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ON PRACTICAL ATTAINMENT OF PREASSIGNED ACCURACY OF RESULTS IN THE PROCESS OF NUMERICAL ANALYSIS OF SOME TWO-DIMENSIONAL INTEGRAL EQUATIONS

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Резюме. У представленій роботі ми досліджуємо різні аспекти побудови наближених схем для розв'язування інтегрального рівняння першого роду зі слабкою особливістю в ядрі, яке є характерним для теорії потенціалу. У зв'язку з цим ми приймаємо до уваги суттєво просторове формулювання початкової проблеми, а також сингулярну поведінку шуканого розв'язку в околі контуру розімкненої граничної поверхні. З метою отримання гарантованої точності результату, використовуючи відомі загальні ідеї методології апостеріорної оцінки похибки, ми пропонуємо таку її версію, що є придатною саме до розглядуваного інтегрального рівняння.

Abstract. In the article we investigate different aspects of approximate schemes construction for the first kind integral equations being used in potential theory. In this connection we take into consideration substantially spatial setting of the problem and specific behavior of desired solution near the contour of unclosed boundary surface. With a view to obtain guaranteed accuracy of results, using known general concept of a posteriori error estimation methodology, we propose such it version applicable precisely to considered integral equation.

1. Introduction

The main object of our analysis is different aspects of approximate schemes construction for the first kind integral equation solving. In addition, we have to do with equations in the form as

$$(A\sigma)(M) \equiv \iint\limits_S \sigma(P)\,|\,M-P\,|^{-1}\mathrm{d}S_P = U(M), \quad M \in S, \tag{1}$$

where in general case S is an open Lipschitz surface, M and P are the points of Euclidean space $\mathbb{R}^3$. The type (1) equations appear at the modelling of potential theory some boundary problems, in particular, electron optics. Ordinary generalization of (1) is a permission that S is formed by the aggregate m of surfaces, so that $S := \bigcup_{i=1}^{m} S_i$. In this case we interpret $\sigma(P)$ as a desired total charge distribution density on S, that is $\sigma(P) := \{\sigma(P),\ P \in S_i;\ i = \overline{1,N}\}$. It is possible to research operator equation (1) in various functional spaces [4, 8]. However, it should be taken into account the specificity of investigated physical phenomenon. Thus, for example, the modelling of electrostatic field in the

Key words. Integral equation, axial symmetry, the collocation method, a posteriori error estimation.

substantially spatial setting foresees the account of desired charge distribution density $\sigma(P)$ behavior near the contour of unclosed surface S and lines of its fracture [7]. In this case we consider that $U(M)$, $M \in S$, is the given boundary value of potential on an electrode which is actually modeled by a surface S ($U(M) \equiv \text{const}$). As to numerical methods applied for initial problem solving we can point out in principle some approaches for integral equations approximate schemes construction [1, 2, 6]. In this connection, taking into account substantially spatial setting of the problem and specific behavior of desired solution, from practical point of view, in the best way, mentioned above questions were solved in [2].

2. Considered set of boundary surfaces and the ways of their specification

At first, using parametric representation of S, we will consider that

$$M := \left\{ x(\alpha_0, \beta_0),\, y(\alpha_0, \beta_0),\, z(\alpha_0, \beta_0);\ (\alpha_0, \beta_0) \in D := (-1, 1)^2 \right\},$$
$$P := \left\{ x(\alpha, \beta),\, y(\alpha, \beta),\, z(\alpha, \beta);\ (\alpha, \beta) \in \overline{D} \right\}.$$

At that time, integral equation (1) will be shown as

$$\iint\limits_{\overline{D}} \sigma(\alpha, \beta)\, K(\alpha, \beta; \alpha_0, \beta_0)\, \mathrm{d}\alpha \mathrm{d}\beta = U(\alpha_0, \beta_0), \quad (\alpha_0, \beta_0) \in D; \tag{2}$$

where

$$K(\alpha, \beta; \alpha_0, \beta_0) := \Big\{ [x(\alpha, \beta) - x(\alpha_0, \beta_0)]^2 + [y(\alpha, \beta) - y(\alpha_0, \beta_0)]^2 + \\ + [z(\alpha, \beta) - z(\alpha_0, \beta_0)]^2 \Big\}^{-1/2} J(\alpha, \beta);$$

