$$\|u(x)-u_{n,N}(x)\| \le \varepsilon_1 + \|\eta_N(\mathcal{F}_n,h)\|\,. \tag{30}$$

It allows us to formulate the main theorem.

Theorem 2. *Let the conditions of theorem 1 be valid. Then (24) represents an approximation to $u(x)$. It provides the convergence of exponential order in the case when $w(x)$ is analytically continuable to the Bernstein ellipse.*

4. Numerical examples

Example 1. Let us consider the problem (1) with the operator A defined by

$$\begin{aligned} &D(A) = \{v(y) \in H^2(0,1): \ v(0) = v(1) = 0\}, \\ &Av = -v''(y) \ \forall v \in D(A), \end{aligned} \tag{31}$$

that generates a homogeneous parabolic equation with boundary conditions

$$\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = 0,$$
$$u(x,0) = u(x,1) = 0.$$

Let us supplement this problem with a boundary condition

$$u(0,y) = 0,$$

and nonlocal integral condition

$$u(1,0) + \int_0^1 \sin(\pi s) u(s,y) ds = \sinh(\pi) \frac{1+2\pi}{2\pi} \sin(\pi y).$$

In this case the exact solution to the problem is $u(x,y) = \sinh(\pi x)\sin(\pi y)$. We have performed calculations using Maple. The errors are presented in Table 1 for different number of quadrature points n (17) and number of Sinc-points N (24). The table clearly demonstrates the exponential decay of error according to the theoretical estimate (30).

Tabl. 1. The error for $x = 0.5$, $y = 0.5$

	n		
N	4	8	16
4	0.869502080695972		
8	0.351883285832682	0.351901023526236257	
16	0.017266161343386	0.017307693141547764	0.0173076931433691542
32	0.000071497193928	0.000038004847747042	0.0000380048260057791
64	0.000033550806875	$6.2833535266435186 * 10^{-13}$	$4.738853014104683 * 10^{-13}$
128		$1.5442654065186853 * 10^{-13}$	$3.847741101629530 * 10^{-24}$
256			$3.806256157045269 * 10^{-34}$

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