and $J(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta$ is an element of surface S in local coordinates (α, β) associated with S. Keeping in (2) notation for σ and U from (1), we will also remark that from functions $x(\alpha, \beta)$, $y(\alpha, \beta)$, and $z(\alpha, \beta)$, which express the Cartessian coordinates (x, y, z) of the points on a surface S, it is required, at least, continuous differentiability in $\overline{D}$ because

$$J(\alpha, \beta) := \left\{ E(\alpha, \beta)\, G(\alpha, \beta) - F^2(\alpha, \beta) \right\}^{1/2};$$

and

$$E(\alpha, \beta) := \left(x'_\alpha\right)^2 + \left(y'_\alpha\right)^2 + \left(z'_\alpha\right)^2,$$
$$G(\alpha, \beta) := \left(x'_\beta\right)^2 + \left(y'_\beta\right)^2 + \left(z'_\beta\right)^2,$$
$$F(\alpha, \beta) := x'_\alpha x'_\beta + y'_\alpha y'_\beta + z'_\alpha z'_\beta.$$

As an example, for arbitrary charged quadrangular plate presentation we make use of the following equations:

$$x(\alpha,\,\beta)=\frac{1}{4}\sum_{j=1}^{4}x_j\,\varphi_j(\alpha,\,\beta),\quad \varphi_j(\alpha,\,\beta):=\Big(1+(-1)^p\alpha\Big)\Big(1+(-1)^q\beta\Big),$$

$$y(\alpha,\,\beta)=\frac{1}{4}\sum_{j=1}^{4}y_j\,\varphi_j(\alpha,\,\beta),\quad p:=\left[\frac{j}{2}\right]+1,\quad q:=\left[\frac{j-1}{2}\right]+1,$$

$$z(\alpha,\,\beta)=\frac{1}{4}\sum_{j=1}^{4}z_j\,\varphi_j(\alpha,\,\beta),\quad (\alpha,\,\beta)\in\overline{D};$$

where $(x_j,\,y_j,\,z_j)$ are coordinates of corresponding plate vertex. It is obvious that we simulate this and similar plates with the help of double-sided infinitely-thin surfaces.

Addressing to such type of boundary surfaces is explained by a possibility of the use for the approximate solving (2) in this case of numerically-analytical methodology introduced by the authors [3].

Another example of parametric equations is related to the necessity of so-called flat diaphragms descriptions. The last ones are components of rather complicated and actual in practice electron-optical systems. The figure 1 represents a projection of a diaphragm on the plane $z=\text{const}$. It is easy to notice that in this case the examined surface S represents the combination of eight elements. In addition, each of them is expressed by a flat curvilinear quadrangle. For every element a unique description it is enough to fix eight points on the corresponding contours as shown at the figure 1.

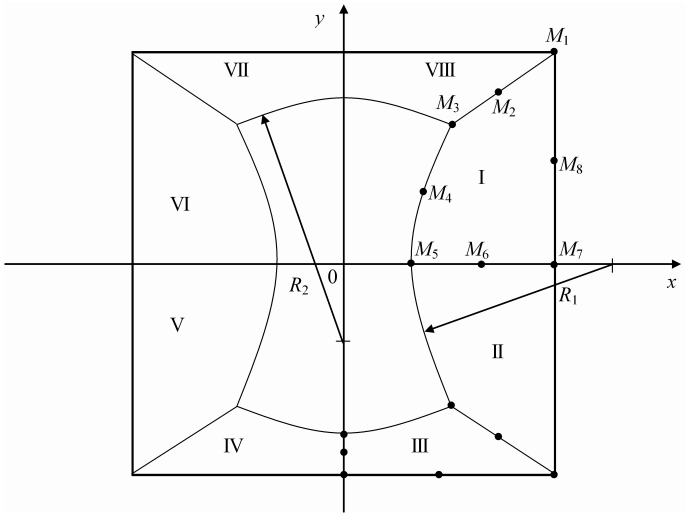

FIG. 1. A projection of flat diaphragm on the plane $z=\text{const}$

In this connection parametric equations have such expression as

$$x(\alpha,\,\beta) = \frac{1}{4}\sum_{j=1}^{4} x_j\,\varphi_j(\alpha,\,\beta),$$

$$y(\alpha,\,\beta) = \frac{1}{4}\sum_{j=1}^{4} y_j\,\varphi_j(\alpha,\,\beta), \quad (\alpha,\,\beta) \in \overline{D};$$

where

$$\varphi_1(\alpha,\,\beta) := (1+\alpha)\,(1+\beta)\,(\alpha+\beta-1),$$
$$\varphi_2(\alpha,\,\beta) := (1-\alpha^2)\,(1+\beta),\ \ldots\ ;$$

and x_j, y_j are coordinates of points M_i $(i = \overline{1,8})$.

3. General remarcs concerned with numerical analysis of type (1) integral equations

Two-dimensional integral equation (2) was solved by the method of collocation with the use of piecewise-constant and bilinear approximation of desired $\sigma(P)$. It is easy to see that (2) belongs to integral equations with weak singularity in the kernel. Therefore, in the process of (2) solving it is necessary to calculate approximately some two-dimensional singular integrals of specific class. In this connection the algorithms of such integrals calculation become substantially complicated through the presence of certain weight functions. The point is that the last ones represent precisely singular behavior of desired solution near the contour of open surface S.

The integral equations of type (1)-(2) were also examined in the context of electrostatic field determination, in the case when the systems of charged electrodes have rather complicated configuration. We will find out some details of initial problem effective solution, based on the integral equations method, in the substantially spatial setting, taking into account present symmetry at geometry of unclosed surfaces-electrodes. The account of symmetry enables to interpret initial problem as a task with finite order abelian group of symmetry. It allows to reduce (2), set on all boundary surface, to the sequence N of independent integral equations, set on one of their congruent constituents. Here N is an order of established group of symmetry. It results in avoidance of numerical instability of the systems of linear algebraic equations solving. With the help of these systems the approximation of corresponding integral equations is realized. In addition, their dimensions excessively increase. There is also a possibility to create pre-conditions for parallelizing an algorithm of the basic problem solving. Choosing a different number of processors, it allows to reach maximal efficiency of their loading and increasing the speed of calculation.

4. Scheme of results refinement obtained in the process of specific model task solving

Illustrating the expediency of the mentioned methodology application, we will consider the problem of electrostatic field calculation of so-called plane-parallel condenser (see Fig. 2).

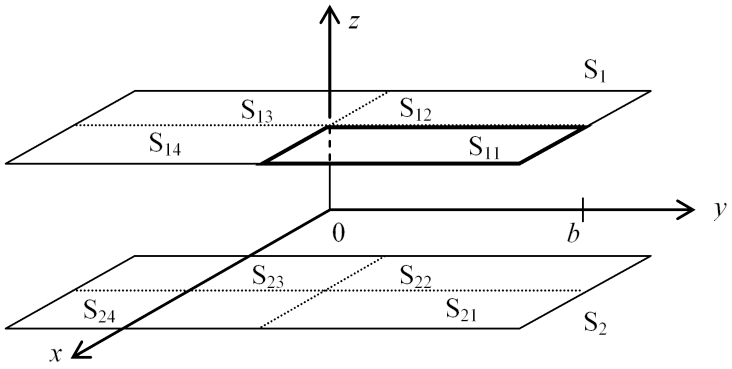

FIG. 2. Investigated plane-parallel condenser

In the course of initial problem mathematical modelling we will represent the corresponding systems of electrodes as an combination of two open surfaces which do not have common points, so that $S := S_1 \bigcup S_2$. It is easy to see that surfaces S_i are bounded to the piecewise-smooth contours of finite length. We consider this task as model one. The point is that the electrostatic field reproduction under the conditions of essential difference of potentials on the plates and step-by-step decrease of the distance between indicated plates is not trivial problem. In this case the results of calculation are especially sensitive with respect to variation of output date.

Returning to our integral equation let us assume that

$$S_l := \Big\{ (x,\, y,\, z) \in \mathbb{R}^3 \,\Big|\, (x,\, y) \in [-a,\, a] \times [-b,\, b]; \\ z = (-1)^{l-1} h;\ l = \overline{1,2};\ a, b, h > 0 \Big\}.$$

Considering the geometric characteristics of total surface S let us represent the last in the form of congruent constituents combination:

$$S = \bigcup_{l=1}^{2} \left(\bigcup_{k=1}^{4} S_{lk} \right).$$

Taking into account such subdivision of S_l $(l = 1, 2)$, integral equations (2), in turn, can be formally represented as

$$\sum_{l=1}^{2} \sum_{k=1}^{4} \int\limits_{S_{lk}} \sigma_{lk}(P)\, |\, P - M\, |^{-1} \mathrm{d}S_P = \\ = U(M) = \begin{cases} U_1, & M \in S_1; \\ \\ U_2, & M \in S_2; \end{cases} \tag{3}$$

where $\sigma_{lk}(P)$ is the projection of $\sigma(P)$ on S_{lk};

$$|\,P-M\,|^{-1}=\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{-1/2};$$
$$M:=(x_0,\,y_0,\,z_0=\pm h);\quad (x,\,y),\,(x_0,\,y_0)\in[-a,\,a]\times[-b,\,b].$$

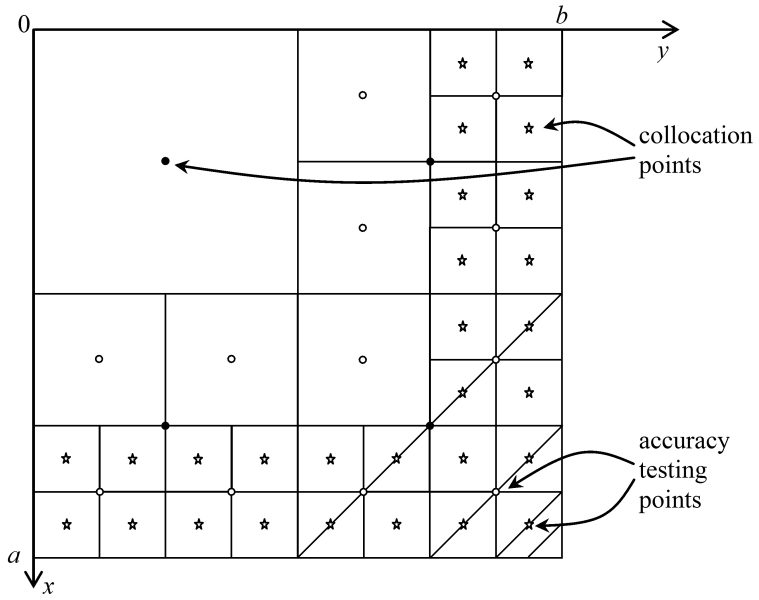

FIG. 3. A projection of congruent component S_{11} on the plane $z=\text{const}$ together with the consequent step-by-step partition into elements for the attainment of desired accuracy of (4) solving

Then, applying in (3) trivial changes of variables, we realize the conversion from integration over total surface S to integration over it congruent constituent S_{11}. In addition, let us note than the point of collocation M is placed also on S_{11}. As a result, we have obtained in fact the system of eight linear integral equations with respect to unknown density $\sigma_j(x,\,y)$ $(j=\overline{1,8})$, according to chosen group of symmetry of surface S:

$$\sum_{j=1}^{8}\iint\limits_{\Delta_1}\sigma_j(x,\,y)\,G_{|\,i-j\,|+1}(x,\,y,\,h;\,x_0,\,y_0,\,z_0)\,\mathrm{d}x\mathrm{d}y=U(M_i)\quad(i=\overline{1,8}).\quad(4)$$

Here $\Delta_1:=[0,\,a]\times[0,\,b]$;

$$M_i:=\left((-1)^{r-1}x_0,\,(-1)^{s-1}y_0,\,(-1)^{p-1}h\right)\in S_{pq};$$

in this case $i:=4(p-1)+2(r-1)+s$, and $q:=2(r-1)+s$ with $p,\,r,\,s=\overline{1,2}$. The point of integration is

$$P:=\left((-1)^{n-1}x,\,(-1)^{m-1}y,\,(-1)^{l-1}h\right)\in S_{lk};$$

in this case $j := 4(l-1)+2(n-1)+m$, and $k := 2(n-1)+m$ with n, $m = \overline{1,2}$ (see Fig. 2); and finally,

$$G_{|\,i-j\,|+1}(x,\, y,\, h;\, x_0,\, y_0,\, z_0) := |\,P - M_i|^{-1}.$$

Now the procedure of splitting (4) into eight independent integral equations may be applied. But, at first, we will observe that the choice of symmetry group with an order of eight is exhaustive from the point of view of electrodes systems design. Another advantage is a possibility to take into account a priori information about desired solution behavior only along a free part of congruent component S_{11} contour. Under these conditions, ignoring the weight function mentioned above, it is possible to use one of the effective methods of accuracy control of the received results. In addition, the correction of required function is carried out by use of special a posteriori error estimation and provided by net condensing in the neighborhood of S_{11} singular points.

TABL. 1. The value of potential at verified points

x	y	z	$U(x,\, y,\, z)$
0.9510	0.9510	0.5000	999.1812
0.8590	0.8590	0.5000	999.9378
1.0000	1.0000	0.4990	780.6304
0.9900	0.9900	0.4990	979.5774
0.9500	0.9500	0.4990	995.8487
0.9000	0.9000	0.4990	996.2100
0.7000	0.7000	0.4990	997.7024
0.5000	0.5000	0.4990	997.9232
0.3000	0.3000	0.4990	997.9784
0.0000	0.0000	0.4990	997.9939
0.0000	0.0000	0.4000	799.4128
0.0000	0.0000	0.2000	399.0503
0.0000	0.0000	0.1000	199.4134
1.0000	1.0000	0.4900	731.5166
1.0000	1.0000	0.4800	688.9154
1.0000	1.0000	0.4000	492.6562
1.0000	1.0000	0.2000	219.5428
1.0000	1.0000	0.1000	107.6509

Using known general concept of a posteriori error estimation methodology [5], we propose such it version applicable precisely to integral equation of type (1). Let $\sigma_h(P)$ is a solution which belongs to chosen approximation space. This solution, taking into consideration it integral representation, generates approximate value of potential in an arbitrary point M of interelectrodes space

$$U_h(M) = (A\sigma_h)(M).$$

At that time, error function e_U is defined with the help of such formula

$$e_U = A\sigma - A\sigma_h = A(\sigma - \sigma_h) = Ae_\sigma;$$

where e_σ is the solution of such integral equation

$$(Ae_\sigma)(M) = U_0 - (A\sigma_h)(M), \quad M \in S;$$

here U_0 is the given value of potential on S. The behavior of this solution is irregular only in the neighborhood of unclosed surface S contour. That is why, we reproduce e_U only over element D^e which appears in the process of S partition and where e_U may be obtained maximal value. We consider such element as "extremal". Then, it is necessary to verify the condition of the accessibility of preassigned accuracy

$$\frac{\| e_\sigma \|_{L_2(D^e)}}{\sqrt{\| \sigma_h \|^2_{L_2(D^e)} + \| e_\sigma \|^2_{L_2(D^e)}}} 100\,\% \leq TOL.$$

If the last condition is not realized, then, it is necessary to repeat stated above procedure, using more dense net, as it was shown at the figure 3. We repeat the described procedure so many times that it needs to obtain the guaranteed accuracy of equation (1) solving.

The considered numerical scheme was applied to solve one typical problem. Computations were realized with the use of some parameters: $a = 1$, $b = 1$, $h = 0.5$, $U_1 = 1000$, $U_2 = -1000$, $TOL = 0.1\%$. The solution of this problem was shows with the help of the Table 1 at some points. In this case the number of iteration to attain preassigned accuracy is 5.

Bibliography

1. Atkinson K. The Numerical Solution of Integral Equations of the Second Kind / K. Atkinson.– Cambridge: Cambridge University Press, 1997.
2. Chapko R. A direct boundary integral equation method for the numerical construction of harmonic functions in three-dimensional layered domains containing a cavity / R. Chapko, B. T. Johansson, O. Protsyuk // International Journal of Computer Mathematics.– 2012.– Vol. 89.– P. 1448-1462.
3. Garasym Ya. S. Identifying effective methods of numerical solution of two-dimensional integral equations of potential theory based on computational experiments // Visnyk of Lviv University. Appl. Math. Comput. Science Series.– 2004.– Vol. 9.– P. 47-53 (in Ukrainian).
4. Hsiao G. C. Boundary Integral Equations / G. C. Hsiao, W. L. Wendland.– Berlin: Springer-Verlag, 2008.
5. Eriksson K. Introduction to Adaptive Method for Differential Equation / K. Eriksson, D. Estep, P. Hansbo, C. Johnson // Acta Numerica.– 1995.– P. 1-54.
6. Kress R. Linear Integral Equations / R. Kress. 2nd. ed. Berlin: Springer-Verlag, 1999.
7. Morrison J. A. Charge singularity at the corner of a flat plate / J. A. Morrison, J. A. Lewis // SIAM J. Appl. Math.– 1976.– Vol. 31, № 2.– P. 233-250.
8. Sybil Yu. M. Three dimensional elliptic boundary value problems for an open Lipschtz surface / Yu. M. Sybil // Матем. студії.– 1997.– Vol. 8, № 2.– P. 79-96.

